

## Asymptotic properties of risks ratios of shrinkage estimators

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### Abstract

We study the estimation of the mean  $\theta$  of a multivariate normal distribution  $N_p(\theta, \sigma^2 I_p)$  in  $\mathbb{R}^p$ ,  $\sigma^2$  is unknown and estimated by the chi-square variable  $S^2 \sim \sigma^2 \chi_n^2$ . In this work we are interested in studying bounds and limits of risk ratios of shrinkage estimators to the maximum likelihood estimator, when  $n$  and  $p$  tend to infinity provided that  $\lim_{p \rightarrow \infty} \frac{\|\theta\|^2}{p\sigma^2} = c$ . The risk ratio for this class of estimators has a lower bound  $B_m = \frac{c}{1+c}$ , when  $n$  and  $p$  tend to infinity provided that  $\lim_{p \rightarrow \infty} \frac{\|\theta\|^2}{p\sigma^2} = c$ . We give simple conditions for shrinkage minimax estimators, to attain the limiting lower bound  $B_m$ . We also show that the risk ratio of James-Stein estimator and those that dominate it, attain this lower bound  $B_m$  (in particularly its positive-part version). We graph the corresponding risk ratios for estimators of James-Stein  $\delta_{JS}$ , its positive part  $\delta_{JS}^+$ , that of a minimax estimator, and an estimator dominating the James-Stein estimator in the sense of the quadratic risk (polynomial estimators proposed by Tze Fen Li and Hou Wen Kuo [13]) for some values of  $n$  and  $p$ .

**Keywords:** James-Stein estimator, multivariate gaussian random variable, non-central chi-square distribution, shrinkage estimator, quadratic risk.

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## 1. Introduction

Since the papers of Stein [10],[11] and James and Stein [6], many studies were carried out in the direction of shrinkage estimators, of the mean  $\theta$  of a multivariate normal distribution  $X \sim N_p(\theta, \sigma^2 I_p)$  in  $\mathbb{R}^p$ . In these works one estimates the mean  $\theta$  of a multivariate normal distribution  $N_p(\theta, \sigma^2 I_p)$  in  $\mathbb{R}^p$  by shrinkage estimators deduced from the empirical mean estimator, which are better in quadratic loss than the empirical mean estimator. A summary of these proceedings is made by Hoffmann [5] who presents an expository development of Stein estimation in several distribution families. He considered both the point estimation and confidence interval cases. Emphasis is laid on the chronological development. In our work we are interested only in the case where the observation  $X$  is Gaussian.

More precisely, if  $X$  represents an observation or a sample of multivariate normal distribution  $N_p(\theta, \sigma^2 I_p)$ , the aim is to estimate  $\theta$  by an estimator  $\delta$  relatively at the quadratic loss function :

$$(1.1) \quad L(\delta, \theta) = \|\delta - \theta\|_p^2$$

where  $\|\cdot\|_p$  is the usual norm in  $\mathbb{R}^p$ . To this loss we associate its risk function:

$$R(\delta, \theta) = E_\theta(L(\delta, \theta)).$$

James and Stein [6] introduced a class of estimators improving  $\delta_0 = X$ , when the dimension of the space of the observations  $p$  is  $\geq 3$ , denoted by

$$(1.2) \quad \delta_{JS} = \left(1 - \frac{(p-2)S^2}{(n+2)\|X\|^2}\right) X,$$

in the case where  $\sigma^2$  is unknown where  $S^2 \sim \sigma^2 \chi_n^2$  is an estimate of  $\sigma^2$ , independent of  $X$ .

Baranchik [1] proposed the positive-part version of the James-Stein estimator, an estimator dominating the James-Stein estimator when  $p \geq 3$ :

$$(1.3) \quad \delta_{JS}^+ = \max\left(0, 1 - \frac{(p-2)S^2}{(n+2)\|X\|^2}\right) X.$$

Robert [9] gives an explicit formula of its quadratic risk. We give a simple demonstration of this domination in Section 4.

Casella and Hwang [4] studied the case where  $\sigma^2$  is known ( $\sigma^2 = 1$ ) and showed that if the limit of the ratio  $\frac{\|\theta\|^2}{p}$ , when  $p$  tends to infinity, is a constant  $c > 0$ , then

$$\lim_{p \rightarrow +\infty} \frac{R(\delta_{JS}(X), \theta)}{R(X, \theta)} = \lim_{p \rightarrow +\infty} \frac{R(\delta_{JS}^+(X), \theta)}{R(X, \theta)} = \frac{c}{1+c}, \quad c > 0.$$

Li Sun [7] has considered the following ANOVA1 model :

$(X_{ij} \mid \theta_j, \sigma^2) \sim N(\theta_j, \sigma^2) \quad i = 1, \dots, n, \quad j = 1, \dots, m$  where  $E(X_{ij}) = \theta_j$  for the group  $j$  and  $var(X_{ij}) = \sigma^2$  is unknown. In this case it is clear that the maximum likelihood estimator, denoted by  $\delta_0$ , has risk  $R(\delta_0, \theta) = \frac{m\sigma^2}{n}$ .

The James-Stein estimators are written in this case

$$\delta_{JS} = (\delta_{JS}^1, \delta_{JS}^2, \dots, \delta_{JS}^m)^t$$

with

$$\delta_{JS}^j = \left(1 - \frac{(m-3)S^2}{(N+2)T^2}\right) (X_{ij} - \bar{X}) + \bar{X}, \quad j = 1, \dots, m$$

and

$$S^2 = \sum_{i=1}^n \sum_{j=1}^m (X_{ij} - \bar{X}_j)^2, \quad T^2 = n \sum_{j=1}^m (\bar{X}_j - \bar{X})^2,$$

$$\bar{X}_j = \frac{\sum_{i=1}^n X_{ij}}{n}, \quad \bar{X} = \frac{\sum_{j=1}^m \bar{X}_j}{m}, \quad N = (n-1)m.$$

He shows that for any estimator of the form

$$\delta = (\delta_1, \dots, \delta_m)^t \quad \text{where} \quad \delta_j = [1 - \psi(S^2, T^2)] (\bar{X}_j - \bar{X}) + \bar{X}, \quad j = 1, \dots, m,$$

if  $\lim_{m \rightarrow +\infty} \frac{1}{m} \left( \sum_{j=1}^m (\theta_j - \bar{\theta})^2 \right) = c$  exists, then  $\lim_{m \rightarrow +\infty} \frac{R(\delta, \theta)}{R(\delta_0, \theta)} \geq \frac{c}{c + \frac{\sigma^2}{n}}$  and also  $\lim_{m \rightarrow +\infty} \frac{R(\delta_{JS}, \theta)}{R(\delta_0, \theta)} = \frac{c}{c + \frac{\sigma^2}{n}}$ . On the other hand  $\frac{c}{c + \frac{\sigma^2}{n}}$  constitutes a lower bound for the ratio  $\lim_{m \rightarrow +\infty} \frac{R(\delta, \theta)}{R(\delta_0, \theta)}$  and is equal to  $\lim_{m \rightarrow +\infty} \frac{R(\delta_{JS}, \theta)}{R(\delta_0, \theta)}$ .

Li Sun [7] also shows that this bound is attained for a class of estimators defined by

$$\delta = (\delta_1, \dots, \delta_m)^t \quad \text{where} \quad \delta_j = \left[ 1 - \psi(S^2, T^2) \frac{S^2}{T^2} \right] (\bar{X}_j - \bar{X}) + \bar{X}, \quad j = 1, \dots, m$$

and  $\psi$  satisfies certain conditions.

This bound is also attained for any estimator dominating the James-Stein estimator, in particular the positive-part version of the James-Stein estimator.

Finally, we note that if  $n$  tends to infinity then the ratio  $\frac{c}{c + \frac{\sigma^2}{n}}$  tends to 1, and thus the risk of the James-Stein estimator is that of  $\delta_0$  (when  $m$  and  $n$  tend to infinity).

Maruyama [8] considered the following model:  $Z \sim N_d(\theta, I_d)$  and the so-called  $l_p$ -norm given by:  $\|z\|_p = \left\{ \sum_{i=1}^{i=d} |z_i|^p \right\}^{\frac{1}{p}}, p > 0$ .

He also notes:  $\|z\|_p^m = \left\{ \sum_{i=1}^{i=d} |z_i|^p \right\}^{\frac{m}{p}}$ . He defined a new class of James-Stein estimators with ' $l_p$ -norm based shrinkage factor, defined as follows:

$\hat{\theta}_\phi = (\hat{\theta}_{1\phi}, \hat{\theta}_{2\phi}, \dots, \hat{\theta}_{d\phi})$  with:  $\hat{\theta}_{i\phi} = \left( 1 - \phi(\|z\|_p) / \|z\|_p^{2-\alpha} |z_i|^\alpha \right) z_i$  where  $0 \leq \alpha < (d-2)/(d-1), p > 0$ . (Since some components of the estimator can be exactly zero, the choice between a full model and reduced models is possible).

When  $d \geq 3$ , he establishes minimaxity and sparsity simultaneously, of this class of estimators with ' $l_p$ -norm based shrinkage factor, under conditions on  $\hat{\theta}_\phi$ , and any positive  $p$ .

Note that the risk functions of these estimators are calculated relatively to the usual quadratic loss function (1.1).

The calculation of risk ratios in this case, and the conditions on the report of the  $l_p$ -norm of  $\theta$  to the dimension of its space, change completely. Extension of our work to this type of estimators presents technical difficulties.

In our work we consider a different model and we obtain for several classes of shrinkage estimators (in particular the James-Stein estimator and its positive-part) that if

$\lim_{p \rightarrow +\infty} \frac{\|\theta\|^2}{\sigma^2 p} = c$  then the risk ratios tend to  $\frac{c}{1+c} < 1$ , when  $n$  and  $p$  tend to infinity.

In the following we denote the general form of a shrinkage estimator as follows:

$$(1.4) \quad \delta = (1 - \psi(S^2, \|X\|^2)) X.$$

We adopt the model  $X \sim N_p(\theta, \sigma^2 I_p)$  and independently of the observations  $X$ , we observe  $S^2 \sim \sigma^2 \chi_n^2$  an estimator of  $\sigma^2$ . Note that  $R(X, \theta) = p\sigma^2$  is the risk of the maximum likelihood estimator.

In Section 2, we recall two results obtained in the paper of Benmansour and Hamdaoui [2]. The authors showed, that if  $\lim_{p \rightarrow \infty} \frac{\|\theta\|^2}{p\sigma^2} = c (> 0)$ , then the risk ratio of James-Stein estimator  $\delta_{JS}$  to the maximum likelihood estimator  $X$ , tends to the value  $\frac{\frac{2}{n+2} + c}{1 + c}$  when  $p$  tends to infinity and  $n$  is fixed. The second result indicates that under the condition  $\lim_{p \rightarrow \infty} \frac{\|\theta\|^2}{p\sigma^2} = c (> 0)$ , the risk ratio of James-Stein estimator  $\delta_{JS}$  to the maximum likelihood estimator  $X$ , tends to the value  $\frac{c}{1 + c}$  when  $n$  and  $p$  tend simultaneously to infinity. We also get the same results with James-Stein positive-part estimator.

In the first part of Section 3 we show that under condition  $\lim_{p \rightarrow +\infty} \frac{\|\theta\|^2}{\sigma^2 p} = c, \lim_{n, p \rightarrow +\infty} \frac{R(\delta, \theta)}{R(X, \theta)} \geq \frac{c}{1 + c}$  and we prove by an argument which is different from the one in Benmansour and Hamdaoui [2], that under the same condition  $\lim_{p \rightarrow +\infty} \frac{\|\theta\|^2}{\sigma^2 p} = c, \lim_{n, p \rightarrow +\infty} \frac{R(\delta_{JS}, \theta)}{R(X, \theta)} = \frac{c}{1 + c}$ . We deduce that any shrinkage estimator defined in (1.4) dominating the James-Stein estimator also satisfies this property. In the second part of this section, we show that if  $\lim_{p \rightarrow +\infty} \frac{\|\theta\|^2}{\sigma^2 p} = c$ , then  $\lim_{n, p \rightarrow +\infty} \frac{R(\delta, \theta)}{R(X, \theta)} \geq \frac{c}{1 + c}$  on the one hand, and for certain forms of  $\psi$ , we show that  $\lim_{n, p \rightarrow +\infty} \frac{R(\delta, \theta)}{R(X, \theta)} = \frac{c}{1 + c}$ .

In Section 4 we consider conditions of minimaxity of an estimator, and show that for certain forms of minimax  $\delta$ , we have the same result as above.

By taking a class of estimators proposed by Benmansour and Mourid [3] (Proposition 4.4), estimators dominating the James-Stein estimator in the case  $\sigma^2$  is known, we propose a simple proof of the domination of the James Stein estimator by its positive-part in the case  $\sigma^2$  is unknown.

Finally, we graph the corresponding risks ratios for estimators of James-Stein  $\delta_{JS}$ , its positive-part  $\delta_{JS}^+$ , that of a minimax estimator, and an estimator dominating the James-Stein estimator in the sense of the quadratic risk ( polynomial estimators proposed by Tze Fen Li and Hou Wen Kuo [13] ) for various values of  $n$  and  $p$ .

## 2. Preliminaries

We recall that if  $X$  is a multivariate Gaussian random  $N_p(\theta, \sigma^2 I_p)$  in  $\mathbb{R}^p$ , then  $U = \frac{\|X\|^2}{\sigma^2} \sim \chi_p^2(\lambda)$  where  $\chi_p^2(\lambda)$  denotes the non-central chi-square distribution with  $p$  degrees of freedom and non-centrality parameter  $\lambda = \frac{\|\theta\|^2}{2\sigma^2}$ .

In this case, for  $\sigma^2 = 1$ , Casella and Hwang [4] have shown the inequalities

$$\frac{1}{(p - 2 + \|\theta\|^2)} \leq E\left(\frac{1}{\|X\|^2}\right) \leq \frac{p}{(p - 2)(p + \|\theta\|^2)}, \quad p \geq 3$$

that we generalize in the following lemma, in the case  $\sigma^2$  is unknown.

**2.1. Lemma.** *Let  $X \sim N_p(\theta, \sigma^2 I_p)$ ; if  $p \geq 3$  then*

$$(2.1) \quad \frac{1}{\sigma^2 \left( p - 2 + \frac{\|\theta\|^2}{\sigma^2} \right)} \leq E \left( \frac{1}{\|X\|^2} \right) \leq \frac{p}{\sigma^2 (p-2) \left( p + \frac{\|\theta\|^2}{\sigma^2} \right)}$$

*Proof.* It follows immediately from the inequalities of Casella and Hwang [4], since  $\frac{X}{\sigma} \sim N_p \left( \frac{\theta}{\sigma}, I_p \right)$   $\square$

From Robert [9], it is clear that the risk of the James-Stein estimator given in (1.2) is

$$R(\delta_{JS}, \theta) = \sigma^2 \left\{ p - \frac{n}{n+2} (p-2)^2 E \left( \frac{1}{p-2+2K} \right) \right\}$$

with  $K \sim P \left( \frac{\|\theta\|^2}{2\sigma^2} \right)$  being the Poisson distribution of parameter  $\frac{\|\theta\|^2}{2\sigma^2}$ .

**2.2. Theorem.** *If  $\lim_{p \rightarrow +\infty} \frac{\|\theta\|^2}{p\sigma^2} = c (> 0)$ , we have*

$$(2.2) \quad \lim_{p \rightarrow +\infty} \frac{R(\delta_{JS}, \theta)}{R(X, \theta)} = \frac{c + \frac{2}{n+2}}{c+1}.$$

*Proof.* See Benmansour and Hamdaoui [2].  $\square$

**2.3. Corollary.** *If  $\lim_{p \rightarrow +\infty} \frac{\|\theta\|^2}{p\sigma^2} = c (> 0)$ , we have*

$$(2.3) \quad \lim_{n, p \rightarrow +\infty} \frac{R(\delta_{JS}, \theta)}{R(X, \theta)} = \frac{c}{c+1}.$$

*Proof.* See Benmansour and Hamdaoui [2].  $\square$

### 3. Lower bound of shrinkage estimators

To calculate the risk function, we recall a lemma similar to Lemma 2.1 of Li Sun [7].

**3.1. Lemma.** *Let  $K \sim P \left( \frac{\|\theta\|^2}{2\sigma^2} \right)$ . Then*

$$(a) \quad E \{ f(S^2, \|X\|^2) \} = E \{ f(\sigma^2 \chi_n^2, \sigma^2 \chi_{p+2K}^2) \}$$

$$(b) \quad E \left\{ g(S^2, \|X\|^2) \sum_{j=1}^p \theta_j X_j \right\} = 2\sigma^2 E \{ K g(\sigma^2 \chi_n^2, \sigma^2 \chi_{p+2K}^2) \}$$

for any functions of two variables such that all expectations of (a) and (b) exist.

*Proof.* Analogous to the proof of Lemma 2.1 of Li Sun [7].  $\square$

In the case of our model, Theorem 2.1 of Li Sun [7] is written as follows:

**3.2. Theorem.** *The risk of the estimator given in (1.4) is*

$$R(\delta, \theta) = \sigma^2 E \{ \psi_K^2 \chi_{p+2K}^2 - 2\psi_K (\chi_{p+2K}^2 - 2K) + p \}$$

where  $\psi_K = \psi(\sigma^2 \chi_n^2, \sigma^2 \chi_{p+2K}^2)$  and  $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$ .

Furthermore  $R(\delta, \theta) \geq B_p(\theta)$  with

$$B_p(\theta) = \sigma^2 \left\{ p - 2 - E \left\{ \frac{(p-2)^2}{p-2+2K} \right\} \right\}.$$

*Proof.* Analogous to the proof of Theorem 2.1 of Li Sun [7], using Lemma 2.1. □

We set  $b_p(\theta) = \frac{B_p(\theta)}{R(\theta, X)}$ , then using Lemma 3.1 of Li Sun [7] and the fact that  $R(\theta, X) = p\sigma^2$ , we have

$$\frac{p-2}{p} - \frac{(p-2)^2}{p^2} \frac{1}{\frac{p-4}{p} + \frac{\|\theta\|^2}{p\sigma^2}} \leq b_p(\theta) \leq \frac{p-2}{p} - \frac{(p-2)^2}{p^2} \frac{1}{\frac{p-2}{p} + \frac{\|\theta\|^2}{p\sigma^2}}.$$

It is clear that if  $\lim_{p \rightarrow \infty} \frac{\|\theta\|^2}{p\sigma^2} = c$ , then

$$(3.1) \quad \lim_{p \rightarrow \infty} b_p(\theta) = \frac{c}{1+c}$$

In the case where  $\psi(S^2, \|X\|^2) = d \frac{S^2}{\|X\|^2}$ , we have  $\delta_d = \left(1 - d \frac{S^2}{\|X\|^2}\right) X$  hence

$$(3.2) \quad R(\delta_d, \theta) = \sigma^2 \left\{ p + n [d^2(n+2) - 2d(p-2)] E \left( \frac{1}{p-2+2K} \right) \right\}.$$

For  $d = \frac{(p-2)}{(n+2)}$  we obtain the James-Stein estimator (defined in (1.2)) which minimizes the risk of  $\delta_d$  whose quadratic risk is

$$(3.3) \quad R(\delta_{JS}, \theta) = \sigma^2 \left\{ p - \frac{n}{n+2} (p-2)^2 E \left( \frac{1}{p-2+2K} \right) \right\}.$$

Next we are interested in the ratios  $\frac{R(\delta, \theta)}{R(X, \theta)}$  in particular when  $n$  and  $p$  tend to infinity.

Casella and Hwang [4], showed in the case  $\sigma^2 = 1$  that if  $\lim_{p \rightarrow \infty} \frac{\|\theta\|^2}{p} = c (c > 0)$  then  $\lim_{p \rightarrow +\infty} \frac{R(\delta_{JS}(X), \theta)}{R(X, \theta)} = \frac{c}{1+c}$ . Li Sun [7] in his case showed that if  $\lim_{p \rightarrow +\infty} \frac{\sum_{j=1}^p (\theta_j - \bar{\theta})^2}{p} = c (c > 0)$ , then  $\lim_{p \rightarrow +\infty} \frac{R(\delta, \theta)}{R(\delta_0, \theta)} \geq \frac{c}{\frac{\sigma^2}{n} + c}$  and also  $\lim_{p \rightarrow \infty} \frac{R(\delta_{JS}, \theta)}{R(\delta_0, \theta)} = \frac{c}{\frac{\sigma^2}{n} + c}$  and therefore  $\lim_{n, p \rightarrow \infty} \frac{R(\delta_{JS}, \theta)}{R(\delta_0, \theta)} = 1$ .

We show in our work that if  $\lim_{p \rightarrow \infty} \frac{\|\theta\|^2}{\sigma^2 p} = c$ , then  $\lim_{n, p \rightarrow \infty} \frac{R(\delta, \theta)}{R(X, \theta)} \geq \frac{c}{1+c}$  on the one hand, and for some forms of  $\delta$ , we show that  $\lim_{n, p \rightarrow \infty} \frac{R(\delta, \theta)}{R(X, \theta)} = \frac{c}{1+c}$ .

Thus we ameliorate the result of Li Sun [7], obtaining a limit strictly less than 1.

**3.3. Proposition.** *If  $\lim_{p \rightarrow \infty} \frac{\|\theta\|^2}{\sigma^2 p} = c$ , then*

$$(3.4) \quad \lim_{n, p \rightarrow \infty} \frac{R(\delta, \theta)}{R(X, \theta)} \geq \frac{c}{1+c},$$

$$(3.5) \quad \lim_{n, p \rightarrow \infty} \frac{R(\delta_{JS}, \theta)}{R(X, \theta)} = \frac{c}{1+c}.$$

*Proof.* Formula (3.4) follows immediately from Theorem 3.2 and Formula (3.1). Formula (3.5) follows immediately from Corollary 2.3. Indeed Theorem 3.2 implies that  $\frac{R(\delta_{JS}, \theta)}{R(X, \theta)} \geq \frac{B_p(\theta)}{R(X, \theta)} = b_p(\theta)$ , and from (3.3), Lemma (2.1) and (3.1) we have

$$\frac{\frac{2}{n+2} + c}{1+c} \geq \lim_{p \rightarrow \infty} \frac{R(\delta_{JS}, \theta)}{R(X, \theta)} \geq \frac{c}{1+c}$$

thus

$$\lim_{n \rightarrow \infty} \frac{\frac{2}{n+2} + c}{1+c} \geq \lim_{n, p \rightarrow \infty} \frac{R(\delta_{JS}, \theta)}{R(X, \theta)} \geq \frac{c}{1+c}$$

hence

$$(3.6) \quad \lim_{n, p \rightarrow \infty} \frac{R(\delta_{JS}, \theta)}{R(X, \theta)} = \frac{c}{1+c}.$$

Thus we find exactly the same limit ratio Casella and Hwang [4], in the case where  $\sigma^2$  is unknown.  $\square$

In the following we study the families of estimators written as follows

$$(3.7) \quad \delta_\psi = \delta_{JS} + l\psi(S^2, \|X\|^2) \frac{S^2}{\|X\|^2} X, \quad l > 0$$

and we give simple conditions on  $\psi$  so that the limiting ratio  $\lim_{n, p \rightarrow \infty} \frac{R(\delta_\psi, \theta)}{R(X, \theta)}$  equals

$\frac{c}{1+c}$ , when  $\lim_{p \rightarrow \infty} \frac{\|\theta\|^2}{\sigma^2 p} = c$ , where  $\psi$  is a measurable function such that  $E[\psi^2(\sigma^2 \chi_n^2, \sigma^2 \chi_p^2(\lambda))] < \infty$ .

In this case, the difference of risks denoted by  $\Delta_{\psi_{JS}} = R(\delta_\psi, \theta) - R(\delta_{JS}, \theta)$  is:

$$(3.8) \quad \Delta_{\psi_{JS}} = E \left[ l^2 \frac{(\sigma^2 \chi_n^2)^2 \psi^2(\sigma^2 \chi_n^2, \sigma^2 \chi_p^2(\lambda))}{\sigma^2 \chi_p^2(\lambda)} + 2l\sigma^2 \chi_n^2 \psi(\sigma^2 \chi_n^2, \sigma^2 \chi_p^2(\lambda)) \right] \\ - 2ldE \left[ \frac{(\sigma^2 \chi_n^2)^2 \psi(\sigma^2 \chi_n^2, \sigma^2 \chi_p^2(\lambda))}{\sigma^2 \chi_p^2(\lambda)} \right] - 4l\lambda E \left[ \frac{\sigma^2 \chi_n^2 (\psi(\sigma^2 \chi_n^2, \sigma^2 \chi_{p+2}^2(\lambda)))}{\chi_{p+2}^2(\lambda)} \right],$$

where  $\lambda = \frac{\|\theta\|^2}{2\sigma^2}$  and  $d = \frac{p-2}{n+2}$ , see ( Benmansour and Mourid [3]).

For estimators of the form (3.7), which are not necessarily minimax we give the following two results which are analogous to Theorem 3.2 of Li Sun [7], with different conditions on  $\psi$  and whose risks ratios attain the lower bound  $B_m$ .

**3.4. Theorem.** Assume that  $\delta_\psi$  is given in (3.7) and that  $\psi(S^2, \|X\|^2)$  satisfies  
 a)  $|\psi(S^2, \|X\|^2)| \leq g(S^2)$  a.s; where  $E\left\{(g^2(S^2))^{1+\gamma}\right\} \leq (M(n))^{1+\gamma}$  for some  $\gamma > 0$ .  
 If  $\lim_{p \rightarrow \infty} \frac{\|\theta\|^2}{p\sigma^2} = c (> 0)$  then

$$(3.9) \quad \lim_{n, p \rightarrow +\infty} \frac{R(\delta_\psi, \theta)}{R(X, \theta)} = \frac{c}{1+c}$$

for all  $l$  such that  $l(M(n))^{1/2} = O\left(\frac{1}{n}\right)$  in the neighborhood of  $+\infty$ . Note that  $l$  may depend on  $n$ .

*Proof.* Relation (3.8) and condition a) give

$$\begin{aligned} \Delta_{\psi_{JS}} \leq & E \left[ l^2 \frac{(S^2)^2 g^2(S^2)}{\|X\|^2} + 2lS^2 g(S^2) + \frac{2ld(S^2)^2 g(S^2)}{\|X\|^2} \right] \\ & + 4l\lambda E(S^2 g(S^2)) E\left(\frac{1}{\chi_{p+2}^2(\lambda)}\right) \end{aligned}$$

thus

$$\begin{aligned} \Delta_{\psi_{JS}} \leq & \frac{l^2}{\sigma^2(p-2)} \left( E \left[ (\sigma^2 \chi_n^2)^{2(1+\gamma)/\gamma} \right] \right)^{\gamma/(1+\gamma)} M(n) \\ & + 2l \left[ E \left( (\sigma^2 \chi_n^2)^2 \right) \right]^{1/2} (M(n))^{1/2} \\ & + \frac{2l}{\sigma^2(n+2)} \left[ E \left( (\sigma^2 \chi_n^2)^4 \right) \right]^{1/2} (M(n))^{1/2} \\ & + 4l\lambda \frac{\left[ E \left( (\sigma^2 \chi_n^2)^2 \right) \right]^{1/2} (M(n))^{1/2}}{p}. \end{aligned}$$

The last inequality follows from Holder inequality, Schwarz inequality, the independence of  $\|X\|^2$  and  $S^2$  and that  $E\left(\frac{1}{\chi^2(p, \lambda)}\right) \leq \frac{1}{p-2}$ . Thus, for  $n$  close to infinity we have

$$\begin{aligned} \Delta_{\psi_{JS}} \leq & \frac{4\sigma^2 l^2 M(n)}{p-2} \left( \frac{\Gamma\left(\frac{n}{2} + \frac{2}{\gamma} + 2\right)}{\Gamma\left(\frac{n}{2}\right)} \right)^{\gamma/(1+\gamma)} + 2l\sigma^2 (M(n))^{1/2} [n(n+2)]^{1/2} \\ & + \frac{2\sigma^2 l (M(n))^{1/2} [(n+6)(n+4)(n+2)n]^{1/2}}{(n+2)} \\ & + \frac{4l\lambda\sigma^2 (M(n))^{1/2} [n(n+2)]^{1/2}}{p}. \end{aligned}$$

Now from Stirling's formula which expresses that in the neighborhood of  $+\infty$ , we have:  $\Gamma(y+1) \simeq \sqrt{2\pi} y^{y+\frac{1}{2}} e^{-y}$  and the fact that  $e^y = \lim_{n \rightarrow +\infty} \left(1 + \frac{y}{n}\right)^n$ , we have

$$(3.10) \quad \left( \frac{\Gamma\left(\frac{n}{2} + \frac{2}{\gamma} + 2\right)}{\Gamma\left(\frac{n}{2}\right)} \right)^{\gamma/(1+\gamma)} \simeq \left( \frac{n}{2} + \frac{2}{\gamma} + 1 \right)^2$$



thus

$$\begin{aligned} \lim_{n,p \rightarrow +\infty} \frac{\Delta_{\psi_{JS}}}{R(X, \theta)} &\leq \frac{4l^2 M(n)}{p(p-2)} \left( \frac{n}{2} + \frac{2}{\gamma} + 1 \right)^2 + \frac{2l(M(n))^{1/2} [n(n+2)]^{1/2}}{p} \\ &\quad + \frac{2l(M(n))^{1/2} [(n+6)(n+4)(n+2)n]^{1/2}}{p(n+2)} \\ &\quad + \frac{4l\lambda(M(n))^{1/2} [n(n+2)]^{1/2}}{p^2} \end{aligned}$$

Since  $\lim_{p \rightarrow \infty} \frac{2\lambda}{p} = c$  and  $l(M(n))^{1/2} = O\left(\frac{1}{n}\right)$  we finally obtain

$$\lim_{n,p \rightarrow +\infty} \frac{\Delta_{\psi_{JS}}}{R(X, \theta)} = \lim_{n,p \rightarrow \infty} \frac{R(\delta_\psi, \theta)}{R(X, \theta)} - \lim_{n,p \rightarrow \infty} \frac{R(\delta_{JS}, \theta)}{R(X, \theta)} \leq 0$$

and thus from (3.4) and (3.5)

$$\lim_{n,p \rightarrow \infty} \frac{R(\delta_\psi, \theta)}{R(X, \theta)} = \frac{c}{1+c},$$

hence the result. □

**3.5. Example.** Let  $\psi_1(S^2, \|X\|^2) = \frac{\|X\|^2}{S^2(\|X\|^2 + 1)}$ . In this case it suffices to take  $g(S^2) = \frac{1}{S^2}$  and to choose  $l = 1$ .

The following proposition gives the same result as Theorem 3.4 for a particular class of the shrinkage function  $\psi(S^2, \|X\|^2)$ . Indeed, we will choose  $g$  in  $L^2$  and not in  $L^{2(1+\gamma)}$  but with the constraint that  $g(S^2)$  is monotone non-increasing.

**3.6. Proposition.** Assume that  $\delta_\psi$  is given in (3.7) and that  $\psi$  satisfies:

a)  $|\psi(S^2, \|X\|^2)| \leq g(S^2)$  a.s where  $g(S^2)$  is monotone non-increasing such that  $E[g^2(S^2)] \leq M(n)$ .

If  $\lim_{p \rightarrow \infty} \frac{\|\theta\|^2}{\sigma^2 p} = c$ , then

$$(3.11) \quad \lim_{n,p \rightarrow \infty} \frac{R(\delta_\psi, \theta)}{R(X, \theta)} = \frac{c}{1+c}$$

for all  $l$  such that  $l(M(n))^{1/2} = O\left(\frac{1}{n}\right)$  in the neighborhood of  $+\infty$  ( $l$  may depend on  $n$ ).

*Proof.* Analogous to the proof of Theorem 3.4, so we give a brief idea. (3.8) and condition a) give

$$\begin{aligned} \Delta_{\psi_{JS}} &\leq \frac{l^2}{\sigma^2} \frac{E[(\sigma^2 \chi_n^2)^2] E[g^2(\sigma^2 \chi_n^2)]}{p-2} + 2l E[\sigma^2 \chi_n^2] E[g(\sigma^2 \chi_n^2)] \\ &\quad + \frac{2l}{\sigma^2(n+2)} E[(\sigma^2 \chi_n^2)^2] E[g(\sigma^2 \chi_n^2)] + 4l\lambda \frac{E(\sigma^2 \chi_n^2) E[g(\sigma^2 \chi_n^2)]}{p}. \end{aligned}$$

The last inequality comes from the fact that  $E\left(\frac{1}{\chi^2(p, \lambda)}\right) \leq \frac{1}{p-2}$  and that the covariance of two functions, one increasing and the other decreasing, is negative. Thus,

$$\begin{aligned} \lim_{n,p \rightarrow \infty} \frac{\Delta_{\psi_{JS}}}{R(X, \theta)} &\leq \lim_{n,p \rightarrow \infty} \frac{nl(M(n))^{1/2}}{p} \left( l \frac{(n+2)(M(n))^{1/2}}{(p-2)} + 4 + \frac{4\lambda}{p} \right) \\ &\leq 0 \end{aligned}$$

because  $\lim_{p \rightarrow \infty} \frac{2\lambda}{p} = c$ , and  $l(M(n))^{1/2} = O\left(\frac{1}{n}\right)$ . We finally obtain

$$\lim_{n,p \rightarrow \infty} \frac{\Delta_{\psi_{JS}}}{R(X, \theta)} = \lim_{n,p \rightarrow \infty} \frac{R(\delta_{\psi}, \theta)}{R(X, \theta)} - \lim_{n,p \rightarrow \infty} \frac{R(\delta_{JS}, \theta)}{R(X, \theta)} \leq 0$$

and from (3.4) and (3.5)

$$\lim_{n,p \rightarrow \infty} \frac{R(\delta_{\psi}, \theta)}{R(X, \theta)} = \frac{c}{1+c},$$

hence the result. □

**3.7. Example.** Let  $\psi_1(S^2, \|X\|^2) = \frac{\|X\|^2}{S^2(\|X\|^2 + 1)}$ , and therefore

$$(3.12) \quad \delta_{\psi_1} = \delta_{JS} + \frac{1}{\|X\|^2 + 1} X.$$

In this case we simply take  $g(S^2) = \frac{1}{S^2}$  and choose  $l = 1$ .

### 4. Minimacity

Now, we recall a result of Strawderman [12] about the minimacity of the following class of estimators. Let:

$$(4.1) \quad \delta_{\phi} = \left(1 - l\phi(S^2, \|X\|^2) \frac{S^2}{\|X\|^2}\right) X, \quad l > 0$$

**4.1. Theorem.** *If :*

- a)  $\phi(S^2, \|X\|^2)$  is monotone non-increasing in  $S^2$  and non-decreasing in  $\|X\|^2$ ,
- b)  $0 \leq \phi(S^2, \|X\|^2) \leq \frac{2(p-2)}{l(n+2)}$ , then  $\delta_{\phi}$  is minimax.

*Proof.* A simple proof of this result is as follows: For  $U = \frac{\|X\|^2}{\sigma^2}$ , we have

$$\begin{aligned} R(\delta_{\phi}, \theta) &= p\sigma^2 + \sigma^2 l E \left[ l \frac{(S^2)^2 \phi^2(S^2, \sigma^2 U)}{U} - 2(p-2) \frac{S^2 \phi(S^2, \sigma^2 U)}{U} \right] \\ &\quad - \sigma^2 E \left[ 4l \frac{S^2 \partial \phi(S^2, \sigma^2 U)}{\partial U} \right], \end{aligned}$$

by using the equality of Stein [11]. Since  $\phi(S^2, \|X\|^2)$  is non-decreasing in  $U$  it suffices to have

$$E \left[ l \frac{(S^2)^2 \phi^2(S^2, \sigma^2 U)}{U} - 2(p-2) \frac{S^2 \phi(S^2, \sigma^2 U)}{U} \right] \leq 0.$$

Setting  $C_0 = \frac{2(p-2)}{l(n+2)}$ , we have

$$\begin{aligned} &E \left[ l \frac{(S^2)^2 \phi^2(S^2, \sigma^2 U)}{U} - 2(p-2) \frac{S^2 \phi(S^2, \sigma^2 U)}{U} \right] \\ &= E \left[ \frac{\phi(S^2, \sigma^2 U)}{U} S^2 [l S^2 \phi(S^2, \sigma^2 U) - 2(p-2)] \right] \end{aligned}$$

$$\leq E \left[ \frac{\phi(S^2, \sigma^2 U)}{U} S^2 [lS^2 C_0 - 2(p-2)] \right].$$

Because  $\phi(S^2, \|X\|^2)$  is non-increasing in  $S^2$ , therefore in both cases where  $\phi(S^2, \|X\|^2) > C_0$  and  $\phi(S^2, \|X\|^2) \leq C_0$ , we have

$$\begin{aligned} & E \left[ l \frac{(S^2)^2 \phi^2(S^2, \sigma^2 U)}{U} - 2(p-2) \frac{S^2 \phi(S^2, \sigma^2 U)}{U} \right] \\ & \leq E \left[ \frac{\phi(C_0, \sigma^2 U)}{U} S^2 [lS^2 C_0 - 2(p-2)] \right]. \end{aligned}$$

As  $S^2$  and  $U$  are independent we obtain

$$\begin{aligned} & E \left[ l \frac{(S^2)^2 \phi^2(S^2, \sigma^2 U)}{U} - 2(p-2) \frac{S^2 \phi(S^2, \sigma^2 U)}{U} \right] \\ & \leq E \left[ \frac{\phi(C_0, \sigma^2 U)}{U} \right] E [l(S^2)^2 C_0 - 2(p-2)S^2] \\ & \leq 0 \end{aligned}$$

hence the result.  $\square$

Note that this class of minimax estimators admits as lower bound  $B_m = \frac{c}{1+c}$  (Proposition 3.3) but does not attain it.

Then we have the following proposition which gives a class of minimax estimators whose risks ratios attains the lower bound.

**4.2. Proposition.** Assume that  $\delta_\psi$  is as given in (3.7), i.e.,

$$\begin{aligned} (4.2) \quad \delta_\psi &= \delta_{JS} + l\psi(S^2, \|X\|^2) \frac{S^2}{\|X\|^2} X \\ &= \left( 1 - \left[ \frac{S^2}{\|X\|^2} \left( \frac{p-2}{n+2} - l\psi(S^2, \|X\|^2) \right) \right] \right) X, \quad l > 0. \end{aligned}$$

If  $\psi$  satisfies the following conditions:

- 1)  $\psi(S^2, \|X\|^2)$  is monotone non-decreasing in  $S^2$  and non-increasing in  $\|X\|^2$ .
- 2)  $|l\psi(S^2, \|X\|^2)| \leq \frac{p-2}{n+2}$ ,

then  $\lim_{p \rightarrow \infty} \frac{\|\theta\|^2}{\sigma^2 p} = c$  implies

$$\lim_{n, p \rightarrow \infty} \frac{R(\delta_\psi, \theta)}{R(X, \theta)} = \frac{c}{1+c}$$

for all  $l$  such that  $\lim_{n \rightarrow \infty} l(p-2) = 0$  ( $l$  depends on  $n$ ).

*Proof.* It follows immediately from Theorems 3.4 and 4.1.  $\square$

**4.3. Example.** Let the estimator

$$(4.3) \quad \delta_{\psi_2} = \delta_{JS} + l\psi_2(S^2, \|X\|^2) \frac{S^2}{\|X\|^2} X$$

such that  $l\psi_2(S^2, \|X\|^2) = \frac{p-2}{n+2} \frac{S^2}{S^2+1} \exp(-\|X\|^2)$ .

Note that the function  $\psi_2$  satisfies the conditions of Proposition 4.2.

We note that the estimators of the form (4.1) are minimax but do not necessarily dominate the James-Stein estimator under the usual quadratic risk.

A class of estimators dominating the James Stein estimator is given as follows:

Let:

$$(4.4) \quad \delta_\phi = \delta_{JS} + m\phi(S^2, \|X\|^2) X \quad m > 0$$

where  $\phi$  is a measurable positive function, such that  $E[\phi^2(S^2, \|X\|^2)] < \infty$ . In this case, the difference of risks denoted by  $\Delta_{\phi_{JS}} = R(\delta_\phi, \theta) - R(\delta_{JS}, \theta)$  is:

$$\begin{aligned} \Delta_{\phi_{JS}} &= E[m^2(\|X\|^2)\phi^2(S^2, \|X\|^2) + 2m(\|X\|^2)\phi(S^2, \|X\|^2)] \\ &\quad - E[2mdS^2\phi(S^2, \|X\|^2) + 4m\lambda(\phi(S^2, \sigma^2\chi_{p+2}^2(\lambda)))] \end{aligned}$$

thus

$$(4.5) \quad \Delta_{\phi_{JS}} \leq E[m\phi(S^2, \|X\|^2) [m\|X\|^2\phi(S^2, \|X\|^2) + 2\|X\|^2 - 2dS^2]],$$

where  $d = \frac{p-2}{n+2}$ .

Then we have the following proposition.

**4.4. Proposition.** *Estimators given in (4.2) dominate the James-Stein estimator if*

- 1)  $0 \leq \phi(S^2, \|X\|^2) \leq \frac{2}{m} \left( d \frac{S^2}{\|X\|^2} - 1 \right) I_{\left(\frac{p-2}{n+2} \frac{s^2}{\|X\|^2} - 1 \geq 0\right)}$ .
- 2) *If in addition,  $\lim_{p \rightarrow \infty} \frac{\|\theta\|^2}{\sigma^2 p} = c$ , then  $\lim_{n, p \rightarrow \infty} \frac{R(\delta_\phi, \theta)}{R(X, \theta)} = \frac{c}{1+c}$ .*

*Proof.* 1) It follows from inequality (4.5). 2) Immediate from (3.4) and (3.5). □

We observe that any estimator dominating the James-Stein estimator satisfies the property 2 of Proposition 4.3. Thus the class of estimators:

$\delta_m = \delta_{JS} + m\phi(S^2, \|X\|^2) = \delta_{JS} + m \left( \frac{p-2}{n+2} \frac{S^2}{\|X\|^2} - 1 \right) I_{\left(\frac{p-2}{n+2} \frac{s^2}{\|X\|^2} - 1 \geq 0\right)} = \delta_{JS} + m\delta_{JS}^-(S^2, \|X\|^2)X$ , dominates the James Stein estimator. And for  $m = 1$  we have  $\delta_\phi = \delta_{JS} + \delta_{JS}^-(S^2, \|X\|^2)X$ , hence  $\delta_\phi = \delta_{JS}^+(S^2, \|X\|^2) X$  dominates  $\delta_{JS}(S^2, \|X\|^2)$  according to Proposition 4.3.

Moreover, its risk is minimal at  $\lambda = 0$ , relative to the whole family of the class of estimators  $\delta_m = \delta_{JS} + m\delta_{JS}^-(S^2, \|X\|^2)X$ .

### 5. Simulation

We recall the form of the estimator introduced by Tze Fen Li and Wen Hou Kuo [13].

Let  $X \sim N_p(\theta, \sigma^2 I_p)$ ,  $Y = \frac{X}{\sigma} \sim N_p\left(\frac{\theta}{\sigma}, I_p\right)$ .

For all  $r \left( 2 < r < \frac{p+2}{2} \right)$ , we consider the family of polynomial estimators:

$$(5.1) \quad \delta_{TZ} = \delta_{JS} + \alpha(S^2)^{\frac{r}{2}} X \|X\|^{-r}$$

where

$$\alpha = \frac{(r-2)(n+p)}{2} \frac{\Gamma\left(\frac{n+r}{2}\right)}{\Gamma\left(\frac{n+2r}{2}\right)} \frac{\Gamma\left(\frac{p-r}{2}\right)}{\Gamma\left(\frac{p-2r+2}{2}\right)}.$$

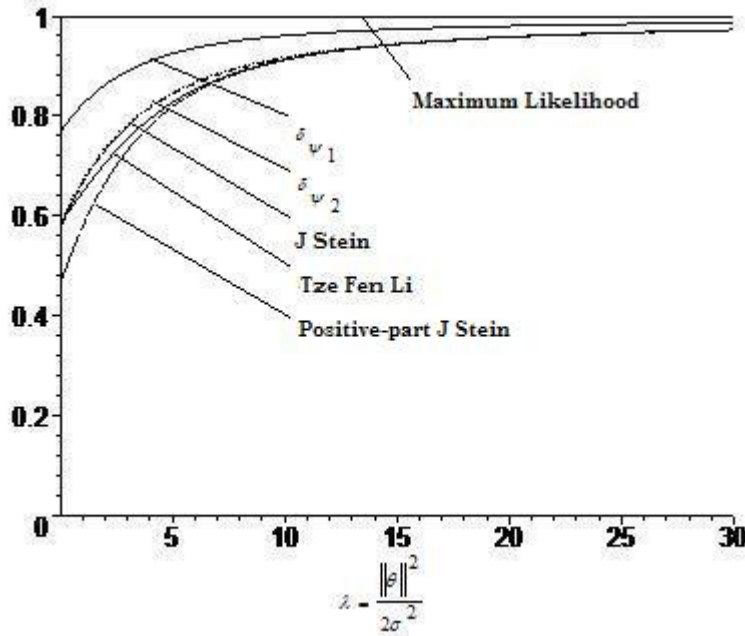
It is known by Tze Fen Li and Wen Hou Kuo [13], that the risk of the estimator  $\delta_{TZ}$  is

$$(5.2) \quad R(\delta_{TZ}, \theta) = R(\delta_{JS}, \theta) + \sigma^2 \frac{2\alpha 2^{\frac{r}{2}} \Gamma(\frac{n+r}{2})}{\Gamma(\frac{n}{2})} \left[ (p-r) - \frac{(p-2)(n+r)}{n+2} \right] E(\|Y\|^{-r}) + \sigma^2 \alpha^2 2^r \frac{\Gamma(\frac{n+2r}{2})}{\Gamma(\frac{n}{2})} E(\|Y\|^{-2r+2}).$$

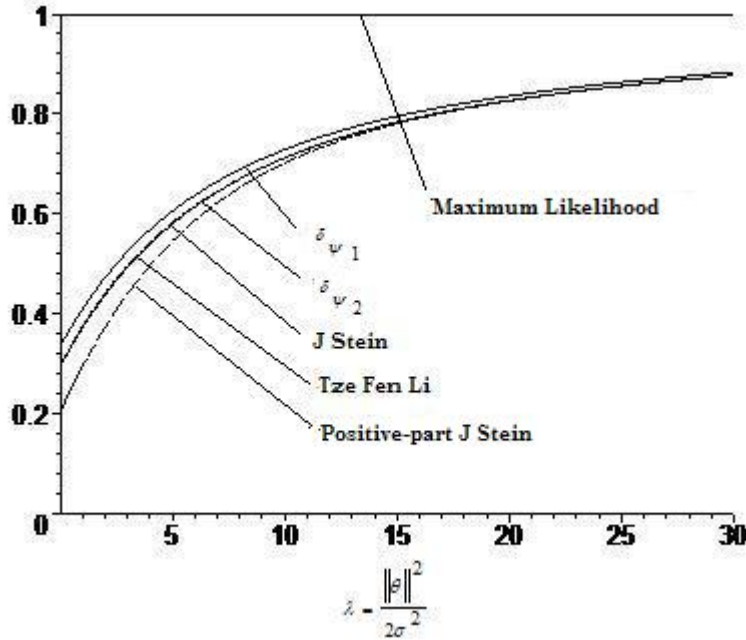
We recall the form of the estimators given in Example 3.7 (3.12), i.e.,  $\delta_{\psi_1} = \delta_{JS} + \frac{\|X\|^2}{S^2(\|X\|^2 + 1)} \frac{S^2}{\|X\|^2} X = \delta_{JS} + \frac{X}{\|X\|^2 + 1}$ , as well as in Example 4.3 (4.3), i.e.,  $\delta_{\psi_2} = \delta_{JS} + \frac{p-2}{n+2} \frac{S^4}{(S^2 + 1)\|X\|^2} \exp(-\|X\|^2) X$ , of which we graph their risks ratios as well as those of Tze Fen Li, James-Stein and the positive part- of James-Stein denoted respectively:

$$\frac{R(\delta_{\psi_1}, \theta)}{R(X, \theta)}, \frac{R(\delta_{\psi_2}, \theta)}{R(X, \theta)}, \frac{R(\delta_{TZ}, \theta)}{R(X, \theta)}, \frac{R(\delta_{JS}, \theta)}{R(X, \theta)}, \frac{R(\delta_{JS}^+, \theta)}{R(X, \theta)},$$

**Fig. 1** Graph of risk ratios  $\frac{R(\delta_{\psi_1}, \theta)}{R(X, \theta)}, \frac{R(\delta_{\psi_2}, \theta)}{R(X, \theta)}, \frac{R(\delta_{TZ}, \theta)}{R(X, \theta)}, \frac{R(\delta_{JS}, \theta)}{R(X, \theta)}, \frac{R(\delta_{JS}^+, \theta)}{R(X, \theta)}$  as functions of  $\lambda = \frac{\|\theta\|^2}{2\sigma^2}$  for  $n = 10$  and  $p = 4$ .



**Fig. 2** Graph of risk ratios  $\frac{R(\delta_{\psi_1, \theta})}{R(X, \theta)}$ ,  $\frac{R(\delta_{\psi_2, \theta})}{R(X, \theta)}$ ,  $\frac{R(\delta_{TZ, \theta})}{R(X, \theta)}$ ,  $\frac{R(\delta_{JS, \theta})}{R(X, \theta)}$ ,  $\frac{R(\delta_{JS}^+, \theta)}{R(X, \theta)}$  as functions of  $\lambda = \frac{\|\theta\|^2}{2\sigma^2}$  for  $n = 30$  and  $p = 8$ .



We note that in both graphs, the risk ratios tend to the same limit less than 1 where  $\lambda$  increases as well as  $n$  and  $p$ .

### 6. Conclusion

In the case of the estimate of the mean  $\theta$  of a multivariate gaussian random  $N_p(\theta, I_p)$  in  $\mathbb{R}^p$ , Casella and Hwang [4] showed that if  $\lim_{p \rightarrow +\infty} \frac{\|\theta\|^2}{p} = c > 0$  then the ratio  $\frac{R(\delta_{JS, \theta})}{R(X, \theta)}$  and  $\frac{R(\delta_{JS}^+, \theta)}{R(X, \theta)}$  tend to  $\frac{c}{1+c}$ . In our work by taking the same model, namely  $X \sim N_p(\theta, \sigma^2 I_p)$  with  $\sigma^2$  unknown, and estimated by the statistic  $S^2 \sim \sigma^2 \chi_n^2$  independent of  $X$ , we have showed that for the shrinkage estimators of the form  $\delta = (1 - \psi(S^2, \|X\|^2)) X$ , we obtain a similar ratio dependent of the sample size  $n$ , as soon as  $\lim_{p \rightarrow +\infty} \frac{\|\theta\|^2}{p\sigma^2} = c > 0$ . Moreover, we obtain a ratio constant less than 1, when  $n$  and  $p$  tend simultaneously to  $+\infty$ , without assuming any order relation or functional relation between  $n$  and  $p$ . We obtained the same result for particular forms of  $\delta$ , which are not necessarily minimax, and for other forms of  $\delta$  which are minimax. Finally we concluded that any shrinkage estimator dominating the James-Stein estimator has a risk ratio tending to  $\frac{c}{1+c}$  when  $n$  and  $p$  tend to infinity.

Li Sun [7] was also interested in the case where  $\sigma^2$  is unknown, but he studied the behaviour of the ratio  $\frac{R(\delta, \theta)}{R(X, \theta)}$ ,  $\frac{R(\delta_{JS, \theta})}{R(X, \theta)}$  and  $\frac{R(\delta_{JS}^+, \theta)}{R(X, \theta)}$ , when only  $p$  tends to infinity.

The simulations in the case of selected examples, show that the asymptotic behaviour of risk ratios are identical and converge to the same limit that is strictly less than 1.

An idea would be to see whether one can obtain similar ratios in the general case of the symmetrical spherical models. Expanding our work to minimax estimators proposed by Maruyama [8] is also an idea that we currently explore.

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