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A study on convergence of non-convolution type double singular integral operators

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Abstract: The aim of this paper is to investigate the pointwise convergence and the rate of convergence of the operators in the following form:

$$L_\lambda(f;x,y) = \iint\limits_{\Omega} f(t,s)K_\lambda(t,s;x,y)\,dsdt,\ (x,y)\in\Omega,$$

where $\Omega = <A,B> \times <C,D>$ is an arbitrary closed, semi-closed or open region in $\mathbb{R}^2$, at a μ-generalized Lebesgue point of $f\in L_p(\Omega)$ as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$.

Keywords: μ-generalized Lebesgue points, pointwise convergence, rate of convergence.

1 Introduction

In [11], some pointwise approximation results for integral operators of the form:

$$U_\lambda(f;x) = \int_{-\pi}^{\pi} f(t)K_\lambda(t-x)\,dt,\ \ x\in(-\pi,\pi), \tag{1}$$

have been studied at a Lebesgue point of integrable functions. Then these type operators, convolution type singular integral operators depending on two parameters, were developed by Gadjiev [3], Rydzewska [7]. In [12], Taberski gave some theorems concerning the Riemann-Stieltjes, Lebesgue and Titchmarsh integrals on a given rectangle. Especially, he formulated a theorem of Faddeev's type ([12], page 246) concerning the convergence of singular integrals of the form:

$$U(x,y;\lambda,f) = \iint\limits_{Q} f(t,s)\,\psi(x,y;\lambda)\,dsdt,\quad (x,y)\in Q, \tag{2}$$

for integrable functions f, here Q denotes a given rectangle and $\psi(x,y;\lambda)$ is the kernel satisfying suitable assumptions. In the same paper, he explored the some approximation theorems by using the following operators:

$$V(x,y;\lambda,f) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(t,s)K(t-x,s-y;\lambda)\,dsdt \tag{3}$$

In the summability theory of double Fourier series the integral operators of type (3) play an important role, where kernel $K(t,s;\lambda)$ are bounded, measurable, even, 2π periodic in each variable x, y separately. After the pioneering work of

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Taberski [12], in [8], Rydzewska estimated the order of convergence of double singular integrals of the form

$$V(x,y;\sigma,f)=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(t,s)\,K(t-x,s-y;\sigma)\,dsdt \tag{4}$$

under various assumptions on $f(s,t)$ and $K(t,s;\sigma)$. Rydzewska considered the convergence of integral operators to a real integrable function $f(t,s)$ at a generalized Lebesgue point. In particular, for further studies on the convergence of double singular integrals at the Lebesgue points, we address the reader to [9], [10], [14],[15] and [16]. For further reading we suggest the following papers : [1], [5], [17], [18], [19], and [21]-[24]. We present the pointwise convergence of non-convolution singular integral operators of the form

$$L_{\lambda}(f;x,y)=\iint_{\Omega} f(t,s)\,K_{\lambda}(t,s;x,y)\,dsdt,\quad (x,y)\in\Omega, \tag{5}$$

where $\Omega:=\langle A,B\rangle\times\langle C,D\rangle$ is an arbitrary closed, semi-closed or open region in $\mathbb{R}^2$, at a $\mu-$generalized Lebesgue point of $f\in L_p(\Omega)$ as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$. Here $L_p(\Omega)$ is the collection of all measurable functions f for which $|f|^p$ is integrable on Ω.

The rest of the paper is organized as follows. Section 2 introduces the terminology used throughout this paper. Section 3 shows the existence of the operators of type (5). In Section 4, we give two theorems concerning the pointwise convergence of $L_{\lambda}(f;x,y)$ on different regions. In Section 5 we estimate the rate of pointwise convergence of the operators of type (5) and give some examples.

2 Preliminaries

Definition 1. *Let $\varphi(x,y)$ be a function defined in the rectangle D, let*

$$P=\begin{cases} a=x_1<x_2<\ldots<x_i<\ldots<x_m<x_{m+1}=b\\ c=y_1<y_2<\ldots<y_j<\ldots<y_n<y_{n+1}=d\end{cases}$$

be a partition of D and

$$\triangle\varphi(x_i,y_j)=\varphi(x_i,y_j)-\varphi(x_{i+1},y_j)-\varphi(x_i,y_{j+1})+\varphi(x_{i+1},y_{j+1}).$$

If $\triangle\varphi(x_i,y_j)\geq 0$ $(i=1,2,\ldots,m;\, j=1,2,\ldots n)$ for any partition of P, then it is said that $\varphi(x,y)$ satisfies the condition Ω in D [12].

In other words, if $\triangle\varphi(x_i,y_j)\geq 0$ for all partitions of D then it is said that $\varphi(x,y)$ is bimonotonically increasing and if $\triangle\varphi(x_i,y_j)\leq 0$ for all partitions of D, then it is said that $\varphi(x,y)$ is bimonotonically decreasing [4].

The definition of the $\mu-$generalized Lebesgue point is the special form of the definition of the Lebesgue point in [8].

Definition 2. *A point $(x_0,y_0)\in D$ is called a $\mu-$generalized Lebesgue point of function $f\in L_p(D)$ if*

$$\lim_{(h,k)\to(0,0)}\frac{1}{\mu_1(h)\mu_2(k)}\int_0^h\int_0^k |f(t+x_0,s+y_0)-f(x_0,y_0)|^p\,dsdt=0$$

where $\mu_1(t)$:$\mathbb{R}\to\mathbb{R}$, absolutely continuous on $[-\delta_0,\delta_0]$, increasing on $[0,\delta_0]$ and $\mu_1(0)=0$ and also $\mu_2(s)$:$\mathbb{R}\to\mathbb{R}$, absolutely continuous on $[-\delta_0,\delta_0]$, increasing on $[0,\delta_0]$ and $\mu_2(0)=0$. Here, $0<h,k<\delta_0$.

The following definition was inspired by the definition of class A in [13].

Definition 3. (*Class A*) *Let $\Omega = <A,B> \times <C,D>$, $\Psi = <a,b> \times <c,d>$, $\Lambda \subset \mathbb{R}_0^+$ be an index set and $\lambda \in \Lambda$ be an accumulation point of it. If the following conditions are satisfied, then $K_\lambda : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_0^+$ belongs to class A; i.e.,*

(a) *For fixed $(x_0,y_0) \in \Omega$, $K_\lambda(x_0,y_0)$ tends to infinity as λ tends to λ_0 for any fixed $(x,y) \in <a,b> \times <c,d>$.*

(b) $\lim\limits_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)} \iint_{\mathbb{R}^2} K_\lambda(t,s;x,y)\,dsdt = 1.$

(c) *For any fixed $(x,y) \in \Psi$ there exists a point $(x_0,y_0) \in \Omega$ such that*

$$\lim_{\lambda\to\lambda_0} \iint_{\mathbb{R}^2\setminus N} K_\lambda(t,s;x,y)\,dsdt = 0,\ \forall N \in \tilde{N}(x_0,y_0),$$

where $\tilde{N}(x_0,y_0)$ stands for the family of all neighborhoods of (x_0,y_0) in $\mathbb{R}^2$.

(d) *For any fixed $(x,y) \in \Psi$ there exists a point $(x_0,y_0) \in \Omega$ such that*

$$\lim_{\lambda\to\lambda_0} \sup_{\mathbb{R}^2\setminus\tilde{N}_\delta(x_0,y_0)} K_\lambda(t,s;x,y) = 0,\ \forall\delta > 0$$

where $\tilde{N}_\delta(x_0,y_0) = (x_0-\delta, x_0+\delta) \times (y_0-\delta, y_0+\delta)$.

(e) *For any fixed $x \in <a,b>$ there exists a point $x_0 \in <A,B>$ such that $K_\lambda(t,s;x,y)$ is monotonically increasing with respect to t on $<x_0-\delta, x_0>$ and monotonically decreasing on $<x_0, x_0+\delta>$ and similarly, for any fixed $y_0 \in <C,D>$ there exists a point $y \in <c,d>$ such that $K_\lambda(t,s;x,y)$ is monotonically increasing with respect to s on $<y_0-\delta, y_0>$ and monotonically decreasing on $<y_0, y_0+\delta>$ for any $\lambda \in \Lambda$. Analogously, for any fixed $(x,y) \in \Psi$ there exists a point $(x_0,y_0) \in \Omega$ such that $K_\lambda(t,s;x,y)$ is bimonotonically increasing with respect to (t,s) on $<x_0,x_0+\delta> \times <y_0,y_0+\delta>$ and $<y_0-\delta,y_0> \times <x_0-\delta,x_0>$ and similarly bimonotonically decreasing with respect to (t,s) on $<x_0,x_0+\delta> \times <y_0-\delta,y_0>$ and $<x_0-\delta,x_0> \times <y_0,y_0+\delta>$ for any $\lambda \in \Lambda$.*

Throughout this paper, we suppose that the kernel $K_\lambda(t,s;x,y)$ belongs to class A.

3 Existence of operator

Lemma 1. *Let $\|K_\lambda(.,.;x,y)\|_{L_1(\mathbb{R}^2)} \le M$, $\forall\lambda \in \Lambda$ and $\forall(x,y) \in \Psi$. If $f \in L_1(\Omega)$ then $L_\lambda(f;x,y)$ defines a continuous transformation acting on $L_1(\Omega)$.*

Proof. By the linearity of the operator $L_\lambda(f;x,y)$, it is sufficient to show that

$$\|L_\lambda\|_1 = \sup_{f\neq0} \frac{\|L_\lambda(f,x,y)\|_{L_1(\Omega)}}{\|f\|_{L_1(\Omega)}} < \infty$$

remains bounded. Now, using Fubini Theorem [2] we can write

$$\begin{aligned}\|L_\lambda(f,x,y)\|_{L_1(\Omega)} &= \iint_\Omega \left(\iint_\Omega f(t,s) K_\lambda(t,s;x,y)\,dsdt \right) dydx \\ &\le \iint_\Omega f(t,s) \left(\iint_{\mathbb{R}^2} K_\lambda(t,s;x,y)\,dydx \right) dsdt \\ &\le M\|f\|_{L_1(\Omega)}.\end{aligned}$$

Thus the proof is completed.

Lemma 2. *Let* $1<p<\infty$ *and* $\|K_\lambda\|_{L_q(\mathbb{R}^2\times\mathbb{R}^2)} \le M$, $\forall\lambda\in\Lambda$ *whenever* $\frac{1}{p}+\frac{1}{q}=1$. *If* $f\in L_p(\Omega)$ *then* $L_\lambda(f;x,y)$ *defines a continuous transformation from* $L_p(\Omega)$ *to* $L_q(\Omega)$.

Proof. We assume that $1<p<\infty$. By the linearity of the operator $L_\lambda(f;x,y)$, it is sufficient to show that

$$\|L_\lambda\|_q = \sup_{f\neq 0}\frac{\|L_\lambda(f,x,y)\|_{L_q(\Omega)}}{\|f\|_{L_p(\Omega)}}$$

is bounded. Let us define a new function by

$$g(t,s)=\begin{cases} f(t,s),(t,s)\in\Omega,\\ 0,(t,s)\in\mathbb{R}^2\backslash\Omega.\end{cases}$$

Rearranging and rewriting the norm as follows

$$\|L_\lambda(f,x,y)\|_{L_q(\Omega)}=\|L_\lambda(g,x,y)\|_{L_q(\Omega)}=\left(\iint_{\Omega}\left|\iint_{\mathbb{R}^2} g(t,s)K_\lambda(t,s;x,y)\,dsdt\right|^q dydx\right)^{\frac{1}{q}}.$$

Applying Hölder's inequality [2] to the last equality we obtain

$$\begin{aligned}\|L_\lambda(f,x,y)\|_{L_q(\Omega)} &\le \left(\iint_{\Omega}\left(\left(\iint_{\mathbb{R}^2}|g(t,s)|^p dsdt\right)^{\frac{1}{p}}\left(\iint_{\mathbb{R}^2}|K_\lambda(t,s;x,y)|^q dsdt\right)^{\frac{1}{q}}\right)^q dydx\right)^{\frac{1}{q}}\\ &\le \left(\iint_{\Omega}\left(\|f\|^q_{L_p(\Omega)}\iint_{\mathbb{R}^2}|K_\lambda(t,s;x,y)|^q dsdt\right)dydx\right)^{\frac{1}{q}}\\ &\le \|f\|_{L_p(\Omega)}\|K_\lambda\|_{L_q(\mathbb{R}^2\times\mathbb{R}^2)}\\ &\le M\|f\|_{L_p(\Omega)}.\end{aligned}$$

Hence the proof is completed.

4 Convergence at characteristic points

We are now ready to prove our first main result.

Theorem 1. *If* (x_0,y_0) *be a generalized Lebesgue point of function* $f\in L_p(\Omega)$ *then*

$$\lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)} L_\lambda(f;x,y)=f(x_0,y_0)$$

on any set Z *on which the function*

$$\int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta} K_\lambda(t,s;x,y)\left|\{\mu_1(|t-x_0|)\}'_t\right|\left|\{\mu_2(|s-y_0|)\}'_s\right|dsdt$$

is bounded as (x,y,λ) *tends to* (x_0,y_0,λ_0).

Proof. Suppose that $\tilde{N}_\delta(x_0,y_0)\subset\Omega$ and (x_0,y_0) be $\mu-$generalized Lebesgue point of $f\in L_p(\Omega)$. For the case $p=1$, the proof is quite similar to for $p>1$, therefore we will prove the theorem for the case $1<p<\infty$. Since $(x_0,y_0)\in\Omega$

be $\mu-$generalized Lebesgue point of $f \in L_p(\Omega)$, given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all h and k satisfying $0 < h,k \leq \delta$, the following inequalities

$$\int_{x_0-\delta}^{x_0}\int_{y_0-\delta}^{y_0} |f(t,s) - f(x_0,y_0)|^p \, dsdt < \varepsilon\mu_1(h)\mu_2(k), \tag{1}$$

$$\int_{x_0}^{x_0+\delta}\int_{y_0-\delta}^{y_0} |f(t,s) - f(x_0,y_0)|^p \, dtdts < \varepsilon\mu_1(h)\mu_2(k), \tag{2}$$

$$\int_{x_0-\delta}^{x_0}\int_{y_0}^{y_0+\delta} |f(t,s) - f(x_0,y_0)|^p \, dsdt < \varepsilon\mu_1(h)\mu_2(k), \tag{3}$$

$$\int_{x_0}^{x_0+\delta}\int_{y_0}^{y_0+\delta} |f(t,s) - f(x_0,y_0)|^p \, dsdt < \varepsilon\mu_1(h)\mu_2(k), \tag{4}$$

hold. Set $I_\lambda(x,y) := |L_\lambda(f;x,y) - f(x_0,y_0)|$. According to condition (c) of class A, we shall write

$$\begin{aligned}
&\left| \iint_{\Omega} f(t,s) K_\lambda(t,s;x,y)\, dsdt - f(x_0,y_0) \right| \\
&\leq \iint_{\Omega} |f(t,s) - f(x_0,y_0)| K_\lambda(t,s;x,y)\, dsdt + |f(x_0,y_0)| \left| \iint_{\mathbb{R}^2} K_\lambda(t,s;x,y)\, dsdt - 1 \right| \\
&+ |f(x_0,y_0)| \iint_{\mathbb{R}^2\setminus\Omega} K_\lambda(t,s;x,y)\, dsdt \\
&= I + II + III.
\end{aligned}$$

By condition (c) of class A, $III \to 0$ as $\lambda \to \lambda_0$ respectively. Now, we investigate the integral $I+II$. Using Hölder's inequality we have the following

$$\begin{aligned}
I + II \leq & \left(\iint_{\Omega} |f(t,s) - f(x_0,y_0)|^p |K_\lambda(t,s;x,y)|\, dsdt \right)^{\frac{1}{p}} \times \left(\iint_{\Omega} K_\lambda(t,s;x,y)\, dsdt \right)^{\frac{1}{q}} \\
& + |f(x_0,y_0)| \left| \iint_{\mathbb{R}^2} K_\lambda(t,s;x,y)\, dsdt - 1 \right|.
\end{aligned}$$

Since whenever m,n positive numbers the inequality $(m+n)^p \leq 2^p(m^p+n^p)$ holds [6], by taking the $p-th$ power of both sides we have

$$\begin{aligned}
(I+II)^p \leq & 2^p \iint_{\Omega} |f(t,s) - f(x_0,y_0)|^p |K_\lambda(t,s;x,y)|\, dsdt \\
& \times \left(\iint_{\Omega} K_\lambda(t,s;x,y)\, dsdt \right)^{\frac{p}{q}} + 2^p |f(x_0,y_0)|^p \left| \iint_{\mathbb{R}^2} K_\lambda(t,s;x,y)\, dsdt - 1 \right|^p \\
& = 2^p(I_1 \times I^* + I_2).
\end{aligned}$$

Observe that by condition (c) of class A, the integral I^{*} tends to 1 as (x,y,λ) *tends to* (x_0,y_0,λ_0) and the integral I_2 tends to zero. Now we investigate the integral I_1.

$$I_1=\left\{\iint\limits_{\Omega\setminus\tilde{N}_{\delta}(x_0,y_0)}+\iint\limits_{\tilde{N}_{\delta}(x_0,y_0)}\right\}\left|f(t,s)-f(x_0,y_0)\right|^{p}K_{\lambda}(t,s;x,y)\,dsdt=I_{11}+I_{12}.$$

The following inequality holds for the integral I_{11} i.e.:

$$I_{11}\leq\sup_{\Omega\setminus\tilde{N}_{\delta}(x_0,y_0)}K_{\lambda}(t,s;x,y)\,2^{p}\left[\|f\|_{L_1(\Omega)}^{p}+\left|f(x_0,y_0)\right|^{p}|B-A|\,|D-C|\right].$$

Hence by condition (d) of class A, $I_{11}\to 0$ as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$.

Next, we can show that I_{12} tends to zero as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$ on $\tilde{N}_{\delta}(x_0,y_0)$.

$$\begin{aligned}I_{12}&=\iint\limits_{\tilde{N}_{\delta}(x_0,y_0)}\left|f(t,s)-f(x_0,y_0)\right|^{p}K_{\lambda}(t,s;x,y)\,dsdt\\&=\int\limits_{x_0-\delta}^{x_0+\delta}\int\limits_{y_0-\delta}^{y_0+\delta}\left|f(t,s)-f(x_0,y_0)\right|^{p}K_{\lambda}(t,s;x,y)\,dsdt\\&=\left\{\int\limits_{x_0-\delta}^{x_0}\int\limits_{y_0-\delta}^{y_0}+\int\limits_{x_0}^{x_0+\delta}\int\limits_{y_0-\delta}^{y_0}\right\}\left|f(t,s)-f(x_0,y_0)\right|^{p}K_{\lambda}(t,s;x,y)\,dsdt\\&+\left\{\int\limits_{x_0-\delta}^{x_0}\int\limits_{y_0}^{y_0+\delta}+\int\limits_{x_0}^{x_0+\delta}\int\limits_{y_0}^{y_0+\delta}\right\}\left|f(t,s)-f(x_0,y_0)\right|^{p}K_{\lambda}(t,s;x,y)\,dsdt\\&=I_{121}+I_{122}+I_{123}+I_{124}.\end{aligned}$$

Hence we can evaluate the integral I_{121}. From [12] (see 2.5 p.101), we can write the following:

$$I_{121}=\int\limits_{x_0-\delta}^{x_0}\int\limits_{y_0-\delta}^{y_0}\left|f(t,s)-f(x_0,y_0)\right|^{p}K_{\lambda}(t,s;x,y)\,dsdt=(S)\int\limits_{x_0-\delta}^{x_0}\int\limits_{y_0-\delta}^{y_0}K_{\lambda}(t,s;x,y)\,dF(t,s)$$

where (S) denotes Stieltjes integral.

Two-dimensional integration by parts (see 2.2 p.100 in [12]) give us

$$\begin{aligned}\int\limits_{x_0-\delta}^{x_0}\int\limits_{y_0-\delta}^{y_0}K_{\lambda}(t,s;x,y)\,dF(t,s)&=\int\limits_{x_0-\delta}^{x_0}\int\limits_{y_0-\delta}^{y_0}F(t,s)\,dK_{\lambda}(t,s;x,y)\\&+\int\limits_{x_0-\delta}^{x_0}F(t,y_0-\delta)\,dK_{\lambda}(t,y_0-\delta;x,y)+\int\limits_{y_0-\delta}^{y_0}F(x_0-\delta,s)\,dK_{\lambda}(x_0-\delta,s;x,y)\\&+F(x_0-\delta,y_0-\delta)\,K_{\lambda}(x_0-\delta,y_0-\delta;x,y).\end{aligned}$$

From (1), we can write

$$\begin{aligned}I_{121} \leq \varepsilon \int_{x_0-\delta}^{x_0}\int_{y_0-\delta}^{y_0} \mu_1(x_0-t)\mu_2(y_0-s)\left|dK_\lambda(t,s;x,y)\right| \\ +\varepsilon\mu_2(\delta)\int_{x_0-\delta}^{x_0}\mu_1(x_0-t)\left|d_tK_\lambda(t,y_0-\delta;x,y)\right| \\ +\varepsilon\mu_1(\delta)\int_{y_0-\delta}^{y_0}\mu_2(y_0-s)\left|d_sK_\lambda(x_0-\delta,s;x,y)\right| \\ +\varepsilon\mu_1(\delta)\mu_2(\delta)K_\lambda(x_0-\delta,y_0-\delta;x,y) \\ = i_1+i_2+i_3+i_4.\end{aligned}$$

After applying integration by parts to i_1, i_2 and i_3 we obtain the following result

$$I_{121} \leq \varepsilon \int_{x_0-\delta}^{x_0}\int_{y_0-\delta}^{y_0} K_\lambda(t,s;x,y)\left|\{\mu_1(x_0-t)\}_t'\right|\left|\{\mu_2(y_0-t)\}_s'\right| dsdt.$$

For the integrals I_{122}, I_{123} and I_{124} the proof is similar to the above one. Thus we obtain the following inequalities:

$$I_{122} \leq -\varepsilon \int_{x_0}^{x_0+\delta}\int_{y_0-\delta}^{y_0} K_\lambda(t,s;x,y)\left|\{\mu_1(t-x_0)\}_t'\right|\left|\{\mu_2(y_0-s)\}_s'\right| dsdt,$$

$$I_{123} \leq -\varepsilon \int_{x_0-\delta}^{x_0}\int_{y_0}^{y_0+\delta} K_\lambda(t,s;x,y)\left|\{\mu_1(x_0-t)\}_t'\right|\left|\{\mu_2(y_0-s)\}_s'\right| dsdt,$$

$$I_{124} \leq \varepsilon \int_{x_0}^{x_0+\delta}\int_{y_0}^{y_0+\delta} K_\lambda(t,s;x,y)\left|\{\mu_1(t-x_0)\}_t'\right|\left|\{\mu_2(s-y_0)\}_s'\right| dsdt.$$

Collecting the estimates $I_{121}, I_{122}, I_{123}$ and I_{124} , we have the following inequality:

$$I_{12} \leq I_{121}+I_{122}+I_{123}+I_{124} = \varepsilon \int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta} K_\lambda(t,s;x,y)\left|\{\mu_1(|t-x_0|)\}_t'\right|\left|\{\mu_2(|s-y_0|)\}_s'\right| dsdt.$$

Therefore, if the points $(x,y,\lambda)\in Z$ are sufficiently near to (x_0,y_0,λ_0) , we have

$$I_{12} \leq \varepsilon K$$

where

$$K = \sup\left\{\int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta} K_\lambda(t,s;x,y)\left|\{\mu_1(|t-x_0|)\}_t'\right|\left|\{\mu_2(|s-y_0|)\}_s'\right| dsdt. : (x,y,\lambda)\in Z\right\}.$$

Thus, the proof is finished.

The following theorem gives a pointwise approximation of the integral operators type (3) to the function f at μ-generalized Lebesgue point of $f\in L_1(\mathbb{R}^2)$ whenever $D=\mathbb{R}^2$.

Theorem 2. *Suppose that the hypothesis of Theorem 1 is satisfied for $\Omega=\mathbb{R}^2$. If (x_0,y_0) is a μ-generalized Lebesgue point of function $f\in L_1(\mathbb{R}^2)$ then*

$$\lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)} L_\lambda(f;x,y)=f(x_0,y_0).$$

Proof. Using the same strategy as in Theorem 1 we obtain

$$\begin{aligned} I_\lambda(x,y) &:= |L_\lambda(f;x,y)-f(x_0,y_0)| \\ &\le \iint_{\mathbb{R}^2} |f(t,s)-f(x_0,y_0)|\,|K_\lambda(t,s;x,y)|\,dsdt + |f(x_0,y_0)| \left| \iint_{\mathbb{R}^2} K_\lambda(t,s;x,y)\,dsdt - 1 \right| \\ &= I+II \end{aligned}$$

$$\begin{aligned} (I+II)^p &\le 2^p \iint_{\mathbb{R}^2} |f(t,s)-f(x_0,y_0)|^p\,|K_\lambda(t,s;x,y)|\,dsdt \\ &\times \left(\iint_{\mathbb{R}^2} K_\lambda(t,s;x,y)\,dsdt \right)^{\frac{p}{q}} + 2^p|f(x_0,y_0)|^p \left| \iint_{\mathbb{R}^2} K_\lambda(t,s;x,y)\,dsdt - 1 \right|^p \\ &= 2^p(I_1\times I^* + I_2) \end{aligned}$$

Observe that by condition (b) of class A, the integral I^* tends to 1 as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$.

$$\begin{aligned} I_1 &= \left\{ \iint_{\mathbb{R}^2\setminus\widetilde{N}_\delta(x_0,y_0)} + \iint_{\widetilde{N}_\delta(x_0,y_0)} \right\} |f(t,s)-f(x_0,y_0)|^p\,|K_\lambda(t,s;x,y)|\,dsdt \\ &= I_{11}+I_{12}. \end{aligned}$$

$$\begin{aligned} I_{11} &= \iint_{\mathbb{R}^2\setminus\widetilde{N}_\delta(x_0,y_0)} |f(t,s)-f(x_0,y_0)|^p\,|K_\lambda(t,s;x,y)|\,dsdt \\ &\le \iint_{\mathbb{R}^2\setminus\widetilde{N}_\delta(x_0,y_0)} 2^p\{|f(t,s)|^p + f|(x_0,y_0)|^p\}\,|K_\lambda(t,s;x,y)|\,dsdt \\ &= \iint_{\mathbb{R}^2\setminus\widetilde{N}_\delta(x_0,y_0)} 2^p|f(t,s)|^p K_\lambda(t,s;x,y)\,dsdt + 2^p|f(x_0,y_0)|^p \iint_{\mathbb{R}^2\setminus\widetilde{N}_\delta(x_0,y_0)} K_\lambda(t,s;x,y)\,dsdt \\ &\le \sup_{\mathbb{R}^2\setminus\widetilde{N}_\delta(x_0,y_0)} K_\lambda(t,s;x,y)\,2^p\|f\|^p_{L_1(\mathbb{R}^2)} + 2^p|f(x_0,y_0)|^p \iint_{\mathbb{R}^2\setminus\widetilde{N}_\delta(x_0,y_0)} K_\lambda(t,s;x,y)\,dsdt \end{aligned}$$

Hence by condition (d) and (c) of class A, $I_{11}\to 0$ as $\lambda\to\lambda_0$. Using the same operations for the integral I_{12} as in Theorem 1, we have the following inequality:

$$\begin{aligned} I_{12} &= \iint_{\widetilde{N}_\delta(x_0,y_0)} |f(t,s)-f(x_0,y_0)|^p\,|K_\lambda(t,s;x,y)|\,dsdt \\ &\le \varepsilon \int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta} |K_\lambda(t,s;x,y)| \left| \{\mu_1(|t-x_0|)\}'_t \right| \left| \{\mu_2(|s-y_0|)\}'_s \right| dsdt. \end{aligned}$$

Therefore, if the points $(x,y,\lambda)\in Z$ are sufficiently near to (x_0,y_0,λ_0), we have

$$I_{12}\leq \varepsilon K$$

where

$$K=\sup\left\{\int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta}|K_\lambda(t,s;x,y)|\left|\{\mu_1(|t-x_0|)\}_t'\right|\left|\{\mu_2(|s-y_0|)\}_s'\right|dsdt:(x,y,\lambda)\in Z\right\}.$$

Thus, the proof is finished.

5 Rate of convergence

In this section, we give a theorem concerning the rate of pointwise convergence.

Theorem 3. *Suppose that the hypothesis of Theorem 1 is satisfied. Let*

$$\Delta(\lambda,\delta,x,y)=\int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta}|K_\lambda(t,s;x,y)|\left|\{\mu_1(|t-x_0|)\}_t'\right|\left|\{\mu_2(|s-y_0|)\}_s'\right|dsdt$$

for $0<\delta<\delta_0$ *and the following assumptions are satisfied:*

(i) $\Delta(\lambda,\delta,x,y)\to 0$ *as* $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$ *for some* $\delta>0$.

(ii) *For any fixed* $(x,y)\in\Psi$ *there exists a point* $(x_0,y_0)\in\Omega$ *such that*

$$\iint_{\mathbb{R}^2\setminus N}K_\lambda(t,s,x,y)\,dsdt=o(\Delta(\lambda,\delta,x,y)),\forall N\in\widetilde{N}(x_0,y_0)$$

as $\lambda\longrightarrow\lambda_0$.

(iii) *For any fixed* $(x,y)\in\Psi$ *there exists a point* $(x_0,y_0)\in\Omega$ *such that*

$$\sup_{\mathbb{R}^2\setminus N}K_\lambda(t,s;x,y)=o(\Delta(\lambda,\delta,x,y))\ \ ,\delta>0,$$

as $\lambda\to\lambda_0$. *Then at each* μ*-generalized Lebesgue point of* $f\in L_1(\Omega)$ *and we have as* $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$

$$|L_\lambda(f;x,y)-f(x_0,y_0)|^p=o(\Delta(\lambda,\delta,x,y)).$$

Proof. The result is clear by Theorem 1 and Theorem 2.

For the one-dimensional counterpart of the following kernel, we refer the reader to see [13].

Example 1. We define $K_\lambda(t,s;x,y)=\begin{cases}\frac{\lambda^2}{|xy|},\ if\ \ (t,s)\in[0\frac{x}{\lambda}]\times[0,\frac{y}{\lambda}],\ \ xy\neq 0,\\ 0,\ \ \ if\ \ (t,s)\in\mathbb{R}^2\setminus[0,\frac{x}{\lambda}]\times[0,\frac{y}{\lambda}]\end{cases}$ and consider the function

$$f(t,s)=\frac{1}{(1+t^2)(1+s^2)},(t,s)\in\mathbb{R}^2.$$

Now first, we compute L_λ and then we have the following:

$$L_\lambda(f;x,y)=\iint_{[0,\frac{x}{\lambda}]\times[0,\frac{y}{\lambda}]}\frac{1}{(1+t^2)(1+s^2)}\frac{\lambda^2}{|xy|}dsdt=\frac{\lambda^2}{|xy|}\arctan\left(\frac{y}{x}\right)\arctan\left(\frac{y}{x}\right),\ \ xy\neq 0.$$

We give the graph of $f(t,s)=\frac{1}{(1+t^2)(1+s^2)},\ \ (t,s)\in\mathbb{R}^2$ as shown in Figure 1:

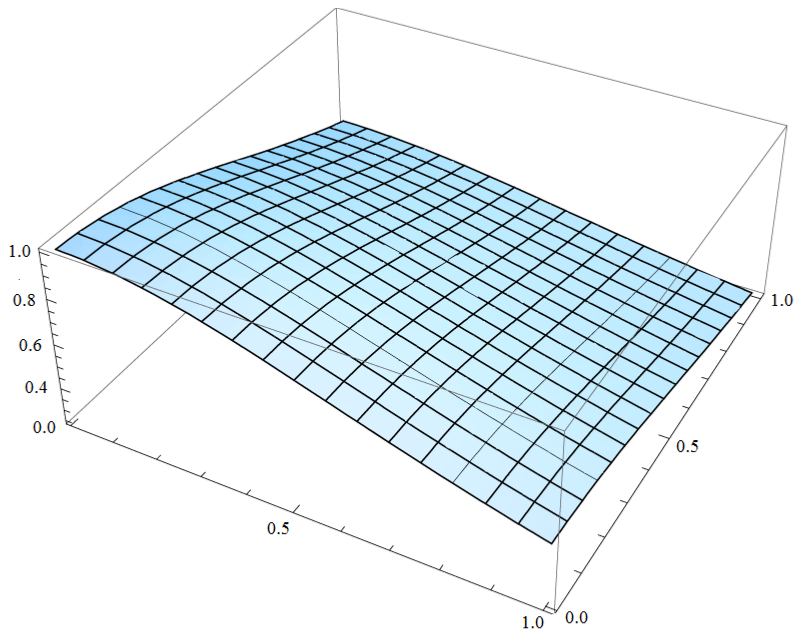

Fig. 1: Graph of $f(t,s) = \frac{1}{(1+t^2)(1+s^2)}$

Now we give the two graph of $L_\lambda(f;x,y)$ as $(x,y,\lambda) \to (0,0,\infty)$. In the first graph we choose $\lambda = 10$ in the second one $\lambda = 100$.

Graph of $L_\lambda(f;x,y) = \frac{\lambda^2}{|xy|}\arctan(\frac{x}{\lambda})\arctan(\frac{y}{\lambda})$, $xy \neq 0$ when $\lambda = 10$, $x \in (0.01,1)$ and $y \in (0.01,1)$.

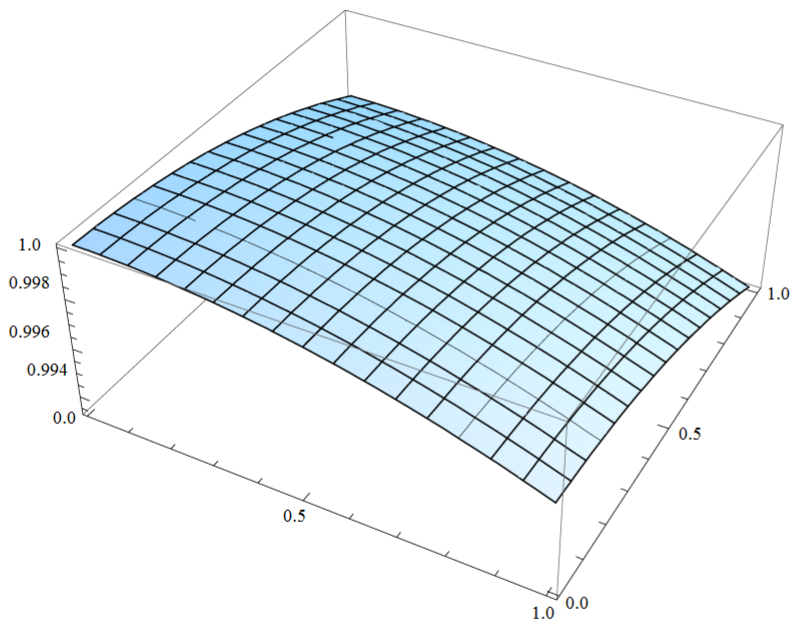

Fig. 2: Graph of $L_\lambda(f;x,y)$ when $\lambda = 10$

Graph of $L_\lambda(f;x,y) = \frac{\lambda^2}{|xy|}\arctan(\frac{x}{\lambda})\arctan(\frac{y}{\lambda})$, $xy \neq 0$ when $\lambda = 100$, $x \in (0.01,1)$ and $y \in (0.01,1)$

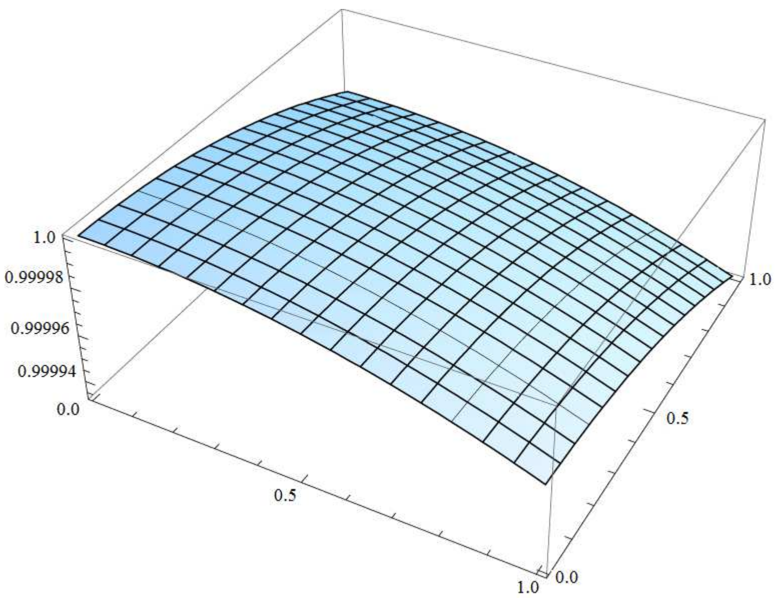

Fig. 3: Graph of $L_\lambda(f;x,y)$ when $\lambda = 100$

Figures are generated by Wolfram Mathematica 7, we refer the reader to see [20].

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