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Family of generalized gamma distributions: Properties and applications

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Abstract

In this paper, a family of generalized gamma distributions, T-gamma family, has been proposed using the T-$R\{Y\}$ framework. The family of distributions is generated using the quantile functions of uniform, exponential, log-logistic, logistic and extreme value distributions. Several general properties of the T-gamma family are studied in details including moments, mean deviations, mode and Shannon's entropy. Three new generalizations of the gamma distribution which are members of the T-gamma family are developed and studied. The distributions in the T-gamma family are very flexible due to their various shapes. The distributions can be symmetric, skewed to the right, skewed to the left, or bimodal. Four data sets with various shapes are fitted by using members of the T-gamma family of distributions.

Keywords: T-$R\{Y\}$ framework, quantile function, Shannon's entropy, bimodality.

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1. Introduction

The origin of gamma distribution, from the book by Johnson et al. (1994, p. 343), can be attributed to Laplace (1836) who obtained a gamma distribution as the distribution of a “precision constant". The gamma distribution has been used to model waiting times. For example in life testing, the waiting time until “death" is a random variable that has a gamma distribution (Hogg et al. 2013, p. 156). The gamma distribution is used in Bayesian statistics, where it is used as a conjugate prior distribution for various types of scale parameters such as the parameter θ in an exponential distribution or a normal distribution with a known mean. Other applications include the size of insurance claims (Boland, 2007), hydrology (Aksoy, 2000), and bacterial gene expression (Friedman et al. 2006). For other types of applications, see for example the works of Costantino and Desharnais (1981), Dennis and Patil (1984), and Johnson et al. (1994, Chapter 17) and the references therein.

The early generalization of gamma distribution can be traced back to Amoroso (1925) who discussed a generalized gamma distribution and applied it to fit income rates. Johnson et al. (1994, Chapter 8) gave a four parameter generalized gamma distribution which reduces to the generalized gamma distribution defined by Stacy (1962) when the location parameter is set to zero. Mudholkar and Srivastava (1993) introduced the exponentiated method to derive a distribution. The generalized gamma defined by Stacy (1962) is a three-parameter exponentiated gamma distribution. Agarwal and Al-Saleh (2001) applied generalized gamma to study hazard rates. Balakrishnan and Peng (2006) applied this distribution to develop generalized gamma frailty model. Cordeiro et al. (2012) derived another generalization of Stacy's generalized gamma distribution using exponentiated method, and applied it to life time and survival analysis. Nadarajah and Gupta (2007) proposed another type of generalized gamma distribution with application to fitting drought data.

Eugene et al. (2002) introduced the beta-generated family of distributions and since then, many variants of this family have been studied. Based on the beta-generated family and its variants, more generalized gamma distributions have been defined and studied. Some examples are the beta-gamma distribution by Kong et al. (2007), the Kumaraswamy-gamma distribution by Cordeiro and de Castro (2011), the Kumaraswamy-generalized gamma distribution by de Pascoa et al. (2011), and the beta generalized gamma distribution by Cordeiro et al. (2013).

The beta-generated family was extended by Alzaatreh et al. (2013) to the T-$R(W)$ family. The cumulative distribution function (CDF) of the T-$R(W)$ distribution is $G(x) = \int_a^{W(F(x))} r(t)dt$, where $r(t)$ is the probability density function (PDF) of a random variable T with support (a, b) for $-\infty \leq a < b \leq \infty$. The function $W(F(x))$ of the CDF $F(x)$ is monotonic and absolutely continuous. Aljarrah et al. (2014) considered the function $W(F(x))$ to be the quantile function of a random variable Y and defined the T-$R\{Y\}$ family. This framework can be applied to derive generalized families of any existing distribution.

Some generalizations of the gamma distribution that fall into the T-$R\{Y\}$ framework include the family of generalized gamma-generated distributions by Zografos and Balakrishnan (2009), the gamma-Pareto distribution by Alzaatreh et al. (2012) and the gamma-normal distribution by Alzaatreh et al. (2014a). These distributions belong to the gamma-$R\{$exponential$\}$ family. Various applications to biological data, lifetime data, hydrological data and others were provided in these literatures. For a review of methods for generating continuous distributions, one may refer to Lee et al. (2013).

Various distributions in the T-$R\{Y\}$ family have been studied in the literature. The distributions, in general, have more parameters which add more flexibility to their usefulness. These distributions have shown their usefulness in many fields. They have been applied in many areas and found to provide better fit to complex real life situations. Examples include the following: the beta-normal (Eugene et al., 2002) was applied to bimodal data; the Kumaraswamy-Weibull (Cordeiro et al., 2010) was applied to model failure time data; the beta-Weibull (Famoye et al., 2005), the beta Pareto (Akinsete et al., 2008) and the beta generalized Pareto (Mahmoudi, 2011) were applied to model flood data.

This article focuses on the generalization of the gamma distribution using the T-gamma$\{Y\}$ framework and studies some new distributions in this family and their applications. Section 2 gives a brief review of the T-$R\{Y\}$ framework, defines several new generalized gamma sub-families. Section 3 gives some general properties of the T-gamma$\{Y\}$ distributions. Section 4 develops several new T-gamma$\{Y\}$ distributions and derives some properties. Section 5 gives some applications. Summary and conclusions are given in section 6.

2. The T-gamma$\{Y\}$ family of distributions

The T-$R\{Y\}$ framework defined in Aljarrah et al. (2014) (see also Alzaatreh et al., 2014b) is briefly described in the following. Let T, R and Y be random variables with CDF $F_T(x) = P(T \leq x)$, $F_R(x) = P(R \leq x)$, $F_Y(x) = P(Y \leq x)$ and corresponding quantile functions $Q_T(p)$, $Q_R(p)$ and $Q_Y(p)$, where the quantile function is defined as $Q_Z(p) = \inf\{z : F_Z(z) \geq p\}$, $0 < p < 1$. If densities exist, we denote them by $f_T(x)$, $f_R(x)$ and $f_Y(x)$. Now assume the random variables T, $Y \in (a, b)$ for $-\infty \leq a < b \leq \infty$. The random variable X in T-$R\{Y\}$ family of distributions is defined as

$$F_X(x) = \int_a^{Q_Y(F_R(x))} f_T(t)dt = F_T(Q_Y(F_R(x))). \tag{2.1}$$

The corresponding PDF associated with (2.1) is

$$f_X(x) = f_T(Q_Y(F_R(x))) \times Q'_Y(F_R(x)) \times f_R(x). \tag{2.2}$$

Alternatively, (2.2) can be written as

$$f_X(x) = f_R(x) \times \frac{f_T(Q_Y(F_R(x)))}{f_Y(Q_Y(F_R(x)))}. \tag{2.3}$$

The hazard function of the random variable X can be written as

$$h_X(x) = h_R(x) \times \frac{h_T(Q_Y(F_R(x)))}{h_Y(Q_Y(F_R(x)))}. \tag{2.4}$$

Alzaatreh et al. (2013) studied the T-$R\{\text{exponential}\}$ distributions. Aljarrah et al. (2014) studied the general framework and some properties of T-$R\{Y\}$.

Let R be a gamma random variable with PDF $f_R(x) = \beta^{-\alpha}(\Gamma(\alpha))^{-1}x^{\alpha-1}e^{-x/\beta}$, $x > 0$ and CDF $F_R(x) = \beta^{-\alpha}(\Gamma(\alpha))^{-1}\int_0^x t^{\alpha-1}e^{-t/\beta}dt$, then (2.2) reduces to

$$\begin{aligned} f_X(x) &= \frac{1}{\beta^\alpha\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta} \times \frac{f_T(Q_Y(F_R(x)))}{f_Y(Q_Y(F_R(x)))} \\ &= \text{gamma}(\alpha, \beta) \times \frac{f_T(Q_Y(F_R(x)))}{f_Y(Q_Y(F_R(x)))}. \end{aligned} \tag{2.5}$$

Gamma(α, β) is the PDF of gamma random variable. Hereafter, the family of distributions in (2.5) will be called the T-gamma$\{Y\}$ family and it will be denoted by T-$G\{Y\}$.

It is clear that the PDF in (2.5) is a generalization of gamma distribution. For consistency, the notation $f_G(x)$ and $F_G(x)$ will respectively be used in place of $f_R(x)$ and $F_R(x)$ for the gamma random variable in the remaining sections. From (2.1), if $T \stackrel{d}{=} Y$, then $X \stackrel{d}{=} \text{gamma}(\alpha\ \beta)$. Also, if $Y \stackrel{d}{=} \text{gamma}(\alpha\ \beta)$, then $X \stackrel{d}{=} T$.

Various existing generalizations of the gamma distributions can be seen as members of T-$G\{Y\}$ family. When $T \sim \text{beta}(a,\ b)$ and $Y \sim \text{uniform}(0,\ 1)$, the T-$G\{Y\}$ reduces to the beta-gamma distribution (Kong et al., 2007). When $T \sim \text{Power}(a)$ and $Y \sim \text{uniform}(0,\ 1)$, the T-$G\{Y\}$ reduces to the exponentiated-gamma distribution (Nadarajah and Kotz, 2006) and when $T \sim \text{Kumaraswamy}(a,\ b)$ and $Y \sim \text{uniform}(0,\ 1)$, the T-$G\{Y\}$ reduces to the Kumaraswamy-gamma distribution (Cordeiro and de Castro, 2011). Table 1 gives five quantile functions of known distributions which will be applied to generate T-$G\{Y\}$ sub-families in the following subsections.

Table 1. Quantile functions for different Y distributions

Y	$Q_Y(p)$
(a) Uniform	p
(b) Exponential	$-b\log(1-p),\ \ b>0$
(c) Log-logistic	$a(p/(1-p))^{1/b},\ \ a,\ b>0$
(d) Logistic	$a+b\log[p/(1-p)],\ \ b>0$
(e) Extreme value	$a+b\log[-\log(1-p)],\ \ b>0$

2.1. T-gamma{uniform} family of distributions (T-G{uniform}). By using the quantile function of the uniform distribution in Table 1, the corresponding CDF to (2.1) is

$$F_X(x) = F_T\left\{F_G(x)\right\}, \tag{2.6}$$

and the corresponding PDF to (2.6) is

$$\begin{aligned} f_X(x) &= \frac{1}{\beta^\alpha\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} \times f_T\left(F_G(x)\right) \\ &= \text{gamma}(\alpha,\beta) \times f_T\left(F_G(x)\right), \quad x>0. \end{aligned} \tag{2.7}$$

2.2. T-gamma{exponential} family of distributions (T-G{exponential}). By using the quantile function of the exponential distribution in Table 1, the corresponding CDF to (2.1) is

$$F_X(x) = F_T\left\{-b\log(1-F_G(x))\right\}, \tag{2.8}$$

and the corresponding PDF to (2.8) is

$$\begin{aligned} f_X(x) &= \frac{b}{\beta^\alpha\Gamma(\alpha)(1-F_G(x))} x^{\alpha-1} e^{-x/\beta} \times f_T\left(-b\log\left(1-F_G(x)\right)\right) \\ &= \text{gamma}(\alpha,\beta) \times \frac{b}{(1-F_G(x))} \times f_T\left(-b\log\left(1-F_G(x)\right)\right), \quad x>0. \end{aligned} \tag{2.9}$$

Note that the CDF and the PDF in (2.8) and (2.9) can be written as $F_X(x) = F_T\left(-bH_G(x)\right)$ and $f_X(x) = bh_G(x)f_T\left(-bH_G(x)\right)$ where $h_G(x)$ and $H_G(x)$ are the hazard and cumulative hazard functions for the gamma distribution, respectively. Therefore, the T-G{exponential} family of distributions arises from the 'hazard function of the gamma distribution'.

2.3. T-gamma{log-logistic} family of distributions (T-G{log-logistic}). By using the quantile function of the log-logistic distribution in Table 1, the corresponding CDF to (2.1) is

$$F_X(x) = F_T\left\{a(F_G(x)/[1-F_G(x)])^{1/b}\right\}, \tag{2.10}$$

and the corresponding PDF is

$$\begin{aligned} f_X(x) &= \frac{a}{b\beta^\alpha\Gamma(\alpha)}\frac{x^{\alpha-1}e^{-x/\beta}}{(1-F_G(x))^2}\left(\frac{F_G(x)}{1-F_G(x)}\right)^{1/b-1} f_T\left(a\left(\frac{F_G(x)}{1-F_G(x)}\right)^{1/b}\right) \\ &= \frac{a\cdot\text{gamma}(\alpha,\beta)}{b(1-F_G(x))^2}\left[\frac{F_G(x)}{1-F_G(x)}\right]^{1/b-1} f_T\left\{a\left[\frac{F_G(x)}{1-F_G(x)}\right]^{1/b}\right\},\ x>0. \end{aligned} \tag{2.11}$$

Note that if $a = b = 1$, (2.11) reduces to

$$f_X(x) = \frac{gamma(\alpha,\beta)}{(1-F_G(x))^2}\times f_T\left(F_G(x)/[1-F_G(x)]\right),\ x>0,$$

which is a family of generalized gamma distributions arising from the 'odds' of the gamma distribution.

2.4. T-gamma{logistic} family of distributions (T-G{logistic}). By using the quantile function of the logistic distribution in Table 1, the corresponding CDF to (2.1) is

$$F_X(x) = F_T\left\{a + b\log\left(F_G(x)/[1-F_G(x)]\right)\right\}, \tag{2.12}$$

and the corresponding PDF is

$$f_X(x) = \frac{bx^{\alpha-1}e^{-x/\beta}}{\beta^\alpha\Gamma(\alpha)F_G(x)[1-F_G(x)]} f_T\left(a + b\log\left(\frac{F_G(x)}{1-F_G(x)}\right)\right),\ x>0. \tag{2.13}$$

Note that if $a = 0$ and $b = 1$, (2.13) reduces to

$$f_X(x) = \frac{h_G(x)}{F_G(x)}\times f_T\left(\log\left(\frac{F_G(x)}{1-F_G(x)}\right)\right),\quad x>0,$$

which is a family of generalized gamma distributions arising from the 'logit function' of the gamma distribution.

2.5. T-gamma{extreme value} family of distributions (T-G{extreme value}). By using the quantile function of the extreme value distribution in Table 1, the corresponding CDF to (2.1) is

$$F_X(x) = F_T\left\{a + b\log(-\log[1-F_G(x)])\right\}, \tag{2.14}$$

and the corresponding PDF is

$$f_X(x) = \frac{bx^{\alpha-1}e^{-x/\beta}f_T\left\{a + b\log[-\log(1-F_G(x))]\right\}}{\beta^\alpha\Gamma(\alpha)[F_G(x)-1]\log(1-F_G(x))},\ x>0. \tag{2.15}$$

The CDF in (2.14) and the PDF in (2.15) can be written as

$$F_X(x) = F_T\left(a + b\log H_G(x)\right)$$

and

$$f_X(x) = b\left\{h_G(x)/H_G(x)\right\}f_T\left(a + b\log H_G(x)\right)$$

respectively.

3. Some properties of the T-$G\{Y\}$ family of distributions

In this section, we discuss some general properties of the T-gamma family of distributions in detail. We omit the proof for some straightforward results.

3.1. Lemma. *Let T be a random variable with PDF $f_T(x)$, then the random variable $X = Q_G(F_Y(T))$, where $Q_G(.)$ is the quantile function of gamma(α, β), follows the T-gamma$\{Y\}$ distribution.*

3.2. Corollary. *Based on Lemma 3.1, we have*

(i) $X = Q_G(T)$ *follows the distribution of T-$G\{$uniform$\}$ family.*
(ii) $X = Q_G(1 - e^{-T/b})$ *follows the distribution of T-$G\{$exponential$\}$ family.*
(iii) $X = Q_G\left([1 + (T/a)^{-b}]^{-1}\right)$ *follows the distribution of T-$G\{$log-logistic$\}$ family.*
(iv) $X = Q_G\left([1 + e^{-(T-a)/b}]^{-1}\right)$ *follows the distribution of T-$G\{$logistic$\}$ family.*
(v) $X = Q_G\left(1 - e^{-e^{(T-a)/b}}\right)$ *follows the distribution of T-$G\{$extreme value$\}$ family.*

3.3. Lemma. *The quantile functions for T-gamma$\{Y\}$ family is given by $Q_X(p) = Q_G(F_Y(Q_T(p)))$.*

3.4. Corollary. *Based on Lemma 3.3, the quantile function for the (i) T-$G\{$uniform$\}$, (ii) T-$G\{$exponential$\}$, (iii) T-$G\{$log-logistic$\}$, (iv) T-$G\{$logistic$\}$ and (v) T-$G\{$extreme value$\}$, are respectively,*

(i) $Q_X(p) = Q_G\left(Q_T(p)\right)$,
(ii) $Q_X(p) = Q_G\left(1 - e^{-b^{-1}Q_T(p)}\right)$,
(iii) $Q_X(p) = Q_G\left([1 + (Q_T(p)/a)^{-b}]^{-1}\right)$,
(iv) $Q_X(p) = Q_G\left([1 + e^{-(Q_T(p)-a)/b}]^{-1}\right)$,
(v) $Q_X(p) = Q_G\left(1 - e^{-e^{(Q_T(p)-a)/b}}\right)$.

3.5. Proposition. *The mode(s) of the T-gamma$\{Y\}$ family are the solutions of the equation*

$$x = \frac{\alpha - 1}{\beta^{-1} - \Psi\{f_T(Q_Y(F_G(x)))\} - \Psi\{Q'_Y(F_G(x))\}}, \tag{3.1}$$

where $\Psi(f) = f'/f$.

Proof. For gamma distribution,

$$f_G(x) = \beta^{-\alpha}(\Gamma(\alpha))^{-1}x^{\alpha-1}e^{-x/\beta},$$

we have $f'_G(x) = [(\alpha - 1)/x - \beta^{-1}]f_G(x)$. Using this fact; one can show the result in (3.1) by equating the derivative of the equation (2.5) to zero and then solving for x. □

The entropy of a random variable X is a measure of variation of uncertainty (Rényi, 1961). Shannon's entropy has been used in many fields such as engineering and information theory. Shannon's entropy (Shannon, 1948) for a random variable X with PDF $f(x)$ is defined as $\eta_X = -E\left\{\log\left(f(X)\right)\right\}$.

3.6. Proposition. *The Shannon's entropy for the T-$G\{Y\}$ family (2.1) is given by*

$$\eta_X = \eta_T + E\left(\log f_Y(T)\right) - E\left\{\log f_G(Q_G(F_Y(T)))\right\}. \tag{3.2}$$

Proof. See Theorem 2 of Aljarrah et al. (2014). □

3.7. Corollary. *The Shannon's entropy for the T-$G\{Y\}$ family can be written as*

$$\eta_X = \eta_T + E\left(\log f_Y(T)\right) + \log\Gamma(\alpha) + \alpha\log(\beta) + (1-\alpha)E\left(\log X\right) + \beta^{-1}\mu_X.$$

Proof. For the T-$G\{Y\}$ family, we have $\log(f_G(x)) = -\log(\Gamma(\alpha)) - \alpha\log(\beta) + (\alpha - 1)\log(x) - x/\beta$. The result follows from Proposition 3.6. □

3.8. Corollary. *Based on Corollary 3.7, the Shannon's entropies for the (i) T-$G\{$uniform$\}$, (ii) T-$G\{$exponential$\}$, (iii) T-$G\{$log-logistic$\}$, (iv) T-$G\{$logistic$\}$ and (v) T-$G\{$extreme value$\}$, distributions, respectively, are given by*

(i) $\eta_X = C_1 + \eta_T + (1-\alpha)E\left(\log X\right) + \beta^{-1}\mu_X$,

(ii) $\eta_X = C_2 + \eta_T - b^{-1}\mu_T + (1-\alpha)E\left(\log X\right) + \beta^{-1}\mu_X$,

(iii) $\eta_X = C_3 + \eta_T + (b-1)E(\log T) - 2E(\log(1+(T/a)^b)) + (1-\alpha)E\left(\log X\right) + \beta^{-1}\mu_X$,

(iv) $\eta_X = C_4 + \eta_T - b^{-1}\mu_T - 2E(\log(1+e^{-(T-a)/b})) + (1-\alpha)E\left(\log X\right) + \beta^{-1}\mu_X$,

(v) $\eta_X = C_5 + \eta_T + b^{-1}\mu_T - E(e^{(T-a)/b}) + (1-\alpha)E\left(\log X\right) + \beta^{-1}\mu_X$,

where $C_1 = \log\Gamma(\alpha) + \alpha\log(\beta)$, $C_2 = -\log b + \log\Gamma(\alpha) + \alpha\log(\beta)$, $C_3 = \log b - b\log a + \log\Gamma(\alpha) + \alpha\log(\beta)$, $C_4 = -\log b + ab^{-1} + \log\Gamma(\alpha) + \alpha\log(\beta)$ *and* $C_5 = -\log b - ab^{-1} + \log\Gamma(\alpha) + \alpha\log(\beta)$.

3.9. Proposition. *The rth moment for the T-gamma$\{Y\}$ family of distributions is given by*

$$E(X^r) = \beta^r \sum_{k=0}^{\infty} c_k E[F_Y(T)]^{k+r}, \tag{3.3}$$

where $c_0 = 1$, $c_m = m^{-1}\sum_{k=1}^{m}(kr - m + k)g_{k+1}c_{m-k}$, $m \geq 1$ *and g_k satisfies the following:*

$$\begin{aligned} g_1 = 1, n(n+\alpha)g_{n+1} &= \sum_{i=1}^{n}\sum_{j=1}^{n-i+1} g_i g_j g_{n-i-j+2}\, j(n-i-j+2) \\ &- \Delta(n)\sum_{i=2}^{n} g_i g_{n-i+2}\, i[i-\alpha-(1-\alpha)(n+2-i)], \\ \textit{and}\quad \Delta(n) &= \begin{cases} 0, & n<2 \\ 1, & n \geq 2. \end{cases} \end{aligned}$$

Proof. From Lemma 3.1, the rth moment for the T-$G\{Y\}$ family can be written as $E(X^r) = E(Q_G(F_Y(T)))^r$, where $Q_G(p)$ is the quantile function of gamma distribution with parameters α and β. Steinbrecher and Shaw (2008) showed that a power series expansion of $Q_G(p)$ is possible and can be written as $Q_G(p) = \beta\sum_{n=1}^{\infty} g_n\, p^n$ where g_n can be obtained from the recurrence relation defined in the statement of Proposition 3.9. For example, the first three terms of g_n are 1, $(\alpha+1)^{-1}$ and $(3\alpha+5)/[2(\alpha+1)^2(\alpha+2)]$. Other terms can be similarly obtained. Therefore, $(Q_G(p))^r = \beta^r\sum_{k=0}^{\infty} c_k\, p^{k+r}$ (see Gradshteyn and Ryzhik, 2007), where c_k can be obtained from the recurrence relation defined in Proposition 3.9. □

3.10. Corollary. *Based on Proposition 3.9, the rth moments for the (i) T-$G\{$uniform$\}$, (ii) T-$G\{$exponential$\}$, (iii) T-$G\{$log-logistic$\}$, (iv) T-$G\{$logistic$\}$ and (v) T-$G\{$extreme value$\}$ distributions, respectively, are given by*

(i) $E(X^r) = \beta^r\sum_{k=0}^{\infty} c_k E(T^{k+r})$,

(ii) $E(X^r) = \beta^r\sum_{k=0}^{\infty}\sum_{j=0}^{k+r}(-1)^j\binom{k+r}{j} c_k M_T(-j/b)$,

(iii) $E(X^r) = \beta^r\sum_{k=0}^{\infty} c_k E\left(1+(T/a)^{-b}\right)^{-k-r}$,

(iv) $E(X^r) = \beta^r\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j c_k M_{T-a}(-j/b)$,

(v) $E(X^r) = \beta^r\sum_{k=0}^{\infty}\sum_{j=0}^{k+r}\sum_{i=0}^{\infty}(-1)^{i+j}\frac{(j)^i}{i!}\binom{k+r}{j} c_k M_{T-a}(i/b)$,

where $M_X(t) = E(e^{tX})$.

3.11. Proposition. *The mean deviations from the mean and the median for the T-gamma{Y} family, respectively, are given by*

$$D(\mu) = 2\mu F_T(Q_Y(F_G(\mu))) - 2\Pi_\mu \text{ and } D(M) = \mu - 2\Pi_M, \tag{3.4}$$

where μ *and* M *are the mean and median for* X*, and*

$$\Pi_c = \beta \sum\nolimits_{k=1}^{\infty} g_k \int_{-\infty}^{Q_Y(F_G(c))} f_T(u)(F_Y(u))^k du.$$

Proof. For a nonnegative random variable X, we have $D(\mu) = 2\mu F_X(\mu) - 2\Pi_\mu$ and $D(M) = \mu - 2\Pi_M$, where $\Pi_c = \int_0^c x f_X(x)dx$. From (2.5) and Lemma 3.1, one can easily see that $\Pi_c = \beta \int_{-\infty}^{Q_Y(F_G(c))} f_T(u) Q_G(F_Y(u))du$. The results in (3.4) can be obtained using the series expansion of $Q_G(.)$ in Proposition 3.9. □

3.12. Corollary. *Based on Proposition 3.11, the* Π_c*'s for (i) T-G{uniform}, (ii) T-G{exponential}, (iii) T-G{log-logistic}, (iv) T-G{logistic} and (v) T-G{extreme value} distributions, are respectively given by*

(i)

$$\Pi_c = \beta \sum_{k=1}^{\infty} g_k S_u(c, 0, k), \tag{3.5}$$

where $S_\xi(c, a, k) = \int_a^{Q_Y(F_G(c))} \xi^k f_T(u)du$ *and* $Q_Y(F_G(c)) = F_G(c)$ *for uniform distribution.*

(ii)

$$\Pi_c = \beta \sum_{k=1}^{\infty} \sum_{j=0}^{k} g_k \binom{k}{j} (-1)^j S_{e^{u/b}}(c, 0, -j), \tag{3.6}$$

where $Q_Y(F_G(c)) = -b \log(1 - F_G(c))$ *for exponential distribution.*

(iii)

$$\Pi_c = \beta \sum_{k=1}^{\infty} \sum_{j=0}^{k} g_k \binom{k}{j} (-1)^j S_{1+(u/a)^b}(c, 0, -j), \tag{3.7}$$

where $Q_Y(F_G(c)) = a[F_G(c)/(1 - F_G(c))]^{1/b}$ *for log-logistic distribution.*

(iv)

$$\Pi_c = \beta \sum_{k=1}^{\infty} g_k S_{1+e^{-(u-a)/b}}(c, -\infty, -j), \tag{3.8}$$

where $Q_Y(F_G(c)) = a + b \log\{F_G(c)/(1 - F_G(c))\}$ *for logistic distribution.*

(v)

$$\Pi_c = \beta \sum_{k=1}^{\infty} \sum_{j=0}^{k} (-1)^j \binom{k}{j} g_k S_{e^{e^{(u-a)/b}}}(c, -\infty, -j), \tag{3.9}$$

where $Q_Y(F_G(c)) = a + b \log\{-\log(1 - F_G(c))\}$ *for extreme value distribution.*

Proposition 3.11 and Corollary 3.12 can be used to obtain the mean deviations for T-G{uniform}, T-G{exponential}, T-G{log-logistic}, T-G{logistic} and T-G{extreme value} distributions.

3.13. Proposition. *Let X be a random variable that follows the T-gamma$\{Y\}$ family in (2.5). Assume that $E(X^n) < \infty$ for all n, then $E(X^n) \le [\beta^n \Gamma(\alpha+n)/\Gamma(\alpha)] \times E(1/[1-F_Y(T)])$.*

Proof. If the random variable R is nonnegative and X follows the T-$R\{Y\}$ family in (2.1) with $E(X^n) < \infty$, one can show that $E(X^n) \le E(R^n)E[1/(1-F_Y(T))]$ (see Theorem 1 in Aljarrah et al., 2014). The result follows by using the fact that R follows a gamma distribution with parameters α and β, and $E(R^n) = \beta^n\Gamma(\alpha+n)/\Gamma(\alpha)$. □

3.14. Corollary. *If $E(X^n) < \infty$ and by using Proposition 3.13, we have the following results:*

(i) If X follows T-$G\{$uniform$\}$, then $E(X^n) \le \left[\frac{\beta^n\Gamma(\alpha+n)}{\Gamma(\alpha)}\right] E\left((1-T)^{-1}\right)$.
(ii) If X follows T-$G\{$exponential$\}$, then $E(X^n) \le \left[\frac{\beta^n\Gamma(\alpha+n)}{\Gamma(\alpha)}\right] M_T(1/b)$.
(iii) If X follows T-$G\{$log-logistic$\}$, then $E(X^n) \le \left[\frac{\beta^n\Gamma(\alpha+n)}{\Gamma(\alpha)}\right] [1+E(T/a)^b]$.
(iv) If X follows T-$G\{$logistic$\}$, then $E(X^n) \le \left[\frac{\beta^n\Gamma(\alpha+n)}{\Gamma(\alpha)}\right] [1+M_{T-a}(1/b)]$.
(v) If X follows T-$G\{$extreme value$\}$, then $E(X^n) \le \left[\frac{\beta^n\Gamma(\alpha+n)}{\Gamma(\alpha)}\right] E(e^{e^{(T-a)/b}})$.

4. Some examples of T-$G\{Y\}$ family of distributions

In this section, we present some members of the T-$G\{Y\}$ family, namely, Weibull-$G\{$exponential$\}$, Weibull-$G\{$log-logistic$\}$ and Cauchy-$G\{$logistic$\}$. For simplicity, we only use the standard form (i.e. no parameters in the distribution of Y) of the quantile functions in Table 1.

4.1. The Weibull-G\{exponential\} distribution. If a random variable T follows the Weibull distribution with parameters c and γ, then

$$f_T(t) = c\gamma^{-1}(t/\gamma)^{c-1}e^{-(t/\gamma)^c},\ c,\gamma > 0.$$

From (2.9), the PDF of the Weibull-$G\{$exponential$\}$ is given by

$$(4.1)\qquad f_X(x) = \frac{c}{\gamma^c\beta^\alpha\Gamma(\alpha)}\frac{x^{\alpha-1}e^{-x/\beta}}{1-F_G(x)}(-\log(1-F_G(x)))^{c-1} \times \exp\left\{-\gamma^{-c}(-\log(1-F_G(x)))^c\right\},\quad x>0.$$

When $c=1$, (4.1) reduces to the exponential-$G\{$exponential$\}$. When $c=\gamma=1$, equation (4.1) reduces to the gamma distribution. From (2.8), the CDF of the Weibull-$G\{$exponential$\}$ is given by

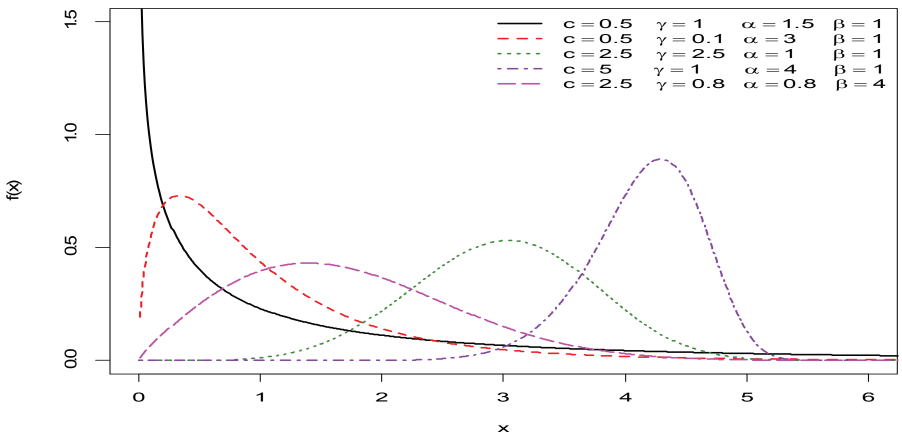
$$F_X(x) = 1-\exp\left\{-\gamma^{-c}(-\log(1-F_G(x)))^c\right\},\quad x>0.$$

In Figure 1, various graphs of Weibull-$G\{$exponential$\}$ PDF for different parameter values are provided. These plots show that the PDF can be left skewed, right skewed, approximately symmetric or have a reversed J-shape.
Some properties of the Weibull-$G\{$exponential$\}$ are obtained in the following by using the general properties discussed in section 3.

(1) Quantile function: By using Lemma 3.3, the quantile function of the Weibull-$G\{$exponential$\}$ distribution is given by

$$Q_X(p) = Q_G\left\{1-e^{-\gamma(-\log(1-p))^{1/c}}\right\}.$$

Figure 1. The PDFs of Weibull-G\{exponential\} for various parameter values



(2) Mode: By using Proposition 3.5, the mode of Weibull-G\{exponential\} distribution can be obtained by solving the following equation numerically

$$x = (\alpha - 1)\left(\beta^{-1} - h_G(x)\left\{\frac{c-1}{(1-F_G(x))H_G(x)} + \gamma^{-c+1}(H_G(x))^{c-1}\right\}\right)^{-1}.$$

(3) Shannon entropy: By using Corollary 3.8 and the fact that $\mu_T = \gamma\Gamma(1+1/c)$ and $\eta_T = 1+\xi(1-1/c)+\log(\gamma/c)$, the Shannon's entropy of Weibull-G\{exponential\} distribution is

$$\eta_X = C + (1-\alpha)E\left(\log X\right) + \beta^{-1}\mu_X,$$

where $C = \log\Gamma(\alpha) + \alpha\log(\beta) + \xi(1-1/c) + \log(\gamma/c) - \gamma\Gamma(1+1/c) + 1$ and $\xi \approx 0.5772$ is the Euler's constant.

(4) Moments: By using Corollary 3.10, the rth moment of the Weibull-G\{exponential\} distribution can be written as

$$E(X^r) = \beta^r \sum\nolimits_{k=0}^{\infty} \sum\nolimits_{j=0}^{k+r} \sum\nolimits_{i=0}^{\infty} \frac{(-1)^{j+i} j^i \gamma^i}{i!} \binom{k+r}{j} c_k \Gamma(1+i/c).$$

(5) Mean deviations: By using Corollary 3.12, the mean deviation from the mean and the mean deviation from the median of Weibull-G\{exponential\} distribution can be obtained from (3.4) where

$$\Pi_c = \beta \sum_{k=1}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{\infty} \frac{(-1)^j k^i \gamma^i}{i!} \binom{k}{j} g_k \Gamma\left[1+i/c, (Q_Y(F_G(c))/\gamma)^c\right],$$

$Q_Y(F_G(c)) = -\log(1-F_G(c))$ and $\Gamma(\alpha, x) = \int_0^x u^{\alpha-1}e^{-u}du$ is the incomplete gamma function.

(6) Moments upper bound: By Corollary 3.14, $E(X^n) \leq [\beta^n\Gamma(\alpha+n)/\Gamma(\alpha)] \times M_T(1)$, where T follows Weibull(c, γ). If $c = 1$ and $\gamma < 1$, one can show that $E(X^n) \leq \frac{\beta^n\Gamma(\alpha+n)}{(1-\gamma)\Gamma(\alpha)}$.

4.2. The Weibull-G\{log-logistic\} distribution. If a random variable T follows the Weibull distribution with parameters c and γ, then

$$f_T(t) = c\gamma^{-1}(t/\gamma)^{c-1}e^{-(t/\gamma)^c}, \ c, \gamma > 0.$$

From (2.11), the PDF of the Weibull-G\{log-logistic\} is given by

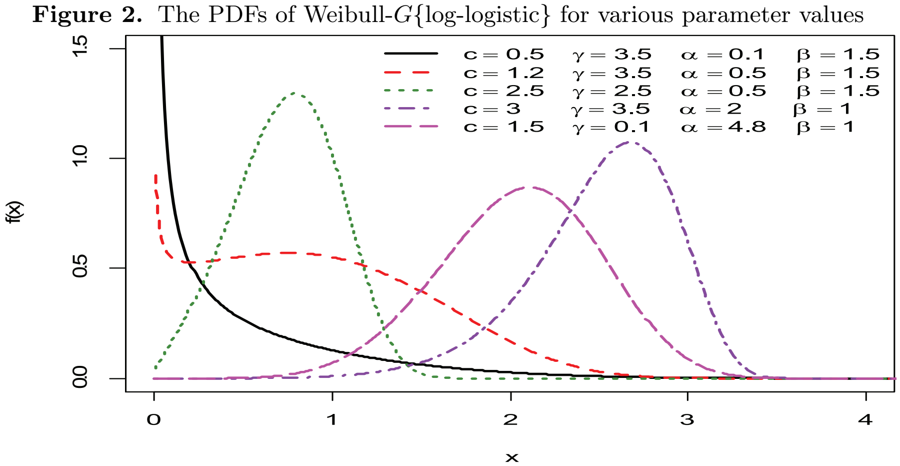
$$f_X(x) = \frac{c}{\gamma^c \beta^\alpha \Gamma(\alpha)} \frac{x^{\alpha-1} e^{-x/\beta}}{(1 - F_G(x))^2} \left(\frac{F_G(x)}{1 - F_G(x)} \right)^{c-1} \times \exp\left\{ -\left(\frac{F_G(x)}{\gamma(1 - F_G(x))} \right)^c \right\}, \quad x > 0. \tag{4.2}$$

When $c = 1$, the Weibull-G\{log-logistic\} reduces to the exponential-G\{log-logistic\}. From (2.10), the CDF of the Weibull-G\{log-logistic\} is given by

$$F_X(x) = 1 - \exp\left\{ -\left(\frac{F_G(x)}{\gamma(1 - F_G(x))} \right)^c \right\}, \ x > 0.$$

Various graphs of Weibull-G\{log-logistic\} PDF for different parameter values are provided in Figures 2 and 3. These plots show the PDF has great shape flexibility. It can be left skewed, right skewed, approximately symmetric or have a reversed J-shape. Also, the distribution can be unimodal or bimodal.

Figure 2. The PDFs of Weibull-G\{log-logistic\} for various parameter values



4.3. The Cauchy-G\{logistic\} distribution. If a random variable T follows the Cauchy distribution with parameters c and γ, then

$$f_T(t) = \pi^{-1}\{1 + [(t-c)/\gamma]^2\}^{-1}, \ \gamma > 0, \, c \in R.$$

From (2.13), the PDF of the Cauchy-G\{logistic\} is defined as

$$f_X(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\pi \gamma \beta^\alpha \Gamma(\alpha) F_G(x)(1 - F_G(x))} \times \left[1 + \gamma^{-2}(\log(F_G(x)/(1 - F_G(x)) - c)^2\right]^{-1}, \quad x > 0. \tag{4.3}$$

In Figure 4, various graphs of the Cauchy-G\{logistic\} distribution for various parameter values are provided. These graphs indicate that the Cauchy-G\{logistic\} distribution can be right skewed, approximately symmetric or have a reversed J-shape.

Figure 3. Some bimodal PDFs of Weibull-G\{log-logistic\} for various parameter values

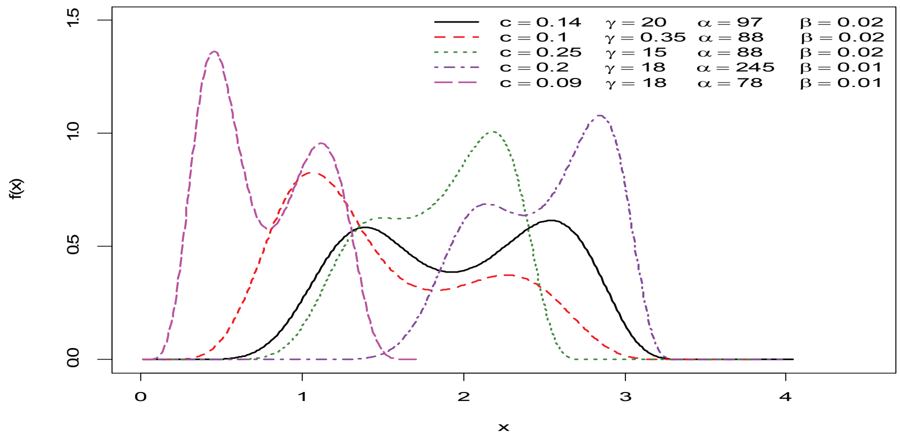
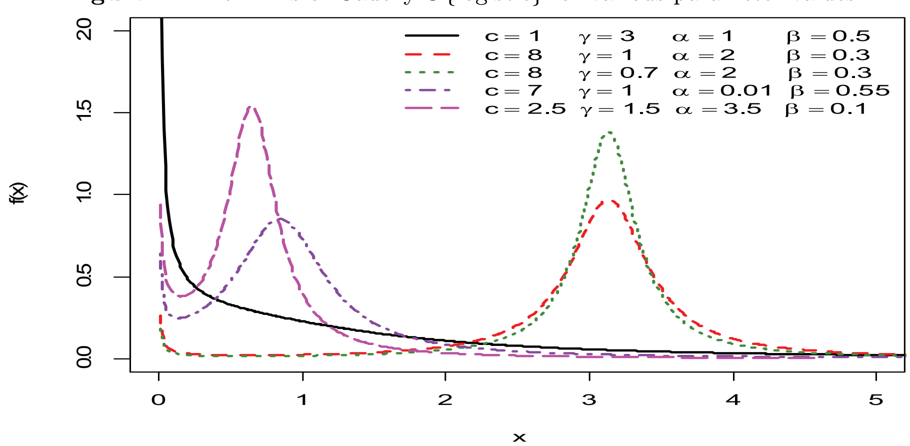


Figure 4. The PDFs of Cauchy-G\{logistic\} for various parameter values



5. Applications

In this section, the applications of the T-gamma distribution are illustrated by fitting some members of the family to different data sets including unimodal and bimodal data sets.

5.1. Unimodal data sets. In this subsection, we fit the Weibull-G\{exponential\}, Weibull-G\{log-logistic\} and Cauchy-G\{logistic\} in equations 4.1, 4.2 and 4.3, respectively, to three data sets with various shapes that are approximately symmetric or left skewed or right skewed. The maximum likelihood method is used to estimate the model parameters. The initial values for the parameters α and β are obtained by assuming the random sample $x_i,\ i = 1, 2, \ldots, n$ is from the gamma distribution with parameters α and β. The moment estimates from the gamma distribution are used as the initial values, which are $\alpha_0 = \bar{x}^2/s^2$ and $\beta_0 = s^2/\bar{x}$. Now, by Lemma 3.1, $t_i = Q_Y(F_G(x_i)),\ i = 1, 2, \ldots, n$ follows the T distribution with parameters c and γ in all the examples in section 4. The moment estimates or the maximum likelihood estimates of the T-distribution can be used as the initial values for c and γ.

The first data set ($n = 80$) in Table 2 represents the annual maximum temperatures at Oxford and Worthing in England for the period of 1901-1980. Chandler and Bate (2007) used the generalized extreme value distribution to model the annual maximum temperatures in Table 2. The summary statistics from the first data set are: $\bar{x} = 85.3250$, $s = 4.2658$, $\gamma_1 = -0.0162$ and $\gamma_2 = 2.7309$, where γ_1 and γ_2 are the sample skewness and kurtosis respectively. The second data set ($n = 202$) in Table 3 is from Weisberg (2005) and it represents the sum of skin folds in 202 athletes collected at the Australian Institute of Sports. The summary statistics from the second data set are: $\bar{x} = 69.0218$, $s = 32.5653$, $\gamma_1 = 1.1660$ and $\gamma_2 = 4.3220$. The third data set ($n = 40$) in Table 4 is from Xu et al. (2003) and it represents the time to failure ($10^3\,h$) of turbocharger of one type of engine. The summary statistics from the third data set are: $\bar{x} = 6.2525$, $s = 1.9555$, $\gamma_1 = -0.6542$ and $\gamma_2 = 2.5750$.

The data sets are fitted to the Weibull-G{exponential}, Weibull-G{log-logistic} and Cauchy-G{logistic} distributions. The maximum likelihood estimates, the log-likelihood value, the Akaike Information Criterion (AIC), the Kolmogorov-Smirnov (K-S) test statistic, and the p-value for the K-S statistic for the fitted distributions to the three data sets are reported in Table 5. The results in Table 5 show that all the generalized gamma distributions provide adequate fit to the data set in Table 2. For the data set in Table 3, the Weibull-G{exponential} provides the best fit followed by the Weibull-G{log-logistic}, while the Cauchy-G{logistic} does not provide an adequate fit. For the data set in Table 4, all the three generalized gamma distributions provide an adequate fit.

On examining the summary statistics of the data sets, it is noticed that the data set in Table 2 is approximately symmetric, the data set in Table 3 is right skewed and the data set in Table 4 is left skewed. This shows the flexibility of these generalized gamma distributions in fitting various data sets with different distribution shapes. We also fit the three data sets to the gamma distribution. The resulting K-S statistics p-values are less than 0.0001 for all data sets. Figure 5 displays the histogram and the fitted density functions for the three data sets, which support the results in Table 5.

Table 2. The annual maximum temperatures data ($n = 80$)

75	92	87	86	85	95	84	87	86	82	77
89	79	83	79	85	89	84	84	82	86	81
84	84	87	89	80	86	85	84	89	80	87
84	85	82	86	87	86	89	90	90	91	81
85	79	83	93	87	83	88	90	83	82	80
81	95	89	86	89	87	92	89	87	87	83
89	88	84	84	77	85	77	91	94	80	80
85	83	88								

5.2. Bimodal data. In this subsection, we fit the Weibull-G{log-logistic} to a bimodal data set obtained from Emlet et al. (1987) on the asteroid and echinoid egg size. The data consists of 88 asteroids species divided into three types; 35 planktotrophic larvae, 36 lecithotrophic larvae, and 17 brooding larvae. The logarithm of the egg diameters of the asteroids data has a bimodal shape. We fit the logarithm of the egg diameters of the asteroids data and compared it with the beta-normal distribution (Famoye et al., 2004) and logistic-normal{logistic} distribution (Alzaatreh et al., 2014b). The results of the

Table 3. The sum of skin folds data ($n = 202$)

28.0	98	89.0	68.9	69.9	109.0	52.3	52.8	46.7	82.7	42.3
109.1	96.8	98.3	103.6	110.2	98.1	57.0	43.1	71.1	29.7	96.3
102.8	80.3	122.1	71.3	200.8	80.6	65.3	78.0	65.9	38.9	56.5
104.6	74.9	90.4	54.6	131.9	68.3	52.0	40.8	34.3	44.8	105.7
126.4	83.0	106.9	88.2	33.8	47.6	42.7	41.5	34.6	30.9	100.7
80.3	91.0	156.6	95.4	43.5	61.9	35.2	50.9	31.8	44.0	56.8
75.2	76.2	101.1	47.5	46.2	38.2	49.2	49.6	34.5	37.5	75.9
87.2	52.6	126.4	55.6	73.9	43.5	61.8	88.9	31.0	37.6	52.8
97.9	111.1	114.0	62.9	36.8	56.8	46.5	48.3	32.6	31.7	47.8
75.1	110.7	70.0	52.5	67	41.6	34.8	61.8	31.5	36.6	76.0
65.1	74.7	77.0	62.6	41.1	58.9	60.2	43.0	32.6	48	61.2
171.1	113.5	148.9	49.9	59.4	44.5	48.1	61.1	31.0	41.9	75.6
76.8	99.8	80.1	57.9	48.4	41.8	44.5	43.8	33.7	30.9	43.3
117.8	80.3	156.6	109.6	50.0	33.7	54.0	54.2	30.3	52.8	49.5
90.2	109.5	115.9	98.5	54.6	50.9	44.7	41.8	38.0	43.2	70.0
97.2	123.6	181.7	136.3	42.3	40.5	64.9	34.1	55.7	113.5	75.7
99.9	91.2	71.6	103.6	46.1	51.2	43.8	30.5	37.5	96.9	57.7
125.9	49.0	143.5	102.8	46.3	54.4	58.3	34.0	112.5	49.3	67.2
56.5	47.6	60.4	34.9							

Table 4. The time to failure of turbocharger data ($n = 40$)

1.6	3.5	4.8	5.4	6.0	6.5	7.0	7.3	7.7	8.0	8.4
2.0	3.9	5.0	5.6	6.1	6.5	7.1	7.3	7.8	8.1	8.4
2.6	4.5	5.1	5.8	6.3	6.7	7.3	7.7	7.9	8.3	8.5
3.0	4.6	5.3	6.0	8.7	8.8	9.0				

maximum likelihood estimates, the log-likelihood value, the AIC, the K-S test statistic, and the p-value for the K-S statistic for the fitted distributions are reported in Table 6. The results in Table 6 show that all distributions provide an adequate fit to the data set. Figure 6 displays the histogram and the fitted density functions for the data.

6. Summary and Conclusions

The gamma distribution is a commonly used distribution for fitting lifetime data, survival data, hydrological data, and others. The generalization of the gamma distribution provides more flexible distributions for these different applications. This article applies the T-$R\{Y\}$ framework proposed by Aljarrah et al. (2014) to define T-gamma$\{Y\}$ family by using the gamma random variable. Some general properties of the family are studied. Five types of generalized gamma sub-families are defined by using five different quantile functions for uniform, exponential, log-logistic, logistic, and extreme value distributions. Various properties for each of these sub-families are studied including moments, modes, entropy, deviation from the mean and deviation from the median. Three generalized gamma distributions, namely, Weibull-G\{exponential\}, Weibull-G\{log-logistic\} and Cauchy-G\{logistic\} are defined and some of their properties investigated. It is noticed that the shapes of T-$G\{Y\}$ distributions can be symmetric, skewed to the right, skewed to the left or bimodal. This shows that the new generalized gamma distributions are very flexible in fitting real world data. For future research, many other types of generalizations of gamma distribution can be derived using the methodology described in this paper.

Table 5. Parameter estimates for the three data sets in Tables 2, 3, and 4

Parameter estimates for the annual maximum temperatures data in Table 2								
Distribution	$\hat{c}$	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\ell}$	AIC	K-S	K-S p-value
Weibull-G{E}	1.4579 (0.7993)*	2.8324 (4.0113)	392.4465 (169.4705)	0.2048 (0.0901)	–228.9830	465.9661	0.0638	0.9006
Weibull-G{LL}	0.4753 (0.1934)	0.0481 (0.1168)	423.0032 (169.6370)	0.2232 (0.0904)	–229.0198	466.0396	0.0635	0.9041
Cauchy-G{L}	0.1349 (0.8799)	1.1913 (0.2113)	534.8853 (90.7804)	0.1591 (0.0269)	–239.0094	486.0187	0.0727	0.7922
Parameter estimates for the sum of skin folds data in Table 3								
Distribution	$\hat{c}$	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\ell}$	AIC	K-S	K-S p-value
Weibull-G{E}	0.7291 (0.0404)	3.9319 (0.4010)	17.3862 (0.0025)	2.6521 (0.0025)	–953.5709	1915.1420	0.0634	0.3921
Weibull-G{LL}	0.3184 (0.0518)	0.0219 (0.0131)	13.4018 (2.7954)	10.4219 (2.8058)	–962.2296	1932.4590	0.0793	0.1578
Cauchy-G{L}	-0.9076 (0.3571)	3.2642 (0.3125)	29.4223 (0.0203)	2.1914 (0.0236)	–977.9650	1963.9300	0.1174	0.0076
Parameter estimates for the time to failure of turbocharger data in Table 4								
Distribution	$\hat{c}$	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\ell}$	AIC	K-S	K-S p-value
Weibull-G{E}	8.3877 (1.8836)	4.2085 (0.9729)	0.1116 (0.0714)	4.9569 (1.6596)	–81.3549	170.7098	0.1114	0.7039
Weibull-G{LL}	0.6094 (0.2749)	28.3338 (43.2957)	7.5745 (5.5233)	0.5396 (0.3396)	–78.9643	165.9286	0.0820	0.9507
Cauchy-G{L}	1.6246 (1.4562)	3.3557 (1.1973)	57.8285 (21.4393)	0.1015 (0.0354)	–85.9245	179.8491	0.1442	0.3766

*standard error

Table 6. Parameter estimates for the asteroids data

Distribution	Weibull-G{LL}	Beta-normal*	Logistic-N{L}
Parameter Estimates	$\hat{\alpha} = 410.7779(16.1145)$ $\hat{\beta} = 0.0151(0.0196)$ $\hat{c} = 0.1390(0.0865)$ $\hat{\gamma} = 3.6233(0.4476)$	$\hat{\alpha} = 0.0129$ $\hat{\beta} = 0.0070$ $\hat{\mu} = 5.7466$ $\hat{\sigma} = 0.0675$	$\hat{\lambda} = 0.1498(0.0185)$ $\hat{\mu} = 6.0348(0.0685)$ $\hat{\sigma} = 0.2604(0.0100)$
Log-likelihood	-111.2091	-109.4800	-111.4287
AIC	230.4182	226.9600	228.4974
K-S statistic	0.1088	0.1233	0.0988
p-value	0.2486	0.1377	0.3572

*From Famoye et al. (2004) and the MLE standard errors were not provided

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Figure 5. PDF for the fitted distributions for the three data sets

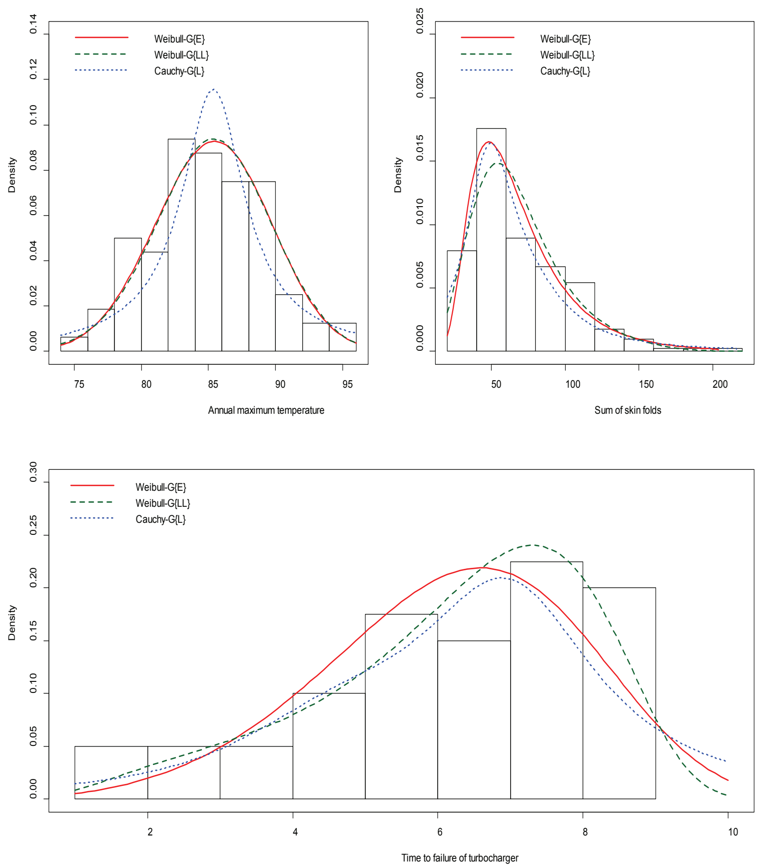

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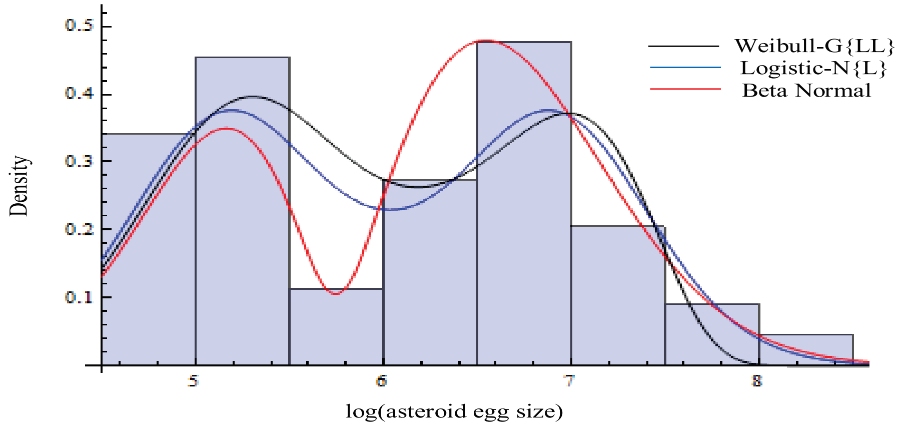
Figure 6. PDF for the fitted distributions for the asteroids data



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