Hacettepe Journal of Mathematics and Statistics
Volume 45 (1) (2016), 95 – 105

Annihilator conditions related to the quasi-Baer condition

A. Taherifar*

Abstract

We call a ring R an *EGE-ring* if for each $I \trianglelefteq R$, which is generated by a subset of right semicentral idempotents there exists an idempotent e such that $r(I) = eR$. The class EGE includes quasi-Baer, semiperfect rings (hence all local rings) and rings with a complete set of orthogonal primitive idempotents (hence all Noetherian rings) and is closed under direct product, full and upper triangular matrix rings, polynomial extensions (including formal power series, Laurent polynomials, and Laurent series) and is Morita invariant. Also we call R an AE-ring if for each $I \trianglelefteq R$, there exists a subset $S \subseteq S_r(R)$ such that $r(I) = r(RSR)$. The class AE includes the principally quasi-Baer ring and is closed under direct products, full and upper triangular matrix rings and is Morita invariant. For a semiprime ring R, it is shown that R is an EGE (resp., AE)-ring if and only if the closure of any union of clopen subsets of $Spec(R)$ is open (resp., $Spec(R)$ is an EZ-space).

Keywords: Quasi-Baer ring, AE-ring, EGE-ring, $Spec(R)$, Semicentral idempotent, EZ-space.

2000 AMS Classification: Primary 16D25, 16N60; Secondary 54G05.

Received : 11.02.2014 *Accepted* : 20.11.2014 *Doi* : 10.15672/HJMS.20164512485

1. Introduction

Throughout this paper, R denotes an associative ring with identity. In this paper, we introduce and investigate the concept of EGE (resp., AE)-ring. We call R an EGE (resp., AE)-ring, if for any ideal I of R which $I = RSR$, $S \subseteq S_r(R)$ (resp., any ideal I of R) there exists an idempotent $e \in R$ (resp., a subset $S \subseteq S_r(R)$) such that $r(I) = eR$ (resp., $r(I) = r(RSR)$), where $r(I)$ (resp., $l(J)$) denotes the right annihilator (resp., left annihilator) of I.

In Section 2, we show that any quasi-Baer ring and any ring with a complete set of right (left) triangulating idempotents are EGE-ring. Hence semiperfect rings (hence all

*Department of Mathematics, Yasouj University, Yasouj, Iran.
Email : ataherifar@mail.yu.ac.ir

local rings) and rings with a complete set of orthogonal primitive idempotents (hence all Noetherian rings) are EGE-ring. We also show that any principally quasi-Baer-ring (hence, biregular rings) is an AE-ring. We provide examples of EGE (resp., AE)-rings which are not quasi-Baer (resp., principally quasi-Baer)-ring.

In Section 3, we consider the closure of the class of EGE (resp., AE)-ring with respect to various ring extensions including matrix, and polynomial extension (including formal power series, Laurent polynomials, and Laurent series). In Theorem 3.3, we obtain a characterization of semicentral idempotents in $\mathbf{M_n(R)}$ (resp., $\mathbf{T_n(R)}$). The EGE (resp., AE) property is shown to be Morita invariant in Theorem 3.6.

Topological equivalency of semiprime EGE (resp., AE)-ring is the focus of Section 4. In Theorem 4.2, we show that a semiprime ring R is an EGE (resp., AE)-ring if and only if the closure of any union of clopen subsets of $Spec(R)$ (i.e., the space of prime ideals of R), is open (resp., $Spec(R)$ is an EZ-space).

Let $\emptyset \neq X \subseteq R$. Then $X \leq R$ and $X \trianglelefteq R$ denote that X is a right ideal and X is an ideal respectively. For any subset S of R, $l(S)$ and $r(S)$ denote the left annihilator and the right annihilator of S in R. The ring of n-by-n (upper triangular) matrices over R is denoted by $\mathbf{M_n(R)}$ ($\mathbf{T_n(R)}$). We use $R[x]$, $R[[x]]$, $R[x, x^{-1}]$ and $R[[x, x^{-1}]]$ for the ring of polynomials over R, the ring of formal power series over R, the skew Laurent polynomial ring over R, and the skew Laurent series ring over R, respectively. A ring R is called (*quasi-*)*Baer* if the left annihilator of every (ideal) nonempty subset of R is generated , as a left ideal, by an idempotent. The (quasi-)Baer conditions are left -right symmetric. It is well known that R is a quasi-Baer if and only if $\mathbf{M_n(R)}$ is quasi-Baer if and only if $\mathbf{T_n(R)}$ is a quasi-Baer ring (see [2], [7], [8], [13] and [18]). An idempotent e of a ring R is called left (resp., right) semicentral if $ae = eae$ (resp., $ea = eae$) for all $a \in R$. It can be easily checked that an idempotent e of R is left (resp., right) semicentral if and only if eR (resp., Re) is an ideal. Also note that an idempotent e is left semicentral if and only if $1 - e$ is right semicentral. See [3] and [5], for more detailed account of semicentral idempotents. Thus for a left (resp., right) ideal I of a ring R, if $l(I) = Re$ (resp., $r(I) = eR$) with an idempotent e, then e is right (resp., left) semicentral, since Re (resp., eR) is an ideal. Thus for a left (resp., right) ideal I of a quasi-Baer ring R with $l(I) = Re$ (resp. $r(I) = eR$) for some idempotent $e \in R$, it follows that e is a right (resp., left) semicentral idempotent. We use $S_l(R)$ ($S_r(R)$) to denote the set of left (right) semicentrel idempotents of R. For an idempotent e of R if $S_r(R) = \{0, e\}$, then e is called *semicentral reduced*. If 1 is semicentral reduced, then we say R is *semicentral reduced*.

2. Preliminary results and examples

2.1. Definition. We call R an *EGE-ring*, if for each ideal $I = RSR$, $S \subseteq S_r(R)$, there exists an idempotent e such that $r(I) = eR$. Since for each $S \subseteq S_r(R)$, $r(RSR) = r(RS) = r(SR) = r(S)$, R is an EGE-ring if and only if for each $S \subseteq S_r(R)$, there exists an idempotent e such that $r(S) = eR$.

2.2. Definition. We call R an *AE-ring*, if for any ideal I of R there exists a subset $S \subseteq S_r(R)$ such that $r(I) = r(RSR) = r(S)$. We know that I is equivalent to J if and only if r(I) = r(J). Then R is an AE-ring if an only if every ideal of R is equivalent to one which is generated by a subset of right semicentral idempotents.

2.3. Lemma. Let e_1 and e_2 be two right semicentral idempotents.

(i) e_1e_2 is a right semicentral idempotent.
(ii) $(e_1 + e_2 - e_1e_2)$ is a right semicentral idempotent.

(iii) If $S \subseteq S_r(R)$ is finite, then there is a right semicentral idempotent e such that $RSR = ReR =< e >$.

Proof. (i) By hypothesis, for any $r \in R$ we have, $e_1e_2r = e_1e_2re_2 = e_1e_2re_1e_2$. On the other hand, $(e_1e_2)^2 = e_1e_2e_1e_2 = e_1e_2^2 = e_1e_2$. Hence $e_1e_2 \in S_r(R)$.

(ii) The routine calculation shows that $(e_1 + e_2 - e_1e_2)^2 = (e_1 + e_2 - e_1e_2)$, and by hypothesis, for any $r \in R$ we have, $(e_1 + e_2 - e_1e_2)r = e_1r + e_2r - e_1e_2r = e_1re_1 + e_2re_2 - e_1e_2re_2 = (e_1 + e_2 - e_1e_2)r(e_1 + e_2 - e_1e_2)$. Hence $(e_1 + e_2 - e_1e_2) \in S_r(R)$.

(iii) We use induction. If $S = \{e_1, e_2\}$, then we have $< e_1, e_2 >=< e_1 + e_2 - e_1e_2 >$. By (ii), $e_1 + e_2 - e_1e_2 \in S_r(R)$. Now let the statement is true for $|S| = n$ and let $S = \{e_1, ..., e_n, e_{n+1}\}$. Then we have $< S >=< \{e_1, ..., e_n\} > + < e_{n+1} >$. By hypothesis, there is a right semicentral idempotent f such that $< \{e_1, ..., e_n\} >=< f >$. Hence $< S >=< f + e_{n+1} - fe_{n+1} >$, where by (ii), we have $e = f + e_{n+1} - fe_{n+1} \in S_r(R)$.□

Recall that an ordered set $\{b_1, ..., b_n\}$ of nonzero distinct idempotents in R is called a set of *right triangulating idempotents* of R if all the following hold:

(i) $1 = b_1 + ... + b_n$;
(ii) $b_1 \in S_r(R)$; and
(iii) $b_{k+1} \in S_r(c_kRc_{k+1})$, where $1 = 1 - (b_1 + ... + b_k)$, for $1 \leq k \leq n$.

Similarly is defined a set of *left triangulating idempotents* of R using (i), $b_1 \in S_l(R)$ and $b_{k+1} \in S_l(c_kRc_k)$. From part (iii) of the above definition, a set of right (left) triangulating idempotents is a set of pairwise orthogonal idempotents.

A set $\{b_1, ..., b_n\}$ of right (left) triangulating idempotents of R is said to be *complete* if each b_i is also semicentral reduced (see [11]).

2.4. Proposition. The following statements hold.

(i) Any ring R with finite triangulating dimension (equivalently, R has a complete set of right (left)triangulating idempotents) is an EGE-ring.
(ii) A ring R is quasi-Baer if and only if R is EGE and AE.

Proof. (i) By [5, Theorem 2.9], R has a complete set of right traingulating idempotents if and only if $\{Rb : b \in S_r(R)\}$ is finite. Now let $I = RSR$ be an ideal of R and $S \subseteq S_r(R)$. Then we have $r(I) = r(RS) = r(\{Rb : b \in S\})$. But $\{Rb : b \in S\}$ is finite say $\{Rb_1, ..., Rb_n\}$. Hence $r(I) = r(\{Rb_1, ..., Rb_n\}) = r(\{b_1, ..., b_n\})$. By Lemma 2.3, there exists a right semicentral idempotent e such that $r(I) = r(\{b_1, ..., b_n\}) = r(eR) = r(Re) = (1 - e)R$. Thus R is an EGE-ring.

(ii) By definition, any quasi-Baer ring is an EGE-ring. If I is an ideal of a quasi-Baer ring R, then there is $e \in S_l(R)$ such that $r(I) = eR = r(R(1 - e))$. On the other hand for each $S \subseteq S_r(R)$ we have $r(RS) = r(SR) = r(RSR)$, hence $r(I) = r(RSR)$, where $S = \{1 - e\}$, and $S \subseteq S_r(R)$. Hence R is an AE-ring. Conversely, let $I \trianglelefteq R$. Then by hypothesis, there exists a subset $S \subseteq S_r(R)$ such that $r(I) = r(RSR)$. Again by hypothesis, there is an idempotent e such that $r(RSR) = eR$. Thus $r(I) = eR$. □

2.5. Example. By Proposition 2.4, all of the rings mentioned in Proposition 2.14 of [5], are EGE-rings. Note that this list includes semiperfect rings (hence all local rings, left or right artinian rings) and rings with a complete set of orthogonal primitive idempotents (hence all Noetherian rings) and many more rings.

Recall that, a ring R is *right* (resp., *left*) *principally quasi-Baer* (or simply right (resp., left) *pq*-Baer) if the right (resp., left) annihilator of a principally right (resp/ left) ideal is generated (as a right (resp., left) ideal) by an idempotent (see [9]).

2.6. Proposition. The following statements hold.

(i) R is an EGE ring if and only if for each $I \trianglelefteq R$, which is generated by a subset $S \subseteq S_l(R)$, we have $l(I) = Re$, for some idempotent $e \in R$.

(ii) R is an AE-ring if and only if for each $a \in R$ there exists a subset $S_a \subseteq S_r(R)$ such that $r(RaR) = r(aR) = r(S_a)$.

(iii) Every right principally quasi-Baer ring is an AE-ring.

Proof. (i) Let $I = RSR$, where $S \subseteq S_l(R)$. Take $J = RKR$, $K = \{1 - s : s \in S\}$. Then $K \subseteq S_r(R)$. By hypothesis and Lemma 2.3, there is $e \in S_l(R)$ such that $r(J) = r(KR) = r(RK) = eR$. Hence for each $s \in S$, $(1 - s)e = 0$, so $e = se$. Therefore $Re = SRe$. This implies that $l(RSR) = l(RS) = l(SR) = l(SRe) = l(Re) = l(eR) = R(1 - e)$. Similarly we can get the converse.

(ii) By definition, $\Rightarrow$ is evident.

$\Leftarrow$ Now let $I \trianglelefteq R$. We have $r(I) = \bigcap_{a\in I} r(RaR)$. By hypothesis, for each $a \in R$ there exists $S_a \subseteq S_r(R)$ such that $r(RaR) = r(RS_aR)$. Hence $r(I) = \bigcap_{a\in I} r(RS_aR) = r(R(\bigcup_{a\in I} S_a)R)$.

(iii) Let $a \in R$. Then there is an idempotent $e \in R$ such that $r(RaR) = r(aR) = eR = r(R(1-e)) = r((1-e)R) = r(R(1-e)R)$. We know that $1-e$ is a right semicentral idempotent. By (ii), R is an AE-ring.

A ring R is called *biregular* if every principal ideal of R is generated by a central idempotent of R (see [8]). Note that a biregular ring is *pq*-Baer. Hence any biregular ring is an AE-ring.

Recall from [20] that a topological space X is an *EZ-space* if for every open subset A of X there exists a collection $\{A_\alpha : \alpha \in S\}$ of clopen (i.e., sets that are simultaneously closed and open) subsets of X such that $cl_X A = cl_X(\bigcup_{\alpha\in S} A_\alpha)$. We denote by $C(X)$, the ring of all real-valued continuous functions on a completely regular Hausdorff space X. For any $f \in C(X)$, $Z(f) = \{x \in X : f(x) = 0\}$ is called a zero-set. A topological space X is called *extremally disconnected* (resp., *basically disconnected*), if the interior of any open set (resp., the interior of any zero-set) is closed. Clearly any extremally disconnected space is an EZ-space, but there exist EZ-spaces which are not extremally disconnected (resp., basically disconnected) (see [20]). It is clear that a subset A of X is clopen if and only if $A = Z(f)$ for some idempotent $f \in C(X)$. For terminology and notations, the reader is referred to [15] and [14]. For any subset A of X we denote by $intA$ the interior of A (i.e., the largest open subset of X contained in A).

In the following, we provide examples of commutative AE and non-commutative EGE rings which are not quasi-Baer. We need the following lemma which is Corollary 2.2 in [1].

2.7. Lemma. For $f, g \in C(X)$, $r(f) = r(g)$ if and only if $intZ(f) = intZ(g)$.

2.8. Example. By [20, Theorem 3.7], $C(X)$ is an AE-ring if and only if X is an EZ-space. On the other hand by [1], we have $C(X)$ is a *pq*-Baer ring if and only if X is a basically disconnected space. So, if X is an EZ-space which is not basically disconnected space (e.g., [20, Example 3.4]), then $C(X)$ is an AE-ring but is not a *pq*-Baer ring. By Proposition 2.4 (ii), $C(X)$ is not an EGE-ring.

2.9. Example. The ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix} = \{\begin{pmatrix} n & a \\ 0 & b \end{pmatrix} : n \in \mathbb{Z}, a, b \in \mathbb{Z}_2\}$ has a finite number of right semicentral idempotents. By Proposition 2.4, R is an EGE-ring. But R is not a quasi-Baer ring. If $I = \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$, then we have $l(I) = \begin{pmatrix} 2\mathbb{Z} & 0 \\ 0 & 0 \end{pmatrix}$, which does not contain any idempotent. By Proposition 2.4 (ii), R is not an AE-ring.

2.10. Theorem. Let $R = \prod_{x\in X} R_x$ be a direct product of rings.

(i) R is an EGE-ring if and only if each R_x is an EGE ring.

(ii) R is an AE-ring if and only if each R_x is an AE ring.

Proof. (i) Assume that R is an EGE-ring. Choose $x \in X$. Let $I_x \trianglelefteq R_x$ and $I_x =< K_x >$, where $K_x \subseteq S_r(R_x)$ and $h_x : R_x \to R$ be the canonical homomorphism. Then $h_x(I_x) \trianglelefteq R$, $h_x(I_x) =< h_x(K_x) >$ and $h_x(K_x) \subseteq S_r(R)$. So there exists an idempotent $e \in R$ such that $r(h_x(I_x)) = eR$. Let $\pi_x : R \to R_x$ denote the canonical projection homomorphism. Then $\pi_x(e)$ is an idempotent in R_x and $r(I_x) = \pi_x(e)R_x$.

Conversely, assume that R_x is an EGE-ring for all $x \in X$. Let $I \trianglelefteq R$ and $I =< K >$, $K \subseteq S_r(R)$. Then $I_x = \pi_x(I) =< \pi_x(K) >=< K_x >$. It is easy to see that $K_x \subseteq S_r(R)$ for each $x \in X$. Hence there exists an idempotent $e_x \in R_x$ such that $r(I_x) = e_xR_x$ for each $x \in X$. Let $e = (e_x)_{x\in X}$. Then e is an idempotent in R and $r(I) = eR$.

(ii) Let R be an AE-ring. For $x \in X$, suppose that $a_x \in R_x$. Then there is $a \in R$ such that $\pi_x(a) = a_x$. By hypothesis, there exists $S \subseteq S_r(R)$ such that $r(RaR) = r(RSR)$. Now we can see that $r(R_xa_xR_x) = r(R_xS_xR_x)$, where $S_x = \pi_x(S) \subseteq S_r(R_x)$. By Proposition 2.6, R_x is an AE-ring. Conversely, suppose that $a \in R$. Then $\pi_x(a) = a_x \in R_x$ for each $x \in X$. By hypothesis, for each $x \in X$ there exists $S_x \subseteq S_r(R_x)$ such that $r(R_xa_xR_x) = r(R_xS_xR_x)$. Now let $S = \prod_{x\in X} S_x$. Then $S \subseteq S_r(R)$ and $r(RaR) = r(RSR)$. By Proposition 2.6, R is an AE-ring. □

3. Extensions of EGE and AE-rings

In this section, we investigate the behavior of the EGE (rep., AE)-ring property with respect to various ring extensions including matrix, polynomial, and formal power series. Also semicentral idempotents in $\mathbf{M_n(R)}$ (resp., $\mathbf{T_n(R)}$) are investigated.

The following Lemma is Lemma 3.1 in [4].

3.1. Lemma. Let R be a ring and $S = \mathbf{M_n(R)}$.

(i) Then $J \trianglelefteq S$ if and only if $J = \mathbf{M_n(I)}$, for some $I \trianglelefteq R$.
(ii) If $I \trianglelefteq R$, then $r_S(\mathbf{M_n(I)}) = \mathbf{M_n(r_R(I))}$.

3.2. Lemma. The following statements hold.

(i) If R is an EGE-ring and e is an idempotent, then eRe is an EGE-ring.
(ii) If R is an AE-ring and e is an idempotent, then eRe is an AE-ring.

Proof. (i) Let $I \trianglelefteq eRe$ and $I = eReKeRe$, where $K \subseteq S_r(eRe)$. For each $exe \in K$ and $r \in R$, we have $(exe)(re) = (exe)(ere) = (exe)(ere)(exe) = (exe)(re)(exe)$. So $K \subseteq S_r(Re)$. Now let $J = ReKRe$. Then $J \trianglelefteq Re$. By hypothesis and Theorem 2.10, Re is an EGE-ring, hence there is an idempotent $f \in Re$ such that $r_{Re}(J) = fRe$. Now we claim that $r_{eRe}(I) = (ef)(eRe)$. For see this, let $exe \in r_{eRe}(eReKeRe)$. Then we have $exe \in r_{eRe}(eKRe) = r_{eRe}(ReKRe)$, so $xe \in r_{Re}(ReKRe)$. This says that $r_{eRe}(I) \subseteq (ef)(eRe)$. Therefore $xe = fse$ for some $s \in R$. But $f = fe$, so $exe = (ef)(ere)$. On the other hand we have $f \in r_{Re}(ReKRe)$. This implies that $Ief = 0$, thus $(ef)(eRe) \subseteq r_{eRe}(I)$.

(ii) Assume that $I \trianglelefteq eRe$. Then $I \leq Re$. By hypothesis and Theorem 2.10, Re is an AE-ring. Hence there exists $S \subseteq S_r(Re)$ such that $r_{Re}(I) = r_{Re}(ReSRe)$. We have $eSe(eRe)eSe = eS(Re) = eSReS = eS(eRe)eS$ and for each $s \in S$, $(es)^2 = eses = es^2 = es$. This shows that $eS = eSe \subseteq S_r(eRe)$. Now we claim that $r_{eRe}(I) = r_{eRe}(eRe(eSe)eRe) = r_{eRe}(eReSRe)$. Let $exe \in r_{eRe}(I)$. Then $Iexe = Ixe = 0$. So $xe \in r_{Re}(I) = r_{Re}(ReSRe)$. Therefore $ReSRexe = 0$. This implies that $exe \in r_{eRe}(ReSRe) \subseteq r_{eRe}(eReSRe)$. Now suppose that $exe \in r_{eRe}(eReSRe)$. Then $exe \in r_{eRe}(eSRe) = r_{eRe}(ReSRe)$. Hence $xe \in r_{Re}(ReSRe) = r_{Re}(I)$. Thus $Iexe = Ixe = 0$. This shows that $exe \in r_{eRe}(I)$. □

In the following Theorem, we characterize semicentral idempotents in $\mathbf{M_n(R)}$ and $\mathbf{T_n(R)}$.

3.3. Theorem. The following statements hold.

(i) $A = [a_{ij}] \in S_r(\mathbf{M_n(R)})$ if and only if we have;

(a) $a_{11} \in S_r(R)$.

(b) $a_{ij} = a_{ij}a_{11}$ for all $1 \leq i, j \leq n$.

(c) For each $1 \leq i \leq n$, $a_{11}a_{ii} = a_{11}$ and $a_{11}a_{ij} = 0$ for all $1 \leq j \neq i \leq n$.

(ii) $A = [a_{ij}] \in S_r(\mathbf{T_n(R)})$ if and only if we have;

(d) For each $1 \leq i \leq n$, $a_{ii} \in S_r(R)$.

(e) For each $1 \leq i \leq n$, $a_{ki} = a_{ki}a_{ii}$ for all $1 \leq k \leq i$ and $a_{ii}a_{ij} = 0$ for all $i < j \leq n$.

Proof. (i)$\Rightarrow$ First we show that (a) holds. Suppose that $r \in R$. Consider $B = [b_{ij}]$, where $b_{11} = r$, and $b_{ij} = 0$ for all $i \neq 1$, $j \neq 1$. Then by hypothesis, $ABA = AB$. This implies that $a_{11}ra_{11} = a_{11}r$, so $a_{11} \in S_r(R)$.

(b) Let $B = [b_{ij}]$, where $b_{j1} = 1$ and $b_{ik} = 0$ for each $i \neq j$ and $k \neq 1$. By hypothesis, $ABA = AB$, so we have $a_{ij}a_{11} = a_{ij}$ for all $1 \leq i, j \leq n$.

(c) For fixed i, consider $B = [b_{ij}]$, where $b_{1i} = 1$ and other entries are zero. Then $ABA = AB$ implies that $a_{11}a_{ii} = a_{11}$ and $a_{11}a_{ij} = 0$ for all $1 \leq j \neq i \leq n$.

(i)$\Leftarrow$ $a_{11} \in S_r(R)$ implies that $D = [d_{ij}] \in S_r(\mathbf{M_n(R)})$, where $d_{ii} = a_{11}$ and other entries are zero. On the other hand, by (b) and (c), we can see that $A = AD$ and $DA = D$. Hence, for $B \in \mathbf{M_n(R)}$ we have $ABA = ADBA = ADBDA = ADBD = ADB = AB$. Therefore $A \in S_r(\mathbf{M_n(R)}$.

(ii)$\Rightarrow$ (d) The proof of this part is analogous to that of part (a).

(e) For $B = [b_{ij}]$, where $b_{ii} = 1$ and other entries are zero. We have $ABA = AB$. Therefore $a_{ki} = a_{ki}a_{ii}$ for all $1 \leq k \leq i$ and $a_{ii}a_{ij} = 0$ for all $i < j \leq n$.

(ii)$\Leftarrow$ If $a_{ii} \in S_r(R)$, then $D = [d_{ij}] \in S_r(\mathbf{T_n(R)})$, where $d_{ii} = a_{ii}$ and other entries are zero. On the other hand, by (e), we can see that $A = AD$ and $DA = D$. Hence for $B \in \mathbf{T_n(R)}$, we have $ABA = ADBA = ADBDA = ADBD = ADB = AB$. Therefore $A \in S_r(\mathbf{T_n(R)}$. □

3.4. Lemma. If $J \trianglelefteq M_n(R)$ and $J =< S >$, where $S \subseteq S_r(\mathbf{M_n(R)})$, then there is $I \trianglelefteq R$ generated by a subset of right semicentral idempotents of R such that $J = M_n(I)$.

Proof. By argument of [16, Theorem 3.1], $J = M_n(I)$, where I is the set of all $(1,1)$-entries of matrices in J. Now let S_{11} be the set of all $(1,1)$-entries of matrices in S. By Theorem 3.3, $S_{11} \subseteq S_r(R)$, and we can see that $I = RS_{11}R$. □

3.5. Proposition. The following statements hold.

(i) R is an EGE-ring if and only if $\mathbf{M_n(R)}$ is an EGE-ring.

(ii) R is an AE-ring if and only if $\mathbf{M_n(R)}$ is an AE-ring.

Proof. (i) Let J be an ideal of $\mathbf{M_n(R)}$ and $J =< S >$, where $S \subseteq S_r(\mathbf{M_n(R)})$. By Lemma 3.4, there exists $I \trianglelefteq R$, where $I =< S_1 >$ for some $S_1 \subseteq S_r(R)$ and $J = \mathbf{M_n(I)}$. By Lemma 3.1 and hypothesis, we have $r(J) = \mathbf{M_n(r(I))} = \mathbf{M_n(eR)}$ for some idempotent e in R. Hence $r(J) = E\mathbf{M_n(R)}$, where in matrix E for each $1 \leq i \leq n$, $E_{ii} = e$ and $E_{ij} = 0$ for all $i \neq j$. Conversely, we have $E\mathbf{M_n(R)E} \simeq \mathbf{R}$, where in matrix E, $E_{11} = 1$ and for each $i \neq 1$ and $j \neq 1$, $E_{ij} = 0$. Now by Lemma 3.2, R is an EGE-ring.

(ii) Let J be an ideal of $\mathbf{M_n(R)}$. By Lemma 3.1, there is an ideal I of R such that $J = M_n(I)$, and $r(J) = r(\mathbf{M_n(I)}) = \mathbf{M_n(r(I))}$. By hypothesis, there exists $S \subseteq S_r(R)$ such that $r(I) = r(RSR)$. Hence $r(J) = \mathbf{M_n(r(RSR))} = \mathbf{r(M_n(RSR))}$. On the other hand, we can see that $\mathbf{M_n(RSR)} = \mathbf{M_n(R)D_n(S)M_n(R)}$, where $\mathbf{D_n(S)}$ is the set of diagonal matrices over S, and $\mathbf{D_n(S)} \subseteq \mathbf{S_r(M_n(R))}$. Thus $r(J) = r(\mathbf{M_n(R)D_n(S)M_n(R)})$. Conversely, by Lemma 3.2, it is obvious. □

3.6. Theorem. The following statements hold.

(i) The EGE property is a Morita invariant.
(ii) The AE property is a Morita invariant.

Proof. These results are consequences of Lemma 3.2, Proposition 3.5 and [17, Corollary 18.35]. □

3.7. Theorem. The following statements hold.

(i) R is an EGE-ring if and only if $\mathbf{T_n(R)}$ is an EGE-ring.
(ii) R is an AE-ring if and only if $\mathbf{T_n(R)}$ is an AE-ring.

Proof. (i)$\Leftarrow$ Assume that $\mathbf{T_n(R)}$ is an EGE-ring. Then we have $E\mathbf{T_n(R)E} \simeq \mathbf{R}$, where in matrix E, $E_{11} = 1$ and other entries are zero. Now by Lemma 3.2, R is an EGE-ring.

(i)$\Rightarrow$ Let I be an ideal of $T_n(R)$ which is generated by $S = \{A_\alpha : \alpha \in K\} \subseteq S_r(\mathbf{T_n(R)})$. By Theorem 3.3, for each $\alpha \in K$ and $1 \leq i \leq n$, we have $(a_{ii})_\alpha \in S_r(R)$, where $(a_{ii})_\alpha$ is the (i,i)-th, entries in A_α. Now for each $1 \leq i \leq n$, let J_i be the ideal generated by $\{(a_{ii})_\alpha : \alpha \in K\}$ in R. By hypothesis, for each $1 \leq i \leq n$ there is an idempotent $e_i \in R$ such that $r(J_i) = e_iR$. We claim that $r(I) = ET_n(R)$ where for each $1 \leq i \leq n$, $E_{ii} = e_i$ and $E_{ij} = 0$, for all $i \neq j$. By Theorem 3.3, we can see that; for each $\alpha \in K$ there exists a diagonal matrix D_α such that $A_\alpha = A_\alpha D_\alpha$, where $(D_\alpha)_{ii} = (A_\alpha)_{ii}$. So, for each $\alpha \in K$ we have $A_\alpha E = A_\alpha D_\alpha E = 0$. Now let $A \in I$. Then we have $A = \sum_{i=1}^n B_iA_iC_i$, where $A_i \in S$ and $B_i, C_i \in \mathbf{T_n(R)}$. Therefore $AE = (\sum_{i=1}^n B_iA_iC_i)E = \sum_{i=1}^n B_iA_iC_iA_iE = 0$. Hence $E \in r(I)$.

Now suppose that $B \in r(I)$ and $x \in J_i =< (a_{ii})_\alpha : \alpha \in S >$. Then $A \in I$ where $a_{ii} = x$ and other entries are zero. So we have

$$AB = \begin{pmatrix} 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ xb_{i1} & xb_{i2} & . & . & . & xb_{in} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 \end{pmatrix}_{n\times n} = 0.$$

This equality implies that $b_{ij} \in r_R < (a_{ii})_\alpha : \alpha \in S >$ for each $1 \leq j \leq n$. Hence for fixed i and each $1 \leq j \leq n$ there is $r_{ij} \in R$ such that $b_{ij} = e_ir_{ij}$. Therefore we have

$$B = \begin{pmatrix} e_1r_{11} & e_1r_{12} & . & . & . & e_1r_{1n} \\ 0 & e_2r_{22} & . & . & . & e_2r_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & e_nr_{nn} \end{pmatrix}_{n\times n} = E \times \begin{pmatrix} r_{11} & r_{12} & . & . & . & r_{1n} \\ 0 & r_{22} & . & . & . & r_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & r_{nn} \end{pmatrix}_{n\times n}.$$

Thus $B \in ET_n(R)$.

(ii) Let $I \trianglelefteq T_n(R)$. Then

$$I = \begin{pmatrix} I_{11} & I_{12} & . & . & . & I_{1n} \\ 0 & I_{22} & . & . & . & I_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & I_{nn} \end{pmatrix},$$

where each $I_{ij} \trianglelefteq R$, $I_{ij} = \{0\}$ for all $i > j$, $I_{ij} \subseteq I_{ik}$ for all $k \geq j$, and $I_{hj} \subseteq I_{ij}$ for all $h \geq i$. Therefore

$$r_{T_n(R)}(I) = \begin{pmatrix} r_R(I_{11}) & r_R(I_{11}) & . & . & . & r_R(I_{11}) \\ 0 & r_R(I_{12}) & . & . & . & r_R(I_{12}) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & r_R(I_{1n}) \end{pmatrix}.$$

By hypothesis, for each $1 \leq i, j \leq n$, there exists $S_{ij} \subseteq S_r(R)$ such that $r_R(I_{ij}) = r_R(S_{ij})$. This implies that

$$r_{T_n(R)}(I) = r_{T_n(R)}(\begin{pmatrix} S_{11} & 0 & 0 & . & . & 0 \\ 0 & S_{12} & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & S_{1n} \end{pmatrix}).$$

On the other hand, it is easy to see that $\begin{pmatrix} S_{11} & 0 & 0 & . & . & 0 \\ 0 & S_{12} & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & S_{1n} \end{pmatrix} \subseteq S_r(T_n(R))$. So we are done. □

We need the following lemma which is Lemma 1.7 in [10].

3.8. Lemma. For a ring R, let T be $R[x, x^{-1}]$ or $R[[x, x^{-1}]]$. If $e(x) \in S_r(T)$ then $e_0 \in S_r(R)$ where e_0 is the constant term of $e(x)$. Moreover, $Te(x) = Te_0$.

Also we need the following lemma which is Proposition 3 in [12].

3.9. Lemma. Let $e(x) = \sum_{i=0}^{\infty} e_i x^i$. Then $e(x) \in S_l(R[[x]])$ if and only if

(i) $e_0 \in S_l(R)$.;

(ii) $e_0 r e_i = r e_i$ and $e_i r e_0 = 0$, for all $r \in R$, $i = 1, 2, ...$;

(iii) $\sum_{\substack{i+j=k \\ i,j \geq 1}} e_i r e_j = 0$, for all $r \in R$ and $k \geq 2$.

3.10. Theorem. Let R be a ring and X an arbitrary nonempty set of not necessarily commuting indeterminates. Then the following conditions are equivalent:

(i) R is EGE;
(ii) $R[X]$ is EGE;
(iii) $R[[X]]$ is EGE;
(iv) $R[x, x^{-1}]$ is EGE;
(v) $R[[x, x^{-1}]]$ is EGE.

Proof. We will prove the equivalency of (i) and (iv). The equivalency of other cases can be shown similarly, by Lemmas 3.8, 3.9 and [6, Proposition 2.4(iv)]. (i)⇒(iv), let $T = R[x, x^{-1}]$ and $I = TST$, where $S \subseteq S_r(T)$. Let S_0 be the set of all constant elements of S. Then by Lemma 3.8, $S_0 \subseteq S_r(R)$ and RS_0R is an ideal of R. By hypothesis, there exists an idempotent $e \in R$ such that $r_R(RS_0R) = eR$. Now we claim that $r_T(TST) = r_T(S) = eT$. Assume that $e(x) \in S$. Then $e_0 \in S_0$, where e_0 is the constant term of $e(x)$. By Lemma 3.8, we have $e(x) = e(x)e_0$, so $e(x)e = e(x)e_0e = 0$. This implies that $eT \subseteq r_T(S)$. Now let $g(x) \in r_T(S)$. For each $f_0 \in S_0$, there exists $f(x) \in S$ such that f_0 is the constant term of $f(x)$. By Lemma 3.8, we have $f_0 = f_0 f(x)$.

Therefore $f_0g(x) = f_0f(x)g(x) = 0$. Thus $f_0g_i = 0$, where g_i is the i-th coefficient in $g(x)$. Hence $g_i \in r_R(S_0) = eR$. This shows that $g(x) \in eT$.

(iv)⇒(i), let $T = R[x, x^{-1}]$ and $I = RSR$, where $S \subseteq S_r(R)$. Then $r_T(TST) = e(x)T$ for some idempotent $e(x) \in T$. Since $Se(x) = 0$, it follows that $Se_0 = 0$ and hence $e_0 \in r_R(S) = r_R(I)$, where e_0 is the constant term of $e(x)$. Conversely, suppose that $b \in r_R(I)$. Then $b \in r_T(TST)$ and hence $b = e(x)b$. Thus $b = e_0b \in e_0R$. Therefore $r_R(I) = e_0R$. Since $e(x) \in S_r(T)$, it follows that e_0 is an idempotent in R by Lemma 3.8. Therefore R is an EGE-ring.

4. Semiprime EGE (resp., AE)-ring

In this section, we show that for a semiprime ring R, the EGE- condition (resp., AE-condition) is equivalent to the closure of any union of clopen subsets of $Spec(R)$ is clopen (resp., $Spec(R)$ is an EZ-space).

For any $a \in R$, let $supp(a) = \{P \in$Spec(R): $a \notin P\}$. Shin [19, Lemma 3.1] proved that for any R, $\{supp(a) : a \in R\}$ forms a basis of open sets on $Spec(R)$. This topology is called *hull-kernel topology*. We mean of $V(I)$ is the set of $P \in Spec(R)$, where $I \subseteq P$. We use $V(I)(V(a))$ to denote the set of $P \in Spec(R)$, where $I \subseteq P(a \in P)$. Note that $V(I) = \bigcap_{a\in I} V(a)$ (resp., $supp(I) = Spec(R) \setminus V(I)$) and $V(a) = Spec(R) \setminus supp(a)$.

For an open subset A of $Spec(R)$, suppose that $O_A = \{a \in R : A \subseteq V(a)\}$. We can see that $O_A = \bigcap_{P\in A} P$ and $V(O_A) = clA$, where clA is the cluster points of A in $Spec(R)$.

4.1. Lemma. Let R be a semiprime ring.

(i) For any $a \in R$, and any ideal I of R, $supp(a) \cap supp(I) = supp(Ia)$.
(ii) If I and J are two ideals of R, then $r(I) \subseteq r(J)$ if and only if $intV(I) \subseteq intV(J)$.
(iii) $A \subseteq Spec(R)$ is a clopen subset if and only if there exists a central idempotent $e \in R$ such that $A = V(e) = supp(1-e)$.
(iv) For open subsets A, B of $Spec(R)$, $O_A = O_B$ if and only if $clB = clA$.
(v) For any ideal I of R, $r(I) = O_{supp(I)}$.

Proof. For the proof of (i), (ii) and (iii) see [4, Lemma 4.2].

(iv) If $O_A = O_B$, then $clA = V(O_A) = V(O_B) = clB$. On the other hand for any subset A of $Spec(R)$ we have $O_{clA} = O_A$, so $clA = clB$ implies that $O_A = O_B$.

(v) If $x \in r(I)$, then $ax = 0$, for all $a \in I$, so $supp(I) \subseteq V(r)$. This shows that $x \in O_{supp(I)}$. Now $x \in O_{supp(I)}$, implies that $supp(I) \subseteq V(x)$. By (i), $supp(Ix) = supp(I) \cap supp(x) = \emptyset$, so $Ix = 0$. This shows that $x \in r(I)$. □

Note that if A is a subset of a topological space X, then $X \setminus intA = cl(X \setminus A)$.

4.2. Theorem. Let R be a semiprime ring.

(i) R is an EGE-ring if and only if the closure of any union of clopen subsets of $X = Spec(R)$ is clopen.
(ii) R is an AE-ring if and only if $X = Spec(R)$ is an EZ-space.

Proof. (i) For each $\alpha \in S$, let A_α be a clopen subset of X. Then by Lemma 4.1, for each $\alpha \in S$ there exists a central idempoten $e_\alpha \in R$ (since in a semiprime ring R semicentral idempotents are central) such that $A_\alpha = Supp(e_\alpha)$. Now let $I =< e_\alpha : \alpha \in S >$. By hypothesis, there is an idempotent $e \in R$ such that $r(I) = eR = r(R(1-e))$. Now by lemma 4.1, $intV(I) = V(1-e)$. Therefore we have $cl(\bigcup_{\alpha\in S} A_\alpha) = cl(\bigcup_{\alpha\in S} supp(e_\alpha)) = X \setminus int(\bigcap_{\alpha\in S} V(e_\alpha)) = X \setminus intV(I) = X \setminus V(1-e) = supp(1-e)$. Hence $cl(\bigcup_{\alpha\in S} A_\alpha)$ is open.

Conversely, let $I =< e_\alpha : \alpha \in S >$, where for each $\alpha \in S$, e_α is a right semicentral idempotent (hence a central idempotent). Then $K = \{V(e_\alpha) : \alpha \in S\}$ is a subset of clopen subsets of X. By hypothesis, $intV(I)$ is a clopen subset, because we have,

$$cl(\bigcup_{\alpha\in S} V(1-e_\alpha)) = X \setminus int(\bigcap_{\alpha\in S} V(e_\alpha)) = X \setminus intV(I).$$

Hence by Lemma 4.1, there is an idempotent $e \in R$ such that $intV(I) = V(e) = V(Re)$. Again by Lemma 4.1, $r(I) = r(Re) = (1-e)R$. Thus R is an EGE-ring.

(ii) Let A be an open subset of $Spec(R)$. Then there exists a subset K of R such that $A = supp[K]$. Now suppose that I be the ideal generated by K in R. Then by hypothesis and Lemma 4.1, there exists a subset E of central idempotents of R such that $O_A = r(I) = r(RER) = O_{supp[E]}$. Therefore, by Lemma 4.1, we have $cl(A) = cl(supp[E])$. Conversely, let I be an ideal of R. Then we have $supp(I)$ is an open subset of $Spec(R)$. By hypothesis, there exists a collection $\{A_\alpha : \alpha \in S\}$ of clopen subsets of $Spec(R)$ such that $cl(supp(I)) = cl(\bigcup_{\alpha\in S} A_\alpha)$. By Lemma 4.1, for each $\alpha \in S$ there exists an idempotent e_α such that $A_\alpha = supp(e_\alpha)$. Therefore, $cl(supp(I)) = cl(\bigcup_{\alpha\in S} supp(e_\alpha))$. Again by Lemma 4.1, we have $r(I) = r(RER) = r(E)$ where $E = \{e_\alpha : \alpha \in S\}$. □

Recall that a ring R is a right SA-ring if for each $I, J \trianglelefteq R$ there exists $K \trianglelefteq R$ such that $r(I) + r(J) = r(K)$ (see [4]). By [4, Theorem 4.4], a semiprime ring R is a right SA-ring if and only if the space of prime ideals of R is an extremally disconnected space if and only if R is a quasi-Baer ring. Hence by Proposition 2.4, R is a right SA if and only if R is EGE and AE.

ACKNOWLEDGEMENTS

The author wishes to thank the referee for her/his thorough reading of this paper and her/his comments which led to a much improved paper. The author also would like to thank Professor Gary F. Birkenmeier for his encouragement and discussion on this paper, particularly for suggestion which led to an improvement of Proposition 2.4.

References

[1] Azarpanah, F and Karamzadeh, O.A. S. *Algebraic characterization of some disconnected spaces, Ital. J. Pure Appl. Math.* 12*,* 155–168*,* 2002.

[2] Berberian, S.K. Baer*-rings, Springer Berlin, (1972).

[3] Birkenmeier, G.F. *Idempotents and completely semiprime ideals, Commun. Algebra.* 11*,* 567–580*,* 1983.

[4] Birkenmeier, G.F. Ghirati, M and Taherifar, A. *When is a sum of annihilator ideals an annihilator ideal? Commun. Algebra.* 43*,* 2690-2702*,* 2015.

[5] Birkenmeier, G.F. Heatherly, H.E. Kim, J.Y and Park, J.K. *Triangular matrix representations, Journal of Algebra* 230*,* 558–595*,* 2000.

[6] Birkenmeier, G.F. and Huang, F.-K. Annihilator conditions on polynomials, Commun. Algebra, 29, 2097–2112, 2001.

[7] Birkenmeier, G.F. Kim, J.Y and Park, J.K. *A sheaf representation of quasi-Baer rings, Journal of Pure and Applied Algebra.* 146*,* 209–223*,* 2000.

[8] Birkenmeier, G.F. Kim , J.Y and Park, J.K. *Quasi-Baer ring extensions and biregular rings, Bull. AUSTRAL. Math. Soc.* 16*,* 39–52*,* 2000.

[9] Birkenmeier, G.F. Kim , J.Y and Park, J.K. *Principally Quasi-Baer Rings, Commun. Algebra,* 29*,* 639–660*,* 2001.

[10] Birkenmeier, G.F. Kim , J.Y and Park, J.K. *Polynomial extensions of Baer and quasi-Baer rings, Journal of Pure and Applied Algebra,* 159*,* 25–42*,* 2001.

[11] Birkenmeier, G.F. Kim , J.Y and Park, J.K. *Triangular matrix representations of semiprimary rings, Journal of Algebra and Its Applications* 1(2)*,* 123–131*,* 2002.

[12] Cheng, Y and Huang, F.K. *A note on extensions of principally quasi-baer rings, Taiwanese journal of mathematics.* 12(7)*,* 1721–1731*,* 2008.

[13] Clark, V. *Twisted matrix units semigroup algebra, Duke math. J.* 34*,* 417–424*,* 1967.

[14] Engelking, R. *General Topology, PWN-Polish Sci. Publ,* (1977).

[15] Gillman, L and Jerison, M. *Rings of Continuous Functions, Springer,* (1976).

[16] Lam, T.Y. *A First Course in Non-Commutative Rings, New York. Springer* (1991).

[17] Lam, T.Y. *Lecture on Modules and Rings, Springer, New York* (1999).

[18] Pollinigher, A and Zaks, A. *On Baer and quasi-Baer rings, Duke math. J.* 37*,* 127–138*,* 1970.

[19] Shin, G. *Prime ideals and sheaf representation of a pseudo symmetric ring, Trans. Amer. Math. Soc.* 184*,* 43–60*,* 1973.

[20] Taherifar, A. *Some new classes of topological spaces and annihilator ideals, Topol. Appl.* 165*,* 84–97*,* 2014.