

## Some results on a cross-section in the tensor bundle

Aydın Gezer<sup>\*†</sup> and Murat Altunbas<sup>‡</sup>

### Abstract

The present paper is devoted to some results concerning with the complete lifts of an almost complex structure and a connection in a manifold to its  $(0, q)$ -tensor bundle along the corresponding cross-section.

**Keywords:** Almost complex structure, Almost analytic tensor, Complete lift, Connection, Tensor bundle.

*2000 AMS Classification:* Primary 53C15; Secondary 53B05.

### 1. Introduction

The behaviour of the lifts of tensor fields and connections on a manifold to its different bundles along the corresponding cross-sections are studied by several authors. For the case tangent and cotangent bundles, see [13, 14, 15] and also tangent bundles of order 2 and order  $r$ , see [3, 11]. In [2], the first author and his collaborator studied the complete lift of an almost complex structure in a manifold on the so-called pure cross-section of its  $(p, q)$ -tensor bundle by means of the Tachibana operator (for diagonal lift to the  $(p, q)$ -tensor bundle see [1] and for the  $(0, q)$ -tensor bundle see [5]). Moreover they proved that if a manifold admits an almost complex structure, then so does on the pure cross-section of its  $(p, q)$ -tensor bundle provided that the almost complex structure is integrable. In [6], the authors give detailed description of geodesics of the  $(p, q)$ -tensor bundle with respect to the complete lift of an affine connection.

The purpose of the present paper is two-fold. Firstly, to show the complete lift of an almost complex structure in a manifold to its  $(0, q)$ -tensor bundle along the corresponding cross-section, when restricted to the cross-section determined by an almost analytic tensor field, is an almost complex structure. Finally, to study the behavior of the complete lift of a connection on the cross-section of the  $(0, q)$ -tensor bundle.

---

<sup>\*</sup>Ataturk University, Faculty of Science, Department of Mathematics, 25240, Erzurum-TURKEY, Email: [agezer@atauni.edu.tr](mailto:agezer@atauni.edu.tr)

<sup>†</sup>Corresponding Author.

<sup>‡</sup>Erzincan University, Faculty of Science and Art, Department of Mathematics, 24030, Erzincan-TURKEY, Email: [maltunbas@erzincan.edu.tr](mailto:maltunbas@erzincan.edu.tr)

Throughout this paper, all manifolds, tensor fields and connections are always assumed to be differentiable of class  $C^\infty$ . Also, we denote by  $\mathfrak{S}_q^p(M)$  the set of all tensor fields of type  $(p, q)$  on  $M$ , and by  $\mathfrak{S}_q^p(T_q^0(M))$  the corresponding set on the  $(0, q)$ -tensor bundle  $T_q^0(M)$ . The Einstein summation convention is used, the range of the indices  $i, j, s$  being always  $\{1, 2, \dots, n\}$ .

## 2. Preliminaries

Let  $M$  be a differentiable manifold of class  $C^\infty$  and finite dimension  $n$ . Then the set  $T_q^0(M) = \cup_{P \in M} T_q^0(P)$ ,  $q > 0$ , is the tensor bundle of type  $(0, q)$  over  $M$ , where  $\cup$  denotes the disjoint union of the tensor spaces  $T_q^0(P)$  for all  $P \in M$ . For any point  $\tilde{P}$  of  $T_q^0(M)$  such that  $\tilde{P} \in T_q^0(M)$ , the surjective correspondence  $\tilde{P} \rightarrow P$  determines the natural projection  $\pi : T_q^0(M) \rightarrow M$ . The projection  $\pi$  defines the natural differentiable manifold structure of  $T_q^0(M)$ , that is,  $T_q^0(M)$  is a  $C^\infty$ -manifold of dimension  $n + n^q$ . If  $x^j$  are local coordinates in a neighborhood  $U$  of  $P \in M$ , then a tensor  $t$  at  $P$  which is an element of  $T_q^0(M)$  is expressible in the form  $(x^j, t_{j_1 \dots j_q})$ , where  $t_{j_1 \dots j_q}$  are components of  $t$  with respect to natural base. We may consider  $(x^j, t_{j_1 \dots j_q}) = (x^j, x^{\bar{j}}) = x^J$ ,  $j = 1, \dots, n$ ,  $\bar{j} = n + 1, \dots, n + n^q$ ,  $J = 1, \dots, n + n^{p+q}$  as local coordinates in a neighborhood  $\pi^{-1}(U)$ .

Let  $V = V^i \frac{\partial}{\partial x^i}$  and  $A = A_{j_1 \dots j_q} dx^{j_1} \otimes \dots \otimes dx^{j_q}$  be the local expressions in  $U$  of a vector field  $V$  and a  $(0, q)$ -tensor field  $A$  on  $M$ , respectively. Then the vertical lift  ${}^V A$  of  $A$  and the complete lift  ${}^C V$  of  $V$  are given, with respect to the induced coordinates, by

$$(2.1) \quad {}^V A = \begin{pmatrix} 0 \\ A_{j_1 \dots j_q} \end{pmatrix}$$

and

$$(2.2) \quad {}^C V = \begin{pmatrix} V^j \\ -\sum_{\lambda=1}^q t_{j_1 \dots m \dots j_q} \partial_{j_\lambda} V^m \end{pmatrix}.$$

Suppose that there is given a tensor field  $\xi \in \mathfrak{S}_q^0(M)$ . Then the correspondence  $x \mapsto \xi_x$ ,  $\xi_x$  being the value of  $\xi$  at  $x \in M$ , determines a mapping  $\sigma_\xi : M \mapsto T_q^0(M)$ , such that  $\pi \circ \sigma_\xi = id_M$ , and the  $n$  dimensional submanifold  $\sigma_\xi(M)$  of  $T_q^0(M)$  is called the cross-section determined by  $\xi$ . If the tensor field  $\xi$  has the local components  $\xi_{k_1 \dots k_q}(x^k)$ , the cross-section  $\sigma_\xi(M)$  is locally expressed by

$$(2.3) \quad \begin{cases} x^k = x^k, \\ x^{\bar{k}} = \xi_{k_1 \dots k_q}(x^k) \end{cases}$$

with respect to the coordinates  $(x^k, x^{\bar{k}})$  in  $T_q^0(M)$ . Differentiating (2.3) by  $x^j$ , we see that  $n$  tangent vector fields  $B_j$  to  $\sigma_\xi(M)$  have components

$$(2.4) \quad (B_j^K) = \left( \frac{\partial x^K}{\partial x^j} \right) = \begin{pmatrix} \delta_j^k \\ \partial_j \xi_{k_1 \dots k_q} \end{pmatrix}$$

with respect to the natural frame  $\{\partial_k, \partial_{\bar{k}}\}$  in  $T_q^0(M)$ .

On the other hand, the fibre is locally expressed by

$$\begin{cases} x^k = \text{const.}, \\ t_{k_1 \dots k_q} = t_{k_1 \dots k_q}, \end{cases}$$

$t_{k_1 \dots k_q}$  being considered as parameters. Thus, on differentiating with respect to  $x^{\bar{j}} = t_{j_1 \dots j_q}$ , we see that  $n^q$  tangent vector fields  $C_{\bar{j}}$  to the fibre have components

$$(2.5) \quad (C_{\bar{j}}^K) = \left( \frac{\partial x^K}{\partial x^{\bar{j}}} \right) = \begin{pmatrix} 0 \\ \delta_{k_1}^{j_1} \dots \delta_{k_q}^{j_q} \end{pmatrix}$$

with respect to the natural frame  $\{\partial_k, \partial_{\bar{k}}\}$  in  $T_q^0(M)$ .

We consider in  $\pi^{-1}(U) \subset T_q^0(M)$ ,  $n + n^q$  local vector fields  $B_j$  and  $C_{\bar{j}}$  along  $\sigma_\xi(M)$ . They form a local family of frames  $[B_j, C_{\bar{j}}]$  along  $\sigma_\xi(M)$ , which is called the adapted  $(B, C)$ -frame of  $\sigma_\xi(M)$  in  $\pi^{-1}(U)$ . Taking account of (2.2) on the cross-section  $\sigma_\xi(M)$ , and also (2.4) and (2.5), we can easily prove that, the complete lift  ${}^C V$  has along  $\sigma_\xi(M)$  components of the form

$$(2.6) \quad {}^C V = \begin{pmatrix} V^j \\ -L_V \xi_{j_1 \dots j_q} \end{pmatrix}$$

with respect to the adapted  $(B, C)$ -frame. From (2.1), (2.4) and (2.5), the vertical lift  ${}^V A$  also has components of the form

$$(2.7) \quad {}^V A = \begin{pmatrix} 0 \\ A_{j_1 \dots j_q} \end{pmatrix}$$

with respect to the adapted  $(B, C)$ -frame.

### 3. Almost complex structures on a pure cross-section in the $(0, q)$ -tensor bundle

A tensor field  $\xi \in \mathfrak{S}_q^0(M)$  is called pure with respect to  $\varphi \in \mathfrak{S}_1^1(M)$ , if [2, 4, 5, 7, 8, 9, 10, 12]:

$$(3.1) \quad \varphi_{j_1}^r \xi_{r \dots j_q} = \dots = \varphi_{j_q}^r \xi_{j_1 \dots r} = \overset{*}{\xi}_{j_1 \dots j_q}.$$

In particular, vector and covector fields will be considered to be pure.

Let  $\overset{*}{\mathfrak{S}}_q^0(M)$  denotes a module of all the tensor fields  $\xi \in \mathfrak{S}_q^0(M)$  which are pure with respect to  $\varphi$ . Now, we consider a pure cross-section  $\sigma_\xi^\varphi(M)$  determined by  $\xi \in \overset{*}{\mathfrak{S}}_q^0(M)$ . The complete lift  ${}^C \varphi$  of  $\varphi$  along the pure cross-section  $\sigma_\xi^\varphi(M)$  to  $T_q^0(M)$  has local components of the form

$${}^C \varphi = \begin{pmatrix} \varphi_l^k & 0 \\ -(\Phi_\varphi \xi)_{lk_1 \dots k_q} & \varphi_{k_1}^{r_1} \delta_{k_2}^{r_2} \dots \delta_{k_q}^{r_q} \end{pmatrix}$$

with respect to the adapted  $(B, C)$ -frame of  $\sigma_\xi^\varphi(M)$ , where  $(\Phi_\varphi \xi)_{lk_1 \dots k_q} = \varphi_l^m \partial_m \xi_{k_1 \dots k_q} - \partial_l \overset{*}{\xi}_{k_1 \dots k_q} + \sum_{a=1}^q (\partial_{k_a} \varphi_l^m) \xi_{k_1 \dots m \dots k_q}$  is the Tachibana operator.

We consider that the local vector fields

$${}^C X_{(i)} = {}^C \left( \frac{\partial}{\partial x^i} \right) = {}^C \left( \delta_i^h \frac{\partial}{\partial x^h} \right) = \begin{pmatrix} \delta_i^h \\ 0 \end{pmatrix}$$

and

$${}^V X^{(\bar{i})} = {}^V (dx^{i_1} \otimes \dots \otimes dx^{i_q}) = {}^V (\delta_{h_1}^{i_1} \dots \delta_{h_q}^{i_q} dx^{h_1} \otimes \dots \otimes dx^{h_q}) = \begin{pmatrix} 0 \\ \delta_{h_1}^{i_1} \dots \delta_{h_q}^{i_q} \end{pmatrix}$$

$i = 1, \dots, n, \bar{i} = n+1, \dots, n+n^q$  span the module of vector fields in  $\pi^{-1}(U)$ . Hence, any tensor fields is determined in  $\pi^{-1}(U)$  by their actions on  ${}^C V$  and  ${}^V A$  for any  $V \in \mathfrak{S}_0^1(M)$  and  $A \in \mathfrak{S}_q^0(M)$ . The complete lift  ${}^C \varphi$  along the pure cross-section  $\sigma_\xi^\varphi(M)$  has the properties

$$(3.2) \quad \begin{cases} {}^C \varphi({}^C V) = {}^C (\varphi(V)) + {}^V ((L_V \varphi) \circ \xi), \forall V \in \mathfrak{S}_0^1(M), (i) \\ {}^C \varphi({}^V A) = {}^V (\varphi(A)), \forall A \in \mathfrak{S}_q^0(M), (ii) \end{cases}$$

which characterize  ${}^C \varphi$ , where  $\varphi(A) \in \mathfrak{S}_q^0(M)$ . Remark that  ${}^V ((L_V \varphi) \circ \xi)$  is a vector field on  $T_q^0(M)$  and locally expressed by

$${}^V ((L_V \varphi) \circ \xi) = \begin{pmatrix} 0 \\ (L_V \varphi)_{i_1}^j \xi_{j i_2 \dots i_q} \end{pmatrix}$$

with respect to the adapted  $(B, C)$ -frame, where  $\xi_{i_1 \dots i_q}$  are local components of  $\xi$  in  $M$  [5].

**3.1. Theorem.** *Let  $M$  be an almost complex manifold with an almost complex structure  $\varphi$ . Then, the complete lift  ${}^C \varphi \in \mathfrak{S}_1^1(T_q^0(M))$ , when restricted to the pure cross-section determined by an almost analytic tensor  $\xi$  on  $M$ , is an almost complex structure.*

*Proof.* If  $V \in \mathfrak{S}_0^1(M)$  and  $A \in \mathfrak{S}_q^0(M)$ , in view of the equations (i) and (ii) of (3.2), we have

$$(3.3) \quad ({}^C \varphi)^2({}^C V) = {}^C (\varphi^2)({}^C V) + {}^V (N_\varphi \circ \xi)({}^C V)$$

and

$$(3.4) \quad ({}^C \varphi)^2({}^V A) = {}^C (\varphi^2)({}^V A),$$

where  $N_{\varphi, X}(Y) = (L_{\varphi X} \varphi - \varphi(L_X \varphi))(Y) = [\varphi X, \varphi Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] + \varphi^2[X, Y] = N_\varphi(X, Y)$  is nothing but the Nijenhuis tensor constructed by  $\varphi$ .

Let  $\varphi \in \mathfrak{S}_1^1(M)$  be an almost complex structure and  $\xi \in \mathfrak{S}_q^0(M)$  be a pure tensor with respect to  $\varphi$ . If  $(\Phi_\varphi \xi) = 0$ , the pure tensor  $\xi$  is called an almost analytic  $(0, q)$ -tensor. In [4, 7, 9], it is proved that  $\xi \circ \varphi \in \mathfrak{S}_q^0(M)$  is an almost analytic tensor if and only if  $\xi \in \mathfrak{S}_q^0(M)$  is an almost analytic tensor. Moreover if  $\xi \in \mathfrak{S}_q^0(M)$  is an almost analytic tensor, then  $N_\varphi \circ \xi = 0$ . When restricted to the pure cross-section determined by an almost analytic tensor  $\xi$  on  $M$ , from (3.3), (3.4) and linearity of the complete lift, we have

$$({}^C \varphi)^2 = {}^C (\varphi^2) = {}^C (-I_M) = -I_{T_q^0(M)}.$$

This completes the proof.  $\square$

#### 4. Complete lift of a symmetric affine connection on a cross-section in the $(0, q)$ -tensor bundle

We now assume that  $\nabla$  is an affine connection (with zero torsion) on  $M$ . Let  $\Gamma_{ij}^h$  be components of  $\nabla$ . The complete lift  ${}^C\nabla$  of  $\nabla$  to  $T_q^0(M)$  has components  ${}^C\Gamma_{MS}^I$  such that

$$\begin{aligned}
(4.1) \quad {}^C\Gamma_{ms}^i &= \Gamma_{ms}^i, \quad {}^C\Gamma_{\bar{m}s}^i = {}^C\Gamma_{m\bar{s}}^i = {}^C\Gamma_{\bar{m}\bar{s}}^i = {}^C\Gamma_{\bar{m}s}^{\bar{i}} = 0, \\
{}^C\Gamma_{\bar{m}\bar{s}}^{\bar{i}} &= -\sum_{c=1}^q \Gamma_{m_i c}^{s_c} \delta_{i_1}^{s_1} \dots \delta_{i_{c-1}}^{s_{c-1}} \delta_{i_{c+1}}^{s_{c+1}} \dots \delta_{i_q}^{s_q}, \\
{}^C\Gamma_{\bar{m}\bar{s}}^{\bar{i}} &= -\sum_{c=1}^q \Gamma_{s_i c}^{m_c} \delta_{i_1}^{m_1} \dots \delta_{i_{c-1}}^{m_{c-1}} \delta_{i_{c+1}}^{m_{c+1}} \dots \delta_{i_q}^{m_q}, \\
{}^C\Gamma_{ms}^{\bar{i}} &= \sum_{c=1}^q (-\partial_m \Gamma_{s_i c}^a + \Gamma_{m_i c}^r \Gamma_{sr}^a + \Gamma_{ms}^r \Gamma_{r i_c}^a) t_{i_1 \dots i_{c-1} a i_{c+1} \dots i_q} \\
&\quad + \frac{1}{2} \sum_{b=1}^q \sum_{c=1}^q (\Gamma_{m_i c}^l \Gamma_{s_i b}^r + \Gamma_{m_i b}^l \Gamma_{s_i c}^r) t_{i_1 \dots i_{b-1} r i_{b+1} \dots i_{c-1} l i_{c+1} \dots i_q} \\
&\quad + \sum_{d=1}^q t_{i_1 \dots l \dots i_q} R_{i_d k m}{}^l
\end{aligned}$$

with respect to the natural frame in  $T_q^0(M)$ , where  $\delta_j^i$ —Kronecker delta and  $R_{ikm}{}^l$  is components of the curvature tensor  $R$  of  $\nabla$  [6].

We now study the affine connection induced from  ${}^C\nabla$  on the cross-section  $\sigma_\xi(M)$  determined by the  $(0, q)$ -tensor field  $\xi$  in  $M$  with respect to the adapted  $(B, C)$ -frame of  $\sigma_\xi(M)$ . The vector fields  $C_{\bar{j}}$  given by (2.5) are linearly independent and not tangent to  $\sigma_\xi(M)$ . We take the vector fields  $C_{\bar{j}}$  as normals to the cross-section  $\sigma_\xi(M)$  and define an affine connection  $\tilde{\nabla}$  induced on the cross-section. The affine connection  $\tilde{\nabla}$  induced  $\sigma_\xi(M)$  from the complete lift  ${}^C\nabla$  of a symmetric affine connection  $\nabla$  in  $M$  has components of the form

$$(4.2) \quad \tilde{\Gamma}_{ji}^h = (\partial_j B_i{}^A + {}^C\Gamma_{CB}^A B_j{}^C B_i{}^B) B^h{}_A,$$

where  $B^h{}_A$  are defined by

$$(B^h{}_A, C^h{}_A) = (B_i{}^A, C_i{}^A)^{-1}$$

and thus

$$(4.3) \quad B^h{}_A = (\delta_i^h, 0), \quad C^h{}_A = (-\partial_j \xi_{k_1 \dots k_q}, \delta_{k_1}^{j_1} \dots \delta_{k_q}^{j_q}).$$

Substituting (4.1), (2.4), (2.5) and (4.3) in (4.2), we get

$$\tilde{\Gamma}_{ji}^h = \Gamma_{ji}^h,$$

where  $\Gamma_{ji}^h$  are components of  $\nabla$  in  $M$ .

From (4.2), we see that the quantity

$$(4.4) \quad \partial_j B_i{}^A + {}^C\Gamma_{CB}^A B_j{}^C B_i{}^B - \Gamma_{ji}^h B^h{}_A$$

is a linear combination of the vectors  $C_{\bar{i}}^A$ . To find the coefficients, we put  $A = \bar{h}$  in (4.4) and find

$$\nabla_j \nabla_i \xi_{h_1 \dots h_q} + \sum_{\lambda=1}^q \xi_{h_1 \dots l \dots h_q} R_{h_\lambda i j}{}^l.$$

Hence, representing (4.4) by  $\tilde{\nabla}_j B_i{}^A$ , we obtain

$$(4.5) \quad \tilde{\nabla}_j B_i{}^A = (\nabla_j \nabla_i \xi_{h_1 \dots h_q} + \sum_{\lambda=1}^q \xi_{h_1 \dots l \dots h_q} R_{h_\lambda i j}{}^l) C_{\bar{h}}^A.$$

The last equation is nothing but the equation of Gauss for the cross-section  $\sigma_\xi(M)$  determined by  $\xi_{h_1 \dots h_q}$ . Hence, we have the following proposition.

**4.1. Proposition.** *The cross-section  $\sigma_\xi(M)$  in  $T_q^0(M)$  determined by a  $(0, q)$  tensor  $\xi$  in  $M$  with symmetric affine connection  $\nabla$  is totally geodesic if and only if  $\xi$  satisfies*

$$\nabla_j \nabla_i \xi_{h_1 \dots h_q} + \sum_{\lambda=1}^q \xi_{h_1 \dots l \dots h_q} R_{h_\lambda i j}{}^l = 0.$$

Now, let us apply the operator  $\tilde{\nabla}_k$  to (4.5), we have

$$(4.6) \quad \tilde{\nabla}_k \tilde{\nabla}_j B_i{}^A = \nabla_k (\nabla_j \nabla_i \xi_{h_1 \dots h_q} + \sum_{\lambda=1}^q \xi_{h_1 \dots l \dots h_q} R_{h_\lambda i j}{}^l) C_{\bar{h}}^A.$$

Recalling that

$$\tilde{\nabla}_k \tilde{\nabla}_j B_i{}^A - \tilde{\nabla}_j \tilde{\nabla}_k B_i{}^A = \tilde{R}_{DCB}{}^A B_k{}^D B_j{}^C B_i{}^B - R_{kji}{}^h B_h{}^A,$$

and using the Ricci identity for a tensor field of type  $(0, q)$ , from (4.6) we get

$$\begin{aligned} & \tilde{R}_{DCB}{}^A B_k{}^D B_j{}^C B_i{}^B - R_{kji}{}^h B_h{}^A \\ = & \left[ \sum_{\lambda=1}^q (\nabla_k R_{h_\lambda i j}{}^l - \nabla_j R_{h_\lambda i k}{}^l) \xi_{h_1 \dots l \dots h_q} - R_{kji}{}^l \nabla_l \xi_{h_1 \dots h_q} \right. \\ & \left. - \sum_{\lambda=1}^q R_{kjh_\lambda}{}^l \nabla_i \xi_{h_1 \dots l \dots h_q} + \sum_{\lambda=1}^q R_{h_\lambda i j}{}^l \nabla_k \xi_{h_1 \dots l \dots h_q} - \sum_{\lambda=1}^q R_{h_\lambda i k}{}^l \nabla_j \xi_{h_1 \dots l \dots h_q} \right] C_{\bar{h}}^A. \end{aligned}$$

Thus we have the result below.

**4.2. Proposition.**  *$\tilde{R}_{DCB}{}^A B_k{}^D B_j{}^C B_i{}^B$  is tangent to the cross-section  $\sigma_\xi(M)$  if and only if*

$$\begin{aligned} & \sum_{\lambda=1}^q (\nabla_k R_{h_\lambda i j}{}^l - \nabla_j R_{h_\lambda i k}{}^l) \xi_{h_1 \dots l \dots h_q} \\ = & R_{kji}{}^l \nabla_l \xi_{h_1 \dots h_q} + \sum_{\lambda=1}^q R_{kjh_\lambda}{}^l \nabla_i \xi_{h_1 \dots l \dots h_q} - \sum_{\lambda=1}^q R_{h_\lambda i j}{}^l \nabla_k \xi_{h_1 \dots l \dots h_q} \\ & + \sum_{\lambda=1}^q R_{h_\lambda i k}{}^l \nabla_j \xi_{h_1 \dots l \dots h_q}. \end{aligned}$$

## References

- [1] Gezer, A. and Salimov, A. *Diagonal lifts of tensor fields of type (1,1) on cross-sections in tensor bundles and its applications*, J. Korean Math. Soc. **45** (2), 367–376, 2008.
- [2] Gezer, A. and Salimov, A.A. *Almost complex structures on the tensor bundles*, Arab. J. Sci. Eng. Sect. A Sci. **33** (2), 283–296, 2008.
- [3] Houh, C. and Ishihara, S. *Tensor fields and connections on a cross-section in the tangent bundle of order  $r$* , Kodai Math. Sem. Rep. **24**, 234–250, 1972.
- [4] Koto, S. *On almost analytic tensors in almost complex spaces*, Tensor (N.S.) **12**, 110–132, 1962.
- [5] Magden, A. and Salimov, A.A. *Complete lifts of tensor fields on a pure cross-section in the tensor bundle*, J. Geom. **93** (1-2), 128–138, 2009.
- [6] Magden, A. and Salimov, A.A. *Geodesics for complete lifts of affine connections in tensor bundles*, Appl. Math. Comput. **151** (3), 863–868, 2004.
- [7] Muto, Y. *On some almost analytic tensor fields in almost complex manifolds*, Kodai Math. Sem. Rep. **19**, 454–469, 1967.
- [8] Salimov, A., Gezer, A. and Aslanci, S. *On almost complex structures in the cotangent bundle*, Turkish J. Math. **35** (3), 487–492, 2011.
- [9] Salimov, A. *On operators associated with tensor fields*, J. Geom. **99** (1-2), 107–145, 2010.
- [10] Tachibana, S. *Analytic tensor and its generalization*, Tohoku Math. J. **12** (2), 208–221, 1960.
- [11] Tani, M. *Tensor fields and connections in cross-sections in the tangent bundle of order 2*, Kodai Math. Sem. Rep. **21**, 310–325, 1969.
- [12] Yano, K. and Ako, M. *On certain operators associated with tensor field*, Kodai Math. Sem. Rep., **20**, 414–436, 1968.
- [13] Yano, K. and Ishihara, S. *Tangent and Cotangent Bundles*, Marcel Dekker, Inc., New York 1973.
- [14] Yano, K. *Tensor fields and connections on cross-sections in the cotangent bundle*, Tohoku Math. J. **19** (2), 32–48, 1967.
- [15] Yano, K. *Tensor fields and connections on cross-sections in the tangent bundle of a differentiable manifold*, Proc. Roy. Soc. Edinburgh Sect. A **67**, 277–288, 1968.