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Application of nonhomogenous Cauchy-Euler differential equation for certain class of analytic functions

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Abstract

In this paper, some new subclasses of analytic functions with complex order are introduced by means of a family of nonhomogenous Cauchy-Euler differential equations as well as some differential operators available in literature. The main object of the paper is to determine coefficient bounds for the classes already mentioned, and obtain the results relevant to well-known work.

Keywords: Analytic Functions, Differential operator, Nonhomogenous Cauchy-Euler differential equation, Coefficient bound.

2000 AMS Classification: 30C45

1. Introduction and preliminaries

Let A denote the class of analytic functions f in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ normalized by $f(0) = f'(0) - 1 = 0$. Thus each $f \in A$ has a Taylor series representation

$$f(z) = z + \sum_{i=2}^{\infty} a_i z^i \cdot \tag{1.1}$$

A function $f \in A$ is said to belong to the class $S^*(\xi)$ if it satisfies

$$\Re\left(1 + \frac{1}{\xi}\left(\frac{zf'(z)}{f(z)} - 1\right)\right) > 0, \quad (z \in \mathbb{U}; \xi \in \mathbb{C} \setminus \{0\}) \cdot \tag{1.2}$$

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In 1936, Roberton [7] proved that if $f(z) = z + \sum_{i=2}^{\infty} a_i z^i$ is in $S^*(1-\beta)$ and $C(1-\beta)$, then

$$|a_i| \leq \frac{\prod_{k=0}^{i-2}[k+2(1-\beta)]}{(i-2)!} \text{ and } |a_i| \leq \frac{\prod_{k=0}^{i-2}[k+2(1-\beta)]}{i!} \ (i \in \mathbb{N}^*; 0 \leq \beta < 1)\cdot$$

In 1983, Nasr and Aouf [8] proved that if $f(z) = z + \sum_{i=2}^{\infty} a_i z^i$ is in $S^*(b)$, then

$$|a_i| \leq \frac{\prod_{k=0}^{i-2}[k+2|b|]}{i!} \ (i \in \mathbb{N}^*; 0 \leq \beta < 1)\cdot$$

A function $f \in A$ is said to be in the class $C^*(\xi_1)$ if it satisfies the following inequality

$$\Re\left(1 + \frac{1}{\xi_1}\frac{zf''(z)}{f'(z)}\right) > 0, \quad (z \in \mathbb{U}; \xi_1 \in \mathbb{C} \setminus \{0\})\cdot \tag{1.3}$$

A function $f \in A$ is said to be in the class $K^*(\lambda, \alpha, \xi_2)$ if it also satisfies the following inequality

$$\Re\left[1 + \frac{1}{\xi_2}\left(\frac{z[\lambda z f'(z) + (1-\lambda)f(z)]'}{\lambda z f'(z) + (1-\lambda)f(z)} - 1\right)\right] > \alpha, 0 \leq \alpha, \lambda \leq 1, z \in \mathbb{U}; \xi_2 \in \mathbb{C} \setminus \{0\}\cdot \tag{1.4}$$

To get more detailed information about the class of function $K^*(\lambda, \alpha, \xi_2)$, we will refer the reader to Altintas et al. (see for example [9]–[16]).

For a function $f \in A$, we define the following differential operator:

$$\begin{aligned}
D^0 f(z) &= f(z),\\
D^1_\lambda(\alpha,\beta,\mu) f(z) &= \left(\frac{\alpha-\mu+\beta-\lambda}{\alpha+\beta}\right) f(z) + \left(\frac{\mu+\lambda}{\alpha+\beta}\right) z f'(z),\\
D^2_\lambda(\alpha,\beta,\mu) f(z) &= D(D^1_\lambda(\alpha,\beta,\mu) f(z)),\\
&\vdots\\
D^n_\lambda(\alpha,\beta,\mu) f(z) &= D(D^{n-1}_\lambda(\alpha,\beta,\mu) f(z))\cdot \qquad (1.5)
\end{aligned}$$

If f is given by (1.1) then from (1.5) we have

$$D^n_\lambda(\alpha,\beta,\mu) f(z) = z + \sum_{i=2}^{\infty} \left(\frac{\alpha + (\mu+\lambda)(i-1) + \beta}{\alpha+\beta}\right)^n a_i z^i \tag{1.6}$$

$$(f \in A, \alpha, \beta, \mu, \lambda \geq 0, \alpha+\beta \neq 0, n \in N_o)$$

By specializing the parameters of $D^n_\lambda(\alpha,\beta,\mu) f(z)$ we get the following differential operators. If we substitute

- $\beta = 1$, $\mu = 0$, we get $D^n_\lambda(\alpha,1,0) f(z) = D^n f(z) = z + \sum_{i=2}^{\infty}(\frac{\alpha+\lambda(i-1)+1}{\alpha+1})^n a_i z^i$ of differential operator given by Aouf, El-Ashwah and El-Deeb [1].
- $\alpha = 1$, $\beta = o$, and $\mu = 0$, we get $D^n_\lambda(1,0,0) f(z) = D^n f(z) = z + \sum_{i=2}^{\infty}(1 + \lambda(i-1))^n a_i z^i$ of differential operator given by Al-Oboudi [2].
- $\alpha = 1$, $\beta = o$, $\mu = 0$ and $\lambda = 1$, we get $D^n_1(1,0,0) f(z) = D^n f(z) = z + \sum_{i=2}^{\infty}(i)^n a_i z^i$ of Sălăgean's differential operator [3].

- $\alpha = 1$, $\beta = 1$, $\lambda = 1$ and $\mu = 0$, we get $D_1^n(1,1,0)f(z) = D^n f(z) = z + \sum_{i=2}^{\infty}(\frac{i+1}{2})^n a_i z^i$ of differential operator given by Uralegaddi and Somanatha [4].
- $\beta = 1$, $\lambda = 1$ and $\mu = 0$, we get $D_1^n(\alpha,1,0)f(z) = D^n f(z) = z + \sum_{i=2}^{\infty}(\frac{i+\alpha}{\alpha+1})^n a_i z^i$ of differential operator given by Cho and Srivastava, and Cho and Kim [5, 6].

By using the operator $D_\lambda^n(\alpha,\beta,\mu)f(z)$ given by (1.6), we now introduce a new subclass of analytic functions defined as follows:

A function $f \in A$ is said to belong to the class $F(n,\alpha,b)$ if it satisfies

$$\Re\left\{1 + \frac{1}{b}\left(\frac{D_\lambda^{n+1}(\alpha,\beta,\mu)f(z)}{D_\lambda^n(\alpha,\beta,\mu)f(z)} - 1\right)\right\} > \alpha, 0 \le \alpha < 1, b \in C^*\cdot$$

A function $f \in A$ is said to belong to the subclass of analytic functions of order γ in $\mathbb{U}$, denoted by $\Psi(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi)$, and is defined by

$$\Psi(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi) =$$

$$\left\{f \in A : \Re\left\{1 + \frac{1}{\xi}\left[\frac{z[\zeta D_\lambda^{n+1}(\alpha,\beta,\mu)f(z) + (1-\zeta)D_\lambda^n(\alpha,\beta,\mu)f(z)]'}{\zeta D_\lambda^{n+1}(\alpha,\beta,\mu)f(z) + (1-\zeta)D_\lambda^n(\alpha,\beta,\mu)f(z)} - 1\right]\right\} > \gamma\right\}, \tag{1.7}$$

$$0 \le \gamma, \zeta \le 1, z \in \mathbb{U}; \xi \in \mathbb{C} \setminus \{0\}\cdot$$

Using the class $\Psi(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi)$, we obtain the following subclasses studied by various authors.

$\Psi(n,1,0,0,1,\lambda,\alpha,b) = B(n,\lambda,\alpha,b)$,
$\Psi(0,1,0,0,1,0,0,b) = S^*(b)$,
$\Psi(0,1,0,0,1,1,0,b) = C(b)$,
$\Psi(0,1,0,0,1,0,0,1-\beta) = S^*(1-\beta)$,
$\Psi(0,1,0,0,1,1,0,1-\beta) = C(1-\beta)$,
$\Psi(0,1,0,0,1,\lambda,\alpha,\xi_2) = K(\lambda,\alpha,\xi_2)$,
$\Psi(n,1,0,0,1,0,\alpha,b) = F(n,\alpha,b)\cdot$

The main object of the present investigation is to derive some coefficient bounds for functions in the subclass $\Phi(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi,\tau)$ of A satisfying the following nonhomogenous Cauchy-Euler differential equation

$$z^2\frac{d^2w}{dz^2} + 2(1+\tau)z\frac{dw}{dz} + \tau(1+\tau)w = (1+\tau)(2+\tau)g(z) \tag{1.8}$$
$$(w = f(z) \in A; g(z) \in \Psi(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi); \tau \in \mathbb{R}\backslash]-\infty,-1])\cdot$$

Also note that

$$\Phi(n,1,0,0,1,\lambda,\alpha,b,\mu) = T(n,\lambda,\alpha,b,\mu),$$
$$\Phi(0,1,0,0,1,\lambda,\alpha,b,\mu) = SK(\lambda,\alpha,b,\mu),$$
$$\Phi(n,1,0,0,1,0,\alpha,b,\mu) = SD(n,\alpha,b,\mu)\cdot$$

To get more detailed information about the above said classes, we will refer the reader to [16] and [17].

2. Coefficient estimates for the function class $\Psi(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi)$

Now we give our first result as follows:

2.1. Theorem. *Let the function $f \in A$ be defined by (1.1). If the function f is in the class $\Psi(n,\alpha,\beta,\mu,\lambda,\zeta,\gamma,\xi)$, then*

$$\left|a_i\right| \leq \frac{\prod_{j=o}^{j-2}[j+2\left|\xi\right|(1-\gamma)][\alpha+\beta]^{n+1}}{i![\alpha+\zeta(\mu+\lambda)(i-1)+\beta][\alpha+(\mu+\lambda)(i-1)+\beta]^n}, j \in \mathbb{N}^*, i \in \mathbb{N}\setminus\{1\}\cdot$$

Proof. Let the function $f \in A$ be given by (1.1). Define a function

$$H(z) = (\zeta)D_\lambda^{n+1}(\alpha,\beta,\mu)f(z) + (1-\zeta)D_\lambda^n(\alpha,\beta,\mu)f(z), \tag{2.1}$$

where $D_\lambda^n(\alpha,\beta,\mu)f(z)$ is differential operator be given in (1.6). We note that the function H is of the form

$$H(z) = z + \sum_{i=2}^{\infty} \mathfrak{T}_i z^i, \mathfrak{T}_i \tag{2.2}$$

$$= \Big(\frac{[\alpha+\zeta(\mu+\lambda)(i-1)+\beta][\alpha+(\mu+\lambda)(i-1)+\beta]^n}{[\alpha+\beta]^{n+1}}\Big)a_i\cdot$$

Using (1.7) and (2.1), we get

$$\Re\left\{1+\frac{1}{\xi}\left(\frac{zH'(z)}{H(z)}-1\right)\right\} > \gamma,\ (z\in\mathbb{U})\cdot \tag{2.3}$$

Now we define a function $h(z)$ by

$$h(z) = \frac{1+\frac{1}{\xi}\left(\frac{zH'(z)}{H(z)}-1\right)-\gamma}{1-\gamma}\cdot \tag{2.4}$$

We also suppose

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots. \tag{2.5}$$

So we obtain

$$1+\frac{1}{\xi}\left(\frac{zH'(z)}{H(z)}-1\right)-\gamma = (1-\gamma)(1+c_1z+c_2z^2+\cdots), \tag{2.6}$$

or, equivalently,

$$zH'(z) - H(z) = H(z)\xi(1-\gamma)(c_1z+c_2z^2+\cdots)\cdot \tag{2.7}$$

Using (2.7), we conclude that

$$(2-1)\mathfrak{T}_2 = \xi(1-\gamma)c_1,$$

$$(3-1)\mathfrak{T}_3 = \xi(1-\gamma)[c_1\mathfrak{T}_2 + c_2],$$

$$(4-1)\mathfrak{T}_4 = \xi(1-\gamma)[c_1\mathfrak{T}_3 + c_2\mathfrak{T}_2 + c_3],$$

$$\vdots$$

$$(i-1)\mathfrak{T}_i = \xi(1-\gamma)[c_1\mathfrak{T}_{i-1} + c_2\mathfrak{T}_{i-2} + \cdots + c_{i-1}]\cdot \tag{2.8}$$

As $|c_i| \leq 2,\ i = \{1, 2, 3, \cdots\}$, so from (2.8) we have

$$\big|\mathfrak{T}_2\big| = \big|\xi(1-\gamma)c_1\big| \leq 2\big|\xi\big|(1-\gamma), \tag{2.9}$$

$$\begin{aligned} 2\big|\mathfrak{T}_3\big| &= \big|\xi(1-\gamma)[c_1\mathfrak{T}_2 + c_2]\big| \leq \big|\xi\big|(1-\gamma)[2\mathfrak{T}_2 + 2] \\ &\leq 2\big|\xi\big|(1-\gamma)[1 + 2\big|\xi\big|(1-\gamma)]. \end{aligned} \tag{2.10}$$

$$3\big|\mathfrak{T}_4\big| = \big|\xi(1-\gamma)[c_1\mathfrak{T}_3 + c_2\mathfrak{T}_2 + c_3]\big|, \tag{2.11}$$

or

$$\begin{aligned} 6\big|\mathfrak{T}_4\big| &\leq 2\big|\xi\big|(1-\gamma)[\mathfrak{T}_3 + \mathfrak{T}_2 + 1]\big| \\ &\leq 2\big|\xi\big|(1-\gamma)[1 + 2\big|\xi\big|(1-\gamma)][2 + 2\big|\xi\big|(1-\gamma)]. \end{aligned} \tag{2.12}$$

Using (2.9), (2.10) and (2.12), we get

$$\big|\mathfrak{T}_2\big| \leq \frac{\prod_j [j + 2\big|\xi\big|(1-\gamma)]}{(2-1)!}, j = o,$$

$$\big|\mathfrak{T}_3\big| \leq \frac{\prod_j [j + 2\big|\xi\big|(1-\gamma)]}{(3-1)!}, j = 0, 1,$$

similarly

$$\big|\mathfrak{T}_4\big| \leq \frac{\prod_j [j + 2\big|\xi\big|(1-\gamma)]}{(3-1)!}, j = 0, 1, 2$$

therefore

$$\big|\mathfrak{T}_i\big| \leq \frac{\prod_{j=o}^{j-2} [j + 2\big|\xi\big|(1-\gamma)]}{(i-1)!}, j \in \mathbb{N}^*\cdot$$

By using the relationship between the functions $f(z)$ and $H(z)$, we have

$$\mathfrak{T}_i = \Big(\frac{[\alpha + \zeta(\mu+\lambda)(i-1) + \beta][\alpha + (\mu+\lambda)(i-1) + \beta]^n}{[\alpha+\beta]^{n+1}}\Big)a_i,$$

implies

$$\big|a_i\big| \leq \frac{\prod_{j=o}^{j-2} [j + 2\big|\xi\big|(1-\gamma)][\alpha+\beta]^{n+1}}{i![\alpha + \zeta(\mu+\lambda)(i-1) + \beta][\alpha + (\mu+\lambda)(i-1) + \beta]^n}, j \in \mathbb{N}^*, i \in \mathbb{N} \setminus \{1\}\cdot$$

Now, by choosing different values of $\Psi(n, \alpha, \beta, \mu, \lambda, \zeta, \gamma, \xi)$, we have the following corollaries:

2.2. Corollary. *If a function $f \in A$ is in the class $\Psi(n, \alpha, \mu, \lambda, \zeta, \gamma, \xi)$, then*

$$\big|a_i\big| \leq \frac{\prod_{j=o}^{j-2} [j + 2\big|\xi\big|(1-\gamma)][\alpha]^{n+1}}{i![\alpha + \zeta(\mu+\lambda)(i-1)][\alpha + (\mu+\lambda)(i-1)]^n}, j \in \mathbb{N}^*, i \in \mathbb{N} \setminus \{1\}\cdot$$

2.3. Corollary. *If a function $f \in A$ is in the class $B(n, \lambda, \alpha, b)$, then*

$$\left|a_i\right| \leq \frac{\prod_{j=o}^{j-2}[j + 2\left|b\right|(1-\alpha)]}{(i-1)![1+\lambda(i-1)][i]^n}, j \in \mathbb{N}^*, i \in \mathbb{N} \setminus \{1\}\cdot$$

2.4. Corollary. *If a function $f \in A$ is in the class $S^*(b)$, then*

$$\left|a_i\right| \leq \frac{\prod_{j=o}^{j-2}[j + 2\left|b\right|]}{(i-1)!}, j \in \mathbb{N}^*, i \in \mathbb{N} \setminus \{1\}\cdot$$

2.5. Corollary. *If a function $f \in A$ is in the class $C(b)$, then*

$$\left|a_i\right| \leq \frac{\prod_{j=o}^{j-2}[j + 2\left|b\right|]}{i!}, j \in \mathbb{N}^*, i \in \mathbb{N} \setminus \{1\}\cdot$$

2.6. Corollary. *If a function $f \in A$ is in the class $S^*(1-\beta)$, then*

$$\left|a_i\right| \leq \frac{\prod_{j=o}^{j-2}[j + 2(1-\beta)]}{(i-1)!}, j \in \mathbb{N}^*, i \in \mathbb{N} \setminus \{1\}\cdot$$

2.7. Corollary. *If a function $f \in A$ is in the class $C(1-\beta)$, then*

$$\left|a_i\right| \leq \frac{\prod_{j=o}^{j-2}[j + 2(1-\beta)]}{i!}, j \in \mathbb{N}^*, i \in \mathbb{N} \setminus \{1\}\cdot$$

2.8. Corollary. *If a function $f \in A$ is in the class $K(\lambda, \alpha, \xi_2)$, then*

$$\left|a_i\right| \leq \frac{\prod_{j=o}^{j-2}[j + 2(1-\xi_2)(1-\alpha)]}{(i-1)![1+\lambda(i-1)]}, j \in \mathbb{N}^*, i \in \mathbb{N} \setminus \{1\}\cdot$$

2.9. Corollary. *If a function $f \in A$ is in the class $F(n, \alpha, b)$, then*

$$\left|a_i\right| \leq \frac{\prod_{j=o}^{j-2}[j + 2\left|b\right|(1-\alpha)]}{(i-1)![i]^n}, j \in \mathbb{N}^*, i \in \mathbb{N} \setminus \{1\}\cdot$$

2.10. Corollary. *If a function $f \in A$ is in the class $B(n, \lambda, b)$, then*

$$\left|a_i\right| \leq \frac{\prod_{j=o}^{j-2}[j + 2\left|b\right|]}{(i-1)![1+\lambda(i-1)][i]^n}, j \in \mathbb{N}^*, i \in \mathbb{N} \setminus \{1\}\cdot$$

3. Coefficient bound for the class $\Phi(n, \alpha, \beta, \mu, \lambda, \zeta, \gamma, \xi, \tau)$

3.1. Theorem. *Let the function $f \in A$ be defined by (1.1). If the function f is in the class $\Phi(n, \alpha, \beta, \mu, \lambda, \zeta, \gamma, \xi, \tau)$, then*

$$\left|a_i\right| \leq \frac{(1+\tau)(2+\tau)\prod_{j=o}^{j-2}[j + 2\left|\xi\right|(1-\gamma)][\alpha+\beta]^{n+1}}{i![\alpha+\zeta(\mu+\lambda)(i-1)+\beta][\alpha+(\mu+\lambda)(i-1)+\beta]^n(i+\tau)(i+1+\tau)},$$

$j \in \mathbb{N}^*, i \in \mathbb{N} \setminus \{1\}\cdot$

Proof. Let the function $f \in A$ be given by (1.1). Also let

$$f(z) = z + \sum_{i=2}^{\infty} v_i z^i \in \Psi(n, \alpha, \beta, \mu, \lambda, \zeta, \gamma, \xi), \text{ implies} \tag{3.1}$$

$$\left|v_i\right| \leq \tag{3.2}$$

$$\frac{\prod_{j=o}^{j-2}[j+2\left|\xi\right|(1-\gamma)][\alpha+\beta]^{n+1}}{i![\alpha+\zeta(\mu+\lambda)(i-1)+\beta][\alpha+(\mu+\lambda)(i-1)+\beta]^{n}}, j \in \mathbb{N}^{*}, i \in \mathbb{N}\setminus\{1\}\cdot$$

Since

$$a_i = \frac{(1+\tau)(2+\tau)}{(i+\tau)(i+1+\tau)}v_i\cdot$$

Using (3.2) we get

$$\left|a_i\right| \leq \frac{(1+\tau)(2+\tau)\prod_{j=o}^{j-2}[j+2\left|\xi\right|(1-\gamma)][\alpha+\beta]^{n+1}}{i![\alpha+\zeta(\mu+\lambda)(i-1)+\beta][\alpha+(\mu+\lambda)(i-1)+\beta]^{n}(i+\tau)(i+1+\tau)},$$

$j \in \mathbb{N}^{*}, i \in \mathbb{N}\setminus\{1\}\cdot$

Next we have the following corollaries:

3.2. Corollary. *If a function $f \in A$ is in the class $\Phi(n,\alpha,\mu,\lambda,\zeta,\gamma,\xi,\tau)$, then*

$$\left|a_i\right| \leq \frac{(1+\tau)(2+\tau)\prod_{j=o}^{j-2}[j+2\left|\xi\right|(1-\gamma)][\alpha]^{n+1}}{i![\alpha+\zeta(\mu+\lambda)(i-1)][\alpha+(\mu+\lambda)(i-1)]^{n}(i+\tau)(i+1+\tau)}, j \in \mathbb{N}^{*}, i \in \mathbb{N}\setminus\{1\}\cdot$$

3.3. Corollary. *If a function $f \in A$ is in the class $\Phi(n,\alpha,\lambda,\zeta,\gamma,\xi,\tau)$, then*

$$\left|a_i\right| \leq \frac{(1+\tau)(2+\tau)\prod_{j=o}^{j-2}[j+2\left|\xi\right|(1-\gamma)][\alpha]^{n+1}}{i![\alpha+\zeta\lambda(i-1)][\alpha+\lambda(i-1)]^{n}(i+\tau)(i+1+\tau)}, j \in \mathbb{N}^{*}, i \in \mathbb{N}\setminus\{1\}\cdot$$

3.4. Corollary. *If a function $f \in A$ is in the class $\Phi(n,\alpha,\zeta,\gamma,\xi,\tau)$, then*

$$\left|a_i\right| \leq \frac{(1+\tau)(2+\tau)\prod_{j=o}^{j-2}[j+2\left|\xi\right|(1-\gamma)][\alpha]^{n+1}}{i![\alpha+\zeta(i-1)][\alpha+(i-1)]^{n}(i+\tau)(i+1+\tau)}, j \in \mathbb{N}^{*}, i \in \mathbb{N}\setminus\{1\}\cdot$$

3.5. Corollary. *If a function $f \in A$ is in the class $SK(\lambda,\gamma,\xi,\tau)$, then*

$$\left|a_i\right| \leq \frac{(1+\tau)(2+\tau)\prod_{j=o}^{j-2}[j+2\left|\xi\right|(1-\gamma)]}{(i-1)!(1-\lambda+\lambda i)(i+\tau)(i+1+\tau)}, j \in \mathbb{N}^{*}, i \in \mathbb{N}\setminus\{1\}\cdot$$

3.6. Corollary. *If a function $f \in A$ is in the class $SD(n,\gamma,\xi,\tau)$, then*

$$\left|a_i\right| \leq \frac{(1+\tau)(2+\tau)\prod_{j=o}^{j-2}[j+2\left|\xi\right|(1-\gamma)]}{(i-1)!(i+\tau)(i+1+\tau)}, j \in \mathbb{N}^{*}, i \in \mathbb{N}\setminus\{1\}\cdot$$

4. Conclusions

There are many different types of operators can be reached in the literature, see for example: [18]- [23], and many more. Some similar results can also be found for different type of classes associated with the many different differential operators.

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