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Some normal subgroups of extended generalized Hecke groups

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In memory of my dear son Can Şahin.

Abstract

Generalized Hecke group $H_{p,\infty}(\lambda)$ is generated by $X(z) = -(z - \lambda_p)^{-1}$ and $Y(z) = -(z + \lambda)^{-1}$ where $\lambda_p = 2\cos\frac{\pi}{p}$, $p \geq 2$ integer and $\lambda \geq 2$. Extended generalized Hecke group $\overline{H}_{p,\infty}(\lambda)$ is obtained by adding the reflection $R(z) = 1/\overline{z}$ to the generators of generalized Hecke group $H_{p,\infty}(\lambda)$. In this paper, we study the commutator subgroups of extended generalized Hecke groups $\overline{H}_{p,\infty}(\lambda)$. Also, we determine the power subgroups of generalized Hecke groups $H_{p,\infty}(\lambda)$ and extended generalized Hecke groups $\overline{H}_{p,\infty}(\lambda)$.

Keywords: Generalized Hecke groups, Extended generalized Hecke groups, Commutator subgroups, Power subgroups.

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1. Introduction

In [1], Hecke introduced the groups $H(\lambda)$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \quad \text{and} \quad U(z) = z + \lambda,$$

where λ is a fixed positive real number. Let $S = TU$, i.e.,

$$S(z) = -\frac{1}{z + \lambda}.$$

Hecke showed that $H(\lambda)$ is discrete if and only if either $\lambda = \lambda_q = 2\cos(\frac{\pi}{q})$, $q \geq 3$ integer, or $\lambda \geq 2$. These groups have come to be known as the *Hecke groups* and we will denote them by H_q, or by $H(\lambda)$, respectively. The first few Hecke groups are $H_3 = PSL(2, \mathbb{Z})$ (the modular group), $H_4 = H(\sqrt{2})$, $H_5 = H(\frac{1+\sqrt{5}}{2})$, and $H_6 = H(\sqrt{3})$ for $q = 3, 4, 5$ and 6, respectively.

It is known that when $\lambda = \lambda_q = 2\cos(\frac{\pi}{q})$, $q \geq 3$ integer, Hecke group H_q is isomorphic to the free product of two finite cyclic groups of orders 2 and q,

$$H_q =< T, S \mid T^2 = S^q = I >\cong C_2 * C_q,$$

and when $\lambda \geq 2$, Hecke group $H(\lambda)$ is a free product of a cyclic group of order 2 and infinity, so all such $H(\lambda)$ have the same algebraic structure, i.e.

$$H(\lambda) =< T,\ S \mid T^2 = I >\cong C_2 * \mathbb{Z}.$$

Also Hecke group H_q or $H(\lambda)$ is the Fuchsian group of the first kind when either $\lambda = \lambda_q = 2\cos(\frac{\pi}{q})$, $q \geq 3$ integer or $\lambda = 2$, and $H(\lambda)$ is the Fuchsian group of the second kind when $\lambda > 2$.

On the other hand, Lehner studied in [2] more general class $H_{p,q}$ of Hecke groups H_q, by taking

$$X = \frac{-1}{z - \lambda_p} \text{ and } V = z + \lambda_p + \lambda_q,$$

where $2 \leq p \leq q \leq \infty$, $p + q > 4$. Here if we take $Y = XV = -\frac{1}{z+\lambda_q}$, then we have the presentation,

$$H_{p,q} =< X, Y \mid X^p = Y^q = I >\cong C_p * C_q. \quad (1.1)$$

We call these groups as *generalized Hecke groups* $H_{p,q}$. We know from [2] that $H_{2,q} = H_q$, $|H_q : H_{q,q}| = 2$, and there is no group $H_{2,2}$. Also, all Hecke groups H_q are included in generalized Hecke groups $H_{p,q}$. Also, generalized Hecke groups $H_{p,q}$ have been studied extensively for many aspects in the literature (for examples, please see, [3], [4], [5], [6], [7] and [8]).

Extended generalized Hecke groups $\overline{H}_{p,q}$ have been defined in [9] and [10], similar to extended Hecke groups $\overline{H}_q$ (please see, [11] and [12]), by adding the reflection $R(z) = 1/\overline{z}$ to the generators of generalized Hecke group $H_{p,q}$. From [9], extended generalized Hecke groups $\overline{H}_{p,q}$ have a presentation

$$\overline{H}_{p,q} =< X, Y, R \mid X^p = Y^q = R^2 = I,\ RX = X^{-1}R, RY = Y^{-1}R >,$$

or

$$\overline{H}_{p,q} =< X, Y, R \mid X^p = Y^q = R^2 = (XR)^2 = (YR)^2 = I >\cong D_p *_{C_2} D_q.$$

The group $H_{p,q}$ is a subgroup of index 2 in $\overline{H}_{p,q}$.

In (1.1), if $q = \infty$, then we have more general class $H_{p,\infty}$, of Hecke groups $H(\lambda)$.

Now we can give the following definitions;

1.1. Definition. Let $\lambda_p = 2\cos\frac{\pi}{p}$, $p \geq 2$ integer and let $\lambda \geq 2$. Generalized Hecke groups $H_{p,\infty}(\lambda)$ are defined as the groups generated by

$$X = \frac{-1}{z-\lambda_p} \text{ and } Y = -\frac{1}{z+\lambda},$$

and have a presentation

$$H_{p,\infty}(\lambda) =< X, Y \mid X^p = Y^\infty = I >\cong C_p * \mathbb{Z}.$$

1.2. Definition. Extended generalized Hecke groups $\overline{H}_{p,\infty}(\lambda)$, are defined by adding reflection $R(z) = 1/\overline{z}$ to the generators of generalized Hecke groups $H_{p,\infty}(\lambda)$ and have a presentation

$$\overline{H}_{p,\infty}(\lambda) =< X, Y, R \mid X^p = Y^\infty = R^2 = I, RX = X^{p-1}R, RY = Y^{-1}R >,$$

or

$$\begin{aligned}\overline{H}_{p,\infty}(\lambda) &= < X, Y, R \mid X^p = Y^\infty = R^2 = (XR)^2 = (YR)^2 = I >,\\ &\cong D_p *_{C_2} D_\infty.\end{aligned}$$

In this paper, we study the commutator subgroups of extended generalized Hecke groups $\overline{H}_{p,\infty}(\lambda)$. Then, we determine the power subgroups of generalized Hecke groups $H_{p,\infty}(\lambda)$ and extended generalized Hecke groups $\overline{H}_{p,\infty}(\lambda)$. We use the Reidemeister-Schreier method to get the generators of all these subgroups.

Let G be a group and N be a normal subgroup of G with finite index. According to the Reidemeister-Schreier method we get the generators of N as follows: We first choose a Schreier transversal Σ for the quotient group G/N such that all certain words of generators including.Note that this transversal is not unique. Then we get the generators of N as following order:

$$\text{(An element of } \Sigma) \times \text{(A generator of } G) \times$$
$$\text{(coset representative of the preceeding product)}^{-1}.$$

For more details please see [13].

Commutator subgroups and power subgroups of Hecke and extended Hecke groups have been studied in, [14], [15], [17], [20], [23], [24] and [25]. Here, our aim is to generalize the results given in [14] and [15] for Hecke groups $H(\lambda)$ and extended Hecke groups $\overline{H}(\lambda)$ to extended generalized Hecke groups $\overline{H}_{p,\infty}(\lambda)$.

2. Commutator Subgroups of Extended Generalized Hecke Groups $\overline{H}_{p,\infty}(\lambda)$

Since the index of the commutator subgroup $H'_{p,\infty}(\lambda)$ in $H_{p,\infty}(\lambda)$ is infinite, we study only the commutator subgroup $\overline{H}'_{p,\infty}(\lambda)$ of extended generalized Hecke groups $\overline{H}_{p,\infty}(\lambda)$.

Here, we investigate the cases of p, odd or even, seperately.

2.1. Theorem. *Let $p \geq 3$ be an odd integer and let $\lambda \geq 2$. Then*

1) $\left|\overline{H}_{p,\infty}(\lambda) : \overline{H}'_{p,\infty}(\lambda)\right| = 4.$

2) $\overline{H}'_{p,\infty}(\lambda) =< X, YXY^{-1}, Y^2 \mid X^p = (YXY^{-1})^p$
$= (Y^2)^\infty = I >\cong C_p * C_p * \mathbb{Z}.$

Proof. 1) Firstly, we set up the quotient group $\overline{H}_{p,\infty}(\lambda)/\overline{H}'_{p,\infty}(\lambda)$ which can be construct by adding the abelianizing relation to the relations of $\overline{H}_{p,\infty}(\lambda)$. Then

$$\begin{aligned}\overline{H}_{p,\infty}(\lambda)/\overline{H}'_{p,\infty}(\lambda) =< X, Y, R \mid X^p = \ Y^\infty = R^2 \ = \ I,\ RX = X^{p-1}R,\\ RY = Y^{-1}R,\ \ XR = RX,\ YR = RY,\ XY = YX > .\end{aligned}$$

Since p is odd and from the relations $RX = X^{p-1}R$ and $RX = XR$, we have $X = I$. Also we get $Y^2 = I$ from the relations $RY = Y^{-1}R$ and $YR = RY$. Thus we have

$$\overline{H}_{p,\infty}(\lambda)/\overline{H}'_{p,\infty}(\lambda) =< Y, R \mid Y^2 = R^2 = (YR)^2 = I >\simeq C_2 \times C_2.$$

2) Now we determine the set of generators for $\overline{H}'_{p,\infty}(\lambda)$. We choose a Schreier transversal for $\overline{H'}_{p,\infty}(\lambda)$ as $\Sigma = \{I, Y, R, YR\}$. According to Reidemeister-Schreier method we can form all possible products;

$$\begin{array}{lll} I.X.(I)^{-1} = X, & I.Y.(Y)^{-1} = I, & I.R.(R)^{-1} = I, \\ Y.X.(Y)^{-1} = YXY^{-1}, & Y.Y.(I)^{-1} = Y^2, & Y.R.(YR)^{-1} = I, \\ R.X.(R)^{-1} = X^{p-1}, & R.Y.(YR)^{-1} = Y^{-2}, & R.R.(I)^{-1} = I, \\ YR.X.(YR)^{-1} = YX^{p-1}Y^{-1}, & YR.Y.(R)^{-1} = I, & YR.R.(Y)^{-1} = I. \end{array}$$

Since $X^{-1} = X^{p-1}, (YXY^{-1})^{-1} = YX^{p-1}Y^{-1}$ and $(Y^2)^{-1} = Y^{-2}$, the generators are X, YXY^{-1} and Y^2. Thus $\overline{H'}_{p,\infty}(\lambda)$ has a presentation

$$\begin{aligned} \overline{H}'_{p,\infty}(\lambda) &= < X, YXY^{-1}, Y^2 \mid X^p = (YXY^{-1})^p \\ &= (Y^2)^\infty = I >\cong C_p * C_p * \mathbb{Z}. \end{aligned}$$

□

2.2. Theorem. *Let $p \geq 2$ be an even integer and let $\lambda \geq 2$. Then*

1) $\left|\overline{H}_{p,\infty}(\lambda) : \overline{H}'_{p,\infty}(\lambda)\right| = 8.$

2)

$$\begin{aligned} \overline{H}'_{p,\infty}(\lambda) &= < X^2, YX^2Y^{-1},\ XYXY^{-1}, Y^2, XY^2X^{-1} \mid (X^2)^{p/2} \\ &= (YX^2Y^{-1})^{p/2} = (XYXY^{-1})^\infty = (Y^2)^\infty = (XY^2X^{-1})^\infty = I > \\ &\cong C_{p/2} * C_{p/2} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}. \end{aligned}$$

Proof. 1) Similar to the previous proof, we have the quotient group $\overline{H}_{p,\infty}(\lambda)/\overline{H}'_{p,\infty}(\lambda)$ as

$$\begin{aligned} \overline{H}_{p,\infty}(\lambda)/\overline{H}'_{p,\infty}(\lambda) =&< X, Y, R \mid X^p = Y^\infty = R^2 = I,\ RX = X^{p-1}R, \\ & RY = Y^{-1}R,\ XR = RX,\ YR = RY,\ XY = YX > . \end{aligned}$$

Since p is even and from the relations $RX = X^{p-1}R$, $XR = RX$, $RY = Y^{-1}R$ and $YR = RY$, we have $X^2 = I$ and $Y^2 = I$. Thus we get

$$\begin{aligned} \overline{H}_{p,\infty}(\lambda)/\overline{H}'_{p,\infty}(\lambda) &= < X, Y, R : X^2 = Y^2 = R^2 = (XY)^2 = (XR)^2 = (YR)^2 = I >, \\ &\cong C_2 \times C_2 \times C_2. \end{aligned}$$

2) Now we can determine the Schreier transversal as $\Sigma = \{I,\ X,\ Y,\ R,\ XR,\ YR,\ XY,\ XYR\}$. From the Reidemeister-Schreier method all possible products are;

$$\begin{array}{rcl} I.X.(X)^{-1} = I, & & I.Y.(Y)^{-1} = I, \\ X.X.(I)^{-1} = X^2, & & X.Y.(XY)^{-1} = I, \\ Y.X.(XY)^{-1} = YXY^{-1}X^{p-1}, & & Y.Y.(I)^{-1} = Y^2, \\ R.X.(XR)^{-1} = X^{p-2}, & & R.Y.(YR)^{-1} = Y^{-2}, \\ XR.X.(R)^{-1} = I, & & XR.Y.(XYR)^{-1} = XY^{-2}X^{-1}, \\ YR.X.(XYR)^{-1} = YX^{-1}Y^{-1}X^{-1}, & & YR.Y.(R)^{-1} = I, \\ XY.X.(Y)^{-1} = XYXY^{-1}, & & XY.Y.(X)^{-1} = XY^2X^{-1}, \\ XYR.X.(YR)^{-1} = XYX^{-1}Y^{-1}, & & XYR.Y.(XR)^{-1} = I, \end{array}$$

$$\begin{array}{c} I.R.(R)^{-1} = I, \\ X.R.(XR)^{-1} = I, \\ Y.R.(YR)^{-1} = I, \\ R.R.(I)^{-1} = I, \\ XR.R.(X)^{-1} = I, \\ YR.R.(Y)^{-1} = I, \\ XY.R.(XYR)^{-1} = I, \\ XYR.R.(XY)^{-1} = I. \end{array}$$

Since $(X^2)^{-1} = X^{p-2}$, $(YXY^{-1}X^{p-1})^{-1} = XYX^{-1}Y^{-1}$, $(YX^{-1}Y^{-1}X^{-1})^{-1} = XYXY^{-1}$, $(Y^2)^{-1} = Y^{-2}$, $(XY^2X^{-1})^{-1} = XY^{-2}X^{-1}$, we have the presentation of $\overline{H}'_{p,\infty}(\lambda)$ as

$$\begin{aligned} \overline{H}'_{p,\infty}(\lambda) &= \ < X^2, YX^2Y^{-1},\ XYXY^{-1}, Y^2, XY^2X^{-1} \mid (X^2)^{p/2} \\ &= (YX^2Y^{-1})^{p/2} = (XYXY^{-1})^{\infty} = (Y^2)^{\infty} = (XY^2X^{-1})^{\infty} = I > \\ &\cong C_{p/2} * C_{p/2} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}. \end{aligned}$$

□

3. Power Subgroups of $H_{p,\infty}(\lambda)$ and $\overline{H}_{p,\infty}(\lambda)$

In this section, we consider the power subgroups of generalized Hecke groups $H_{p,\infty}(\lambda)$ and extended generalized Hecke groups $\overline{H}_{p,\infty}(\lambda)$. Here, we note that the power subgroups of Hecke groups H_q, or $H(\lambda)$ and extended Hecke groups $\overline{H}_q$, or $\overline{H}(\lambda)$ have been studied by many authors in [6], [7], [10], [11], [12], [14], [16], [18], [19], [21], [22].

Now we give some information about the power subgroups.

Let m be a positive integer. Let us define G^m to be the subgroup generated by the m^{th} powers of all elements of $G = H_{p,\infty}(\lambda)$ or $\overline{H}_{p,\infty}(\lambda)$. The subgroup G^m is called the m^{th} *– power subgroup* of G. As fully invariant subgroups, they are normal in G.

From the definition, it is easy to see that

$$G^{mk} < G^m$$

and

$$G^{mk} < (G^m)^k.$$

We now discuss the group theoretical structure of these subgroups. We find a presentation for the quotient G/G^m by adding the relation $A^m = I$ to the presentation of G. The order of G/G^m gives us the index. Thus we use the Reidemeister-Schreier process to find the presentation of the power subgroups G^m.

Let us start with $H_{p,\infty}(\lambda)$.

3.1. Theorem. 1) *Let $p > 2$ be an odd integer and $\lambda \geq 2$. Then,*
$H^2_{p,\infty}(\lambda) = < X, YXY^{-1},\ Y^2 \mid X^p = (YXY^{-1})^p = (Y^2)^{\infty} = I > \cong C_p * C_p * \mathbb{Z}$.
2) *Let $p \geq 2$ be an even integer and $\lambda \geq 2$. Then,*

$$H^2_{p,\infty}(\lambda) =< X^2, YX^2Y^{-1},\ XYXY^{-1}, Y^2, XY^2X^{-1} \mid (X^2)^{p/2}$$
$$= (YX^2Y^{-1})^{p/2} = (XYXY^{-1})^{\infty} = (Y^2)^{\infty} = (XY^2X^{-1})^{\infty} = I > .$$

Proof. 1) The quotient group $H_{p,\infty}(\lambda)/H^2_{p,\infty}(\lambda)$ is

$$H_{p,\infty}(\lambda)/H^2_{p,\infty}(\lambda) =< X, Y \mid X^p = Y^{\infty} = (XY)^{\infty} = X^2 = Y^2 = (XY)^2 = \cdots = I > .$$

Since $p > 2$ is an odd integer and from the relations $X^2 = X^p = I$ and $Y^2 = Y^{\infty} = I$, we have $X = Y^2 = I$. Thus we get

$$H_{p,\infty}(\lambda)/H^2_{p,\infty}(\lambda) =< Y \mid Y^2 = I >\cong C_2.$$

If we choose a Schreier transversal as $\{I,\ Y\}$ and use the Reidemeister-Schreier method, we obtain all possible products;

$$\begin{array}{ll} I.X.(I)^{-1} = X, & I.Y.(Y)^{-1} = I, \\ Y.X.(Y)^{-1} = YXY^{-1}, & Y.Y.(I)^{-1} = Y^2. \end{array}$$

So we get the presentation of $H^2_{p,\infty}(\lambda)$ as

$$H^2_{p,\infty}(\lambda) =< X, YXY^{-1},\ Y^2 \mid X^p = (YXY^{-1})^p = (Y^2)^{\infty} = I >\cong C_p * C_p * \mathbb{Z}.$$

2) The quotient group $H_{p,\infty}(\lambda)/H^2_{p,\infty}(\lambda)$ is

$$\begin{aligned} H_{p,\infty}(\lambda)/H^2_{p,\infty}(\lambda) &= < X, Y \mid X^p = Y^{\infty} = (XY)^{\infty} \\ &= X^2 = Y^2 = (XY)^2 = \cdots = I > . \end{aligned}$$

Since $p \geq 2$ is an even integer and from the relations $X^2 = X^p = I$ and $Y^2 = Y^{\infty} = I$, we obtain $X^2 = Y^2 = I$. Thus we have

$$H_{p,\infty}(\lambda)/H^2_{p,\infty}(\lambda) =< X, Y \mid X^2 = Y^2 = (XY)^2 = I >\cong D_2.$$

Now we choose a Schreier transversal as $\{I,\ X,\ Y,\ XY\}$ for $H^2_{p,\infty}(\lambda)$. According to the Reidemeister-Schreier method, we can form all possible products;

$$\begin{array}{ll} I.X.(X)^{-1} = I, & I.Y.(Y)^{-1} = I, \\ X.X.(I)^{-1} = X^2, & X.Y.(XY)^{-1} = I, \\ Y.X.(XY)^{-1} = YXY^{-1}X^{-1}, & Y.Y.(I)^{-1} = Y^2, \\ XY.X.(Y)^{-1} = XYXY^{-1}, & XY.Y.(X)^{-1} = XY^2X^{-1}. \end{array}$$

Thus we obtain a presentation of $H^2_{p,\infty}(\lambda)$ as

$$\begin{aligned} H^2_{p,\infty}(\lambda) &= < X^2, YX^2Y^{-1},\ XYXY^{-1}, Y^2, XY^2X^{-1} \mid (X^2)^{p/2} \\ &= (YX^2Y^{-1})^{p/2} = (XYXY^{-1})^{\infty} = (Y^2)^{\infty} = (XY^2X^{-1})^{\infty} = I > \\ &\cong C_{p/2} * C_{p/2} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}. \end{aligned}$$

□

3.2. Theorem. *Let $\lambda \geq 2$. If m and p are positive integers such that $(m, p) = 1$, then*

$$\begin{aligned} H^m_{p,\infty}(\lambda) &= < X, YXY^{-1},\ Y^2XY^{-2}, \cdots, Y^{m-1}XY^{1-m}, Y^m \mid X^p \\ &= (YXY^{-1})^p = (Y^2XY^{-2})^p = \cdots = (Y^{m-1}XY^{1-m})^p = (Y^m)^{\infty} = I > \\ &\cong \underbrace{C_p * C_p * \cdots * C_p}_{m\ times} * \mathbb{Z}. \end{aligned}$$

Proof. The quotient group $H_{p,\infty}(\lambda)/H^m_{p,\infty}(\lambda)$ is

$$H_{p,\infty}(\lambda)/H^m_{p,\infty}(\lambda) =< X, Y \mid X^p = Y^{\infty} = (XY)^{\infty} = X^m = Y^m = (XY)^m = \cdots = I > .$$

Since $(m, p) = 1$ and from the relations $X^p = X^m = I$, we find $X = I$. Thus we have

$$H_{p,\infty}(\lambda)/H^m_{p,\infty}(\lambda) =< Y : Y^m = I >\cong C_m.$$

Then we choose the Schreier transversal as $\Sigma = \{I, Y, Y^2, ..., Y^{m-1}\}$. According to the Reidemeister-Schreier method, we get the following products;

$$\begin{array}{ll} I.X.(I)^{-1} = X, & I.Y.(Y)^{-1} = I, \\ Y.X.(Y)^{-1} = YXY^{-1}, & Y.Y.(Y^2)^{-1} = I, \\ Y^2.X.(Y^2)^{-1} = Y^2XY^{-2}, & Y^2.Y.(Y^3)^{-1} = I, \\ Y^3.X.(Y^3)^{-1} = Y^3XY^{-3}, & Y^3.Y.(Y^4)^{-1} = I, \\ \vdots & \vdots \\ Y^{m-1}.X.(Y^{m-1})^{-1} = Y^{m-1}XY^{1-m}, & Y^{m-1}.Y.(I)^{-1} = Y^m. \end{array}$$

So we have a presentation of $H^2_{p,\infty}(\lambda)$ as

$$\begin{aligned} H^m_{p,\infty}(\lambda) &= < X, YXY^{-1},\ Y^2XY^{-2}, \cdots, Y^{m-1}XY^{1-m}, Y^m \mid X^p \\ &= (YXY^{-1})^p = (Y^2XY^{-2})^p = \cdots = (Y^{m-1}XY^{1-m})^p = (Y^m)^\infty = I > \\ &\cong \underbrace{C_p * C_p * \cdots * C_p}_{m \text{ times}} * \mathbb{Z}. \end{aligned}$$

□

The case $(m,p) = d > 1$, except of $m = 2$ and p even, is more complex, since the index of quotient group $H_{p,\infty}(\lambda)/H^m_{p,\infty}(\lambda)$ is unknown. In this case, we have the relations $X^d = Y^m = (XY)^m = \cdots = I$ and can not say anything about the power subgroups $H^m_{p,\infty}(\lambda)$.

Now we consider the power subgroups $\overline{H}^m_{p,\infty}(\lambda)$ of extended generalized Hecke groups $\overline{H}_{p,\infty}(\lambda)$. Here, we interest with the cases such that the index of the quotient group $\overline{H}_{p,\infty}(\lambda)/\overline{H}^m_{p,\infty}(\lambda)$ is finite.

3.3. Theorem. 1) *Let $p > 2$ be an odd integer and $\lambda \geq 2$. Then,*
$\overline{H}^2_{p,\infty}(\lambda) =< X, YXY^{-1},\ Y^2 \mid X^p = (YXY^{-1})^p = (Y^2)^\infty = I >\cong C_p * C_p * \mathbb{Z}.$
2) *Let $p \geq 2$ be an even integer and $\lambda \geq 2$. Then,*
$\overline{H}^2_{p,\infty}(\lambda) =< X^2, YX^2Y^{-1},\ XYXY^{-1}, Y^2, XY^2X^{-1} \mid (X^2)^{p/2}$
$= (YX^2Y^{-1})^{p/2} = (XYXY^{-1})^\infty = (Y^2)^\infty = (XY^2X^{-1})^\infty = I >.$

Proof. The quotient group $\overline{H}_{p,\infty}(\lambda)/\overline{H}^2_{p,\infty}(\lambda)$ is

$$\begin{aligned} \overline{H}_{p,\infty}(\lambda)/\overline{H}^2_{p,\infty}(\lambda) =< X, Y, R \mid X^p = Y^\infty = R^2 = (XR)^2 = (YR)^2 \\ = X^2 = Y^2 = (XY)^2 = \cdots = I >. \end{aligned}$$

The rest of the proof is similar to the proof of the Theorems 1 and 2. □

By using the Theorems 1, 2, 3 and 5, we can give the following.

3.4. Corollary. $\overline{H}^2_{p,\infty}(\lambda) = H^2_{p,\infty}(\lambda) = \overline{H}'_{p,\infty}(\lambda)$.

3.5. Theorem. 1) *Let $\lambda \geq 2$ and let $p \geq 3$ be an odd number. If m is an even positive integer such that $(m,p) = 1$, then*

$$\begin{aligned} \overline{H}^m_{p,\infty}(\lambda) &= < X, YXY^{-1},\ Y^2XY^{-2}, \cdots, Y^{m-1}XY^{1-m}, Y^m \mid X^p \\ &= (YXY^{-1})^p = (Y^2XY^{-2})^p = \cdots = (Y^{m-1}XY^{1-m})^p = (Y^m)^\infty = I > \\ &\cong \underbrace{C_p * C_p * \cdots * C_p}_{m \text{ times}} * \mathbb{Z}. \end{aligned}$$

2) *Let $\lambda \geq 2$. If $m > 0$ is odd integer such that $(m,p) = 1$, then $\overline{H}^m_{p,\infty}(\lambda) = \overline{H}_{p,\infty}(\lambda)$.*

Proof. 1) The quotient group $\overline{H}_{p,\infty}(\lambda)/\overline{H}^{m}_{p,\infty}(\lambda)$ is

$$\begin{aligned}\overline{H}_{p,\infty}(\lambda)/\overline{H}^{m}_{p,\infty}(\lambda) =&< X,Y,R \mid X^p = Y^\infty = R^2 = (XR)^2 = (YR)^2 \\ &= X^m = Y^m = (XY)^m = \cdots = I > .\end{aligned}$$

Since $(m,p)=1$ and m is even, we have $X = I$.

$$\overline{H}_{p,\infty}(\lambda)/\overline{H}^{m}_{p,\infty}(\lambda) =< Y,R : Y^m = R^2 = (YR)^2 = ... = I >\cong D_m.$$

Considering the presentation of quotient group we can choose Schreier transversal as $\Sigma = \{I,\ Y,\ Y^2, ...,\ Y^{m-1},\ R,\ RY,\ RY^2, ..., RY^{m-1}\}$. Then the process as following;

$$\begin{array}{ll}
I.X.(I)^{-1} = X, & I.Y.(Y)^{-1} = I,\\
Y.X.(Y)^{-1} = YXY^{-1}, & Y.Y.(Y^2)^{-1} = I,\\
Y^2.X.(Y^2)^{-1} = Y^2XY^{-2}, & Y^2.Y.(Y^3)^{-1} = I,\\
\vdots & \vdots\\
Y^{m-1}.X.(Y^{m-1})^{-1} = Y^{m-1}XY^{1-m}, & Y^{m-1}.Y.(I)^{-1} = Y^m,\\
R.X.(R)^{-1} = X^{p-1}, & R.Y.(RY)^{-1} = I,\\
RY.X.(RY)^{-1} = Y^{-1}X^{p-1}Y, & RY.Y.(RY^2)^{-1} = I,\\
RY^2.X.(RY^2)^{-1} = Y^{-2}X^{p-1}Y^{-2}, & RY^2.Y.(RY^3)^{-1} = I,\\
\vdots & \vdots\\
RY^{m-1}.X.(RY^{m-1})^{-1} = Y^{1-m}X^{p-1}Y^{m-1}, & RY^{m-1}.Y.(R)^{-1} = Y^{-m},
\end{array}$$

$$\begin{array}{l}
I.R.(R)^{-1} = I,\\
Y.R.(RY^{m-1})^{-1} = Y^m,\\
Y^2.R.(RY^{m-2})^{-1} = Y^m,\\
\vdots\\
Y^{m-1}.R.(RY)^{-1} = Y^m,\\
R.R.(I)^{-1} = I,\\
RY.R.(Y^{m-1})^{-1} = Y^{-m},\\
RY^2.R.(Y^{m-2})^{-1} = Y^{-m},\\
\vdots\\
RY^{m-1}.R.(Y)^{-1} = Y^{-m},
\end{array}$$

After required calculations, we have a presentation of $\overline{H}^{m}_{p,\infty}(\lambda)$ as

$$\begin{aligned}
\overline{H}^{m}_{p,\infty}(\lambda) &= < X, YXY^{-1},\ Y^2XY^{-2}, \cdots, Y^{m-1}XY^{1-m}, Y^m \mid X^p = (YXY^{-1})^p\\
&= (Y^2XY^{-2})^p = \cdots = (Y^{m-1}XY^{1-m})^p = (Y^m)^\infty = I >\\
&\cong \underbrace{C_p * C_p * \cdots * C_p}_{m \text{ times}} * \mathbb{Z}.
\end{aligned}$$

2) The quotient group $\overline{H}_{p,\infty}(\lambda)/\overline{H}^{m}_{p,\infty}(\lambda)$ is

$$\begin{aligned}
\overline{H}_{p,\infty}(\lambda)/\overline{H}^{m}_{p,\infty}(\lambda) &= < X,Y,R \mid X^p = Y^\infty = R^2\\
&= (XR)^2 = (YR)^2 = X^m = Y^m = (XY)^m = \cdots = I >
\end{aligned}$$

Since $m > 0$ is an odd integer and from the relations $X^m = X^p = I$, $Y^m = (YR)^2 = I$ and $R^2 = R^m = I$, we have $X = Y = R = I$. Obviously we have $X = I$. As a result, we obtain

$$\overline{H}_{p,\infty}(\lambda)/\overline{H}^{m}_{p,\infty}(\lambda) \cong \{I\},$$

and so $\overline{H}^{m}_{p,\infty}(\lambda) = \overline{H}_{p,\infty}(\lambda)$. □

3.6. Corollary. *Let $p \geq 3$ be an odd integer and let $\lambda \geq 2$. If m is an even positive integer such that $(m,p)=1$, then $\overline{H}^m_{p,\infty}(\lambda) = H^m_{p,\infty}(\lambda)$.*

The case $(m,p) = d > 1$,except of $m = 2$ and p even, is unknown and so we can not say anything about the power subgroups $\overline{H}^m_{p,\infty}(\lambda)$, similar to $H^m_{p,\infty}(\lambda)$.

3.7. Remark. In this paper, if we take $p = 2$, then our results coincide with the ones given in [14] and [15].

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