New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2016422372

A new characterization between osculating strip curves and ruled surfaces in Lorentzian space

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Received: 15 February 2016, Accepted: 30 September 2016
Published online: 24 October 2016.

Abstract: In this work, we study the conditions between osculating strip curves and ruled surfaces in Lorentzian space. For this study, we establish a system of differential equations characterizing both spacelike and timelike ruled surfaces in Lorentzian space by using the invariant quantities of osculating strip curves on the given ruled surfaces. We obtain the solutions of these systems for special cases. Regarding to these special solutions, we give some results of relations between osculating strip curves and ruled surfaces in Lorentzian space.

Keywords: Lorentz space, osculating strip curve, normal curve,Blaschke frame, Darboux frame.

1 Introduction

The theory of Klein geometry is of invariants of a transitive transformation group. The structure of the underlying Lie group is an essential element in geometry. This fact can be nicely seen in the theory of ruled surfaces. The first studies from this perspective were made by W. Blaschke and E. Study. The main idea is to replace points by lines as fundamental building blocks of geometric beings [6].

The relations between curves and ruled surfaces were first given by Blaschke in [3]. Blaschke established this relation by means of Study theorem which works in dual space. By Study map, ruled surfaces are considered as spherical curves in dual space. The conditions between curves and ruled surfaces were studied by establishing a system of differential equations which determined ruled surfaces in Euclidean space in [7].

Based on the thought "curve-surface strip", the conditions between geodesic curves and ruled surfaces were dealt with by Pekmen [10] by means of Blaschke invariants in [3]. Later, Şişman wrote an MA dissertation about this problem. He tried to solve the system of differential equations determining ruled surfaces in dual Euclidean space by using the invariants of the surface strip curves, he couldn't obtain the general solution of this system. But he had the solutions by provided that the surface strip curves were osculating, curvature and geodesic strip curves [12]. After Şişman's MA thesis, Pasinli took the subject into consideration by not using the vanishing of Darboux invariants [9]. Also, Ayyıldız studied the conditions between curves and semi-ruled surfaces in Lorentz 3-space [2]. Then the conditions between geodesic curves and ruled surfaces were studied in Lorentzian space by Ayyıldız [1]. Some interesting results between curves and ruled surfaces were also obtained in dual Lorentzian space in [4,5].

In this work, we study the conditions between osculating strip curves and ruled surfaces in dual Lorentz space $\mathbb{D}^3_1$. For

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this study, we establish a system of differential equations characterizing both spacelike and timelike ruled surfaces in dual Lorentz space $\mathbb{D}_1^3$ by using the invariant quantities of osculating strip curves on the given ruled surfaces. We obtain the solutions of these systems for special cases. Regarding to these special solutions, we give some results of relations between osculating strip curves and ruled surfaces in dual Lorentz space $\mathbb{D}_1^3$.

2 Preliminaries

Let $\mathbb{R}_1^3$ be the three-dimensional Minkowski space, that is, the three-dimensional real vector space $\mathbb{R}^3$ with the metric

$$\langle dx, dx \rangle = dx_1^2 + dx_2^2 - dx_3^2,$$

where (x_1, x_2, x_3) denotes the canonical coordinates in R^3. An arbitrary vector x of E_1^3 is said to be spacelike if $\langle x, x \rangle > 0$ or $x = 0$, timelike if $\langle x, x \rangle < 0$ and lightlike or null if $\langle x, x \rangle = 0$ and $x \neq 0$. A timelike or light-like vector in E_1^3 is said to be causal. For $x \in E_1^3$ the norm is defined by $\|x\| = \sqrt{|\langle x, x \rangle|}$, then the vector x is called a spacelike unit vector if $\langle x, x \rangle = 1$ and a timelike unit vector if $\langle x, x \rangle = -1$. Similarly, a regular curve in E_1^3 can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors are spacelike, timelike or null (lightlike), respectively [8].

Dual numbers are given by the set

$$\mathbb{D} = \{\widehat{x} = x + \xi x^*; x, x^* \in \mathbb{R}\},$$

where the symbol ξ designates the dual unit with the property $\xi^2 = 0$ for $\xi \neq 0$. Dual angle is defined as $\widehat{\theta} = \theta + \xi \theta^*$, where θ is the projected angle between two spears and θ^* is the shortest distance between them. The set $\mathbb{D}$ of dual numbers is a commutative ring the the operations $+$ and $\cdot$. The set

$$\mathbb{D}^3 = \mathbb{D} \times \mathbb{D} \times \mathbb{D} = \left\{\widehat{\varphi} = \varphi + \xi \varphi^*; \varphi, \varphi^* \in \mathbb{E}^3\right\}$$

is a module over the ring $\mathbb{D}$ [15].

For any $\widehat{a} = a + \xi a^*$, $\widehat{b} = b + \xi b^* \in \mathbb{D}^3$, the Lorentzian inner product of $\widehat{a}$ and $\widehat{b}$ is defined by

$$< \widehat{a}, \widehat{b} > = < a, b > + \xi(< a^*, b > + < a, b^* >).$$

The dual space $\mathbb{D}^3$ together with this Lorentzian inner product is called the dual Lorentzian space and is denoted by $\mathbb{D}_1^3$ [14]. For $\widehat{\varphi} \neq 0$, the norm $\|\widehat{\varphi}\|$ of $\widehat{\varphi}$ is defined by

$$\|\widehat{\varphi}\| = \sqrt{< \widehat{\varphi}, \widehat{\varphi} >}.$$

A dual vector $\widehat{\omega} = \omega + \xi \omega^*$ is called dual spacelike vector if $\langle \widehat{\omega}, \widehat{\omega} \rangle > 0$ or $\widehat{\omega} = 0$, dual timelike vector if $\langle \widehat{\omega}, \widehat{\omega} \rangle < 0$ and dual null (lightlike) vector if $\langle \widehat{\omega}, \widehat{\omega} \rangle = 0$ for $\widehat{\omega} \neq 0$. Therefore, an arbitrary dual curve which is a differential mapping onto $\mathbb{D}_1^3$ can locally be dual spacelike, dual timelike or dual null if its velocity vector is dual spacelike, dual timelike or dual null, respectively. Also, for the dual vectors $\widehat{a}, \widehat{b} \in \mathbb{D}_1^3$, Lorentzian vector product of these dual vectors is defined by

$$\widehat{a} \times \widehat{b} = a \times b + \xi(a^* \times b + a \times b^*),$$

where $a \times b$ is the classical cross product according to the signature $(+, +, -)$ [14].

In $\mathbb{D}_1^3$, the dual Lorentzian unit sphere is given as

$$S_1^2 = \{\widehat{\varphi} = \varphi + \xi\varphi^*; \|\widehat{\varphi}\| = (1,0); \varphi, \varphi^* \in \mathbb{R}_1^3, \varphi \text{ spacelike}\},$$

and the dual hyperbolic unit sphere is defined as

$$H_0^2 = \{\widehat{\varphi} = \varphi + \xi\varphi^*; \|\widehat{\varphi}\| = (1,0); \varphi, \varphi^* \in \mathbb{R}_1^3, \varphi \text{ timelike}\}.$$

The oriented timelike and spacelike lines in $\mathbb{E}_1^3$ are presented by timelike and spacelike unit vectors with three components in $\mathbb{D}_1^3$, respectively. A differentiable curve on H_0^2 corresponds to a timelike ruled surface while a differentiable curve on S_1^2 corresponds to any ruled surface [14].

Theorem 1. *(E. Study) The oriented lines in R^3 are in one-to-one correspondence with the points of the dual unit sphere $A.A = 1$ in D^3 [11].*

A differentiable curve A(u) on the dual unit sphere, depending on a real unit parameter u, represents a differentiable family of straight lines in R^3 : a ruled surface. The lines A(u) are the generators or rulings of the surface.

The distribution parameter of the ruled surface determined by

$$X(t) = x(t) + \xi x^*(t)$$

is defined by

$$\delta = \frac{\langle x'(t), x^{*\prime}(t)\rangle}{\langle x'(t), x'(t)\rangle} = \frac{p^*}{p}. \tag{1}$$

If $\delta = 0, p^* = 0$ then the ruled surface is a developable surface [3].

Let a curve $\alpha(s)$ on a surface M in Lorentz space be given by the arc-length parameter. The Frenet trihedron which belongs to this curve differs as follows:

The Frenet formulae of a spacelike curve with timelike principal normal are given as

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2}$$

where $\langle T,T\rangle = 1$, $\langle N,N\rangle = -1$, $\langle B,B\rangle = 1$, $\langle T,N\rangle = \langle T,B\rangle = \langle N,B\rangle = 0$ and κ and τ are curvature and torsion of the spacelike curve, respectively [14].

The Frenet formulae of a timelike curve are given as

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{3}$$

where $\langle T,T\rangle = -1$, $\langle N,N\rangle = 1$, $\langle B,B\rangle = 1$, $\langle T,N\rangle = \langle T,B\rangle = \langle N,B\rangle = 0$ and κ and τ are curvature and torsion of the spacelike curve, respectively [14].

According to Lorentzian causal characters of the surface S and the curve $\alpha(s)$ on S, the Darboux derivative formulae are given as follows:

(i) If both the curve and surface are spacelike ones, then the formulae are

$$\begin{bmatrix} \eta_1'(s) \\ \eta_2'(s) \\ \eta_3'(s) \end{bmatrix} = \begin{bmatrix} 0 & \rho_g & \rho_n \\ -\rho_g & 0 & \tau_g \\ \rho_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} \eta_1(s) \\ \eta_2(s) \\ \eta_3(s) \end{bmatrix}, \tag{4}$$

(ii) If both the curve and surface are timelike ones, then the formulae are

$$\begin{bmatrix} \eta_1'(s) \\ \eta_2'(s) \\ \eta_3'(s) \end{bmatrix} = \begin{bmatrix} 0 & \rho_g & \rho_n \\ \rho_g & 0 & -\tau_g \\ \rho_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} \eta_1(s) \\ \eta_2(s) \\ \eta_3(s) \end{bmatrix}. \tag{5}$$

In these formulae, ρ_g, ρ_n and τ_g are called the geodesic curvature, the normal curvature, and the geodesic torsion, respectively [14].

A ruled surface in dual space is represented by the following dual unit vectorial fuction

$$\overrightarrow{X}(t) = x(t) + \xi x^*(t), \tag{6}$$

where $\xi = (0,1)$ is the dual unit. Let the vectors

$$\begin{aligned} X_1(t) &= x_1(t) + \xi x_1^*(t), \\ X_2(t) &= \frac{X_1'(t)}{\|X_1'(t)\|} = \frac{X_1'(t)}{P}, \\ X_3(t) &= X_1(t) \times X_2(t) \end{aligned} \tag{7}$$

be a trihedron depending on the ruled surface in $\mathbb{D}_1^3$. This trihedron is called as Blaschke trihedron. It differs according to the causal character of ruled surface as follows:

(i) If the ruled surface is a spacelike surface, then the Blaschke derivative formulae are

$$\begin{bmatrix} X_1'(t) \\ X_2'(t) \\ X_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & P & 0 \\ -P & 0 & Q \\ 0 & Q & 0 \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{bmatrix}, \tag{8}$$

(ii) If the ruled surface is a timelike surface, then the Blaschke derivative formulae are

$$\begin{bmatrix} X_1'(t) \\ X_2'(t) \\ X_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & P & 0 \\ P & 0 & Q \\ 0 & -Q & 0 \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{bmatrix}. \tag{9}$$

In equations (8), (9), the dual invariants of ruled surfaces are

$$P = p(t) + \xi p^*(t),\ Q = q(t) + \xi q^*(t),$$

where $P = \|X_1'(t)\|, Q = \langle X_2'(t), X_3(t)\rangle$ [1].

Let S and α be a surface and a curve, respectively, in E_1^3. The points of the curve α and the tangent vectors of the surface

S at the same points which constitutes a geometric being are called as strip [13].

Strips which satisfy the condition $\rho_n = \langle \eta_1', \eta_3 \rangle = 0$ are called osculating strips [3].

If the moments of the Darboux vector at the point $\alpha(s)$ are taken with respect to origin in the coordinate system $(O;x,y,z)$, the dual unit vectors which are defined as

$$X_i(s) = \eta_i(s) + \xi \eta_i^*(s) = \eta_i(s) + \xi(\alpha(s) \times \eta_i(s)), 1 \leq i,j \leq 3 \tag{10}$$

form the base $\{X_1(s), X_2(s), X_3(s)\}$ in dual Lorentz space and these vectors have the following property

$$\langle X_i(s), X_j(s) \rangle = \begin{cases} \varepsilon(X_i) \text{ if } i = j \\ 0 \text{ if } i \neq j. \end{cases}$$

When the point $\alpha(s)$ traces the curve α in Lorentz space, the dual unit vector $X_1(s)$ generates the surface in dual Lorentz space. The frame

$$\{X_1(s), X_2(s), X_3(s)\}$$

belonging to the generator $X_1(s)$ is the Blaschke trihedron of the surface.

3 The characterizations between spacelike osculating strip curves and spacelike ruled surfaces in $\mathbb{D}_1^3$

In this condition, we choose both the curve α and the ruled surface X_1 as a spacelike curve and a spacelike surface, respectively. The vector $\alpha(s)$ is written with respect to Darboux vector is

$$\alpha(s) = m(s)\eta_1(s) + n(s)\eta_2(s) + k(s)\eta_3(s), \tag{11}$$

where $m = m(s), n = n(s), k = k(s)$.

Hence we obtain the following relations

$$\begin{aligned} X_1(s) &= \eta_1(s) + \xi[-k(s)\eta_2(s) - n(s)\eta_3(s)], \\ X_2(s) &= \eta_2(s) + \xi[k(s)\eta_1(s) + m(s)\eta_3(s)], \\ X_3(s) &= \eta_3(s) + \xi[-n(s)\eta_1(s) + m(s)\eta_2(s)]. \end{aligned} \tag{12}$$

If the coefficients $m(s), n(s), k(s)$ are found, then the spacelike ruled surface $X_1(s)$ can be determined with respect to the invariants of the curve $\alpha(s)$. Differentiating (11) with respect to s, and then substituting (4) in it gives

$$\begin{cases} m'(s) + \rho_g n(s) + \rho_n k(s) = 1, \\ m(s)\rho_g + n'(s) + k(s)\tau_g = 0, \\ m(s)\rho_n + n(s)\tau_g + k'(s) = 0. \end{cases} \tag{13}$$

Since the condition for the curve α to be an osculating strip curve, that is $\rho_n = 0$, the equation (13) becomes

$$\begin{cases} m'(s) + \rho_g n(s) = 1, \\ m(s)\rho_g + n'(s) + k(s)\tau_g = 0, \\ n(s)\tau_g + k'(s) = 0. \end{cases} \tag{14}$$

We will study the sytem of differential equations (14) for certain special cases as follows:

Case 1. If $m(s)=0$, the curve α is located in affine subspace combined with the vector space $S_p\{\eta_2(s),\eta_3(s)\}$ at the point $\alpha(s)$, then the system (14) turns into

$$\begin{cases} \rho_g n(s)=1, \\ n'(s)+k(s)\tau_g=0, \\ n(s)\tau_g+k'(s)=0. \end{cases} \tag{15}$$

From $(15)_2$ and $(15)_3$, we have

$$k''(s)-\frac{\tau_g'}{\tau_g}k'(s)-\tau_g^2 k(s)=0. \tag{16}$$

If we change the parameter as $t=\int\limits_0^s \tau_g ds$, we obtain $\dfrac{d^2k}{dt^2}-k=0$. Thus the solution of k is as follows:

$$k=c_1 e^{(\int\limits_0^s \tau_g ds)}+c_2 e^{-(\int\limits_0^s \tau_g ds)}, \tag{17}$$

where c_1 and c_2 are real constants. Also from $(15)_1$, it is clear that

$$n(s)=\frac{1}{\rho_g}. \tag{18}$$

Rewriting (17) and (18) in (11), we obtain the curve α as

$$\alpha(s)=\frac{1}{\rho_g}\eta_2(s)+[c_1 e^{(\int\limits_0^s \tau_g ds)}+c_2 e^{-(\int\limits_0^s \tau_g ds)}]\eta_3(s). \tag{19}$$

Thereby, Blaschke vectors of the spacelike ruled surface $X_1(s)$ are determined by the invariants of Darboux trihedron as follows:

$$\begin{aligned} X_1(s)&=\eta_1(s)+\xi[-[c_1 e^{(\int\limits_0^s \tau_g ds)}+c_2 e^{-(\int\limits_0^s \tau_g ds)}]\eta_2(s)-\frac{1}{\rho_g}\eta_3(s)], \\ X_2(s)&=\eta_2(s)+\xi[c_1 e^{(\int\limits_0^s \tau_g ds)}+c_2 e^{-(\int\limits_0^s \tau_g ds)}]\eta_1(s), \\ X_3(s)&=\eta_3(s)+\xi[-\frac{1}{\rho_g}\eta_1(s)]. \end{aligned} \tag{20}$$

Thus we can give the following result.

Corollary 1. *The curve α is determined by (19) and the Blaschke vectors of spacelike ruled surface $X_1(s)$ are determined by the invariants of Darboux trihedron as in (20).*

If $c_1=c_2=0$, the equation (20) becomes

$$\begin{aligned} X_1(s)&=\eta_1(s)+\xi[-\frac{1}{\rho_g}\eta_3(s)], \\ X_2(s)&=\eta_2(s), \\ X_3(s)&=\eta_3(s)+\xi[-\frac{1}{\rho_g}\eta_1(s)]. \end{aligned} \tag{21}$$

Differentiating (21)$_1$ and (21)$_3$ with respect to s, *and then using (8), we have*

$$p=\rho_g,\ p^*=-\frac{\tau_g}{\rho_g},\ q=\tau_g,\ q^*=1. \tag{22}$$

Corollary 2. *The real and dual parts of the dual invariants of spacelike ruled surface* X_1 *are obtained as in (22).*

Case 2. *If* $n(s)=0$, *the curve* α *is located in affine subspace combined with the vector space* $S_p\{\eta_1(s),\eta_3(s)\}$ *at the point* $\alpha(s)$, *hence the system (14) turns into*

$$\begin{cases} m'(s)=1, \\ m(s)\rho_g+k(s)\tau_g=0, \\ k'(s)=0. \end{cases} \tag{23}$$

From (23)$_1$ and (23)$_3$, we have

$$m=s+c_1,\ k=c_2 \tag{24}$$

and also from (23)$_2$, we obtain a ratio between the geodesic torsion and the geodesic curvature as

$$\frac{\rho_g}{\tau_g}=-\frac{c_2}{s+c_1}.$$

Rewriting (24) in (11), we obtain the curve α *as*

$$\alpha(s)=(s+c_1)\eta_1(s)+c_2\eta_3(s). \tag{25}$$

Thereby, Blaschke vectors of the spacelike ruled surface $X_1(s)$ *are determined by the invariants of Darboux trihedron as follows:*

$$\begin{aligned} X_1(s)&=\eta_1(s)+\xi[-c_2\eta_2(s)], \\ X_2(s)&=\eta_2(s)+\xi[c_2\eta_1(s)+(s+c_1)\eta_3(s)], \\ X_3(s)&=\eta_3(s)+\xi[(s+c_1)\eta_2(s)]. \end{aligned} \tag{26}$$

Thus we can give the following result:

Corollary 3. *The curve* α *is determined by (25) and the Blaschke vectors of spacelike ruled surface* $X_1(s)$ *are determined by the invariants of Darboux trihedron as in (26).*

If $c_1=c_2=0$, *the equation (20) becomes*

$$\begin{aligned} X_1(s)&=\eta_1(s), \\ X_2(s)&=\eta_2(s)+\xi(s\eta_3(s)), \\ X_3(s)&=\eta_3(s)+\xi(s\eta_2(s)). \end{aligned} \tag{27}$$

Differentiating (27)$_1$ and (27)$_3$ with respect to s, *and then using (8), we have*

$$p=\rho_g,\ p^*=0,\ q=\tau_g,\ q^*=1. \tag{28}$$

Thus we can give the following result.

Corollary 4. *The real and dual parts of the dual invariants of spacelike ruled surface* X_1 *are obtained as in (28).*

From (28) and (1), we get the following result.

Corollary 5. *If $c_1 = c_2 = 0$, then the spacelike ruled surface generated by the line X_1 is a developable surface.*

Case 3. *If $k(s) = 0$, the curve α is located in affine subspace combined with the vector space $S_p\{\eta_1(s), \eta_2(s)\}$ at the point $\alpha(s)$, then the system (14) turns into*

$$\begin{cases} m'(s) + \rho_g n(s) = 1, \\ m(s)\rho_g + n'(s) = 0, \\ n(s)\tau_g = 0. \end{cases} \tag{29}$$

From (29)$_1$ and (29)$_3$, we have

$$m''(s) + \frac{\rho_g'}{\rho_g}(1 - m'(s)) - \rho_g^2 m(s) = 0. \tag{30}$$

If we change the parameter as $t = \int\limits_0^s \rho_g ds$ in (30), we obtain

$$\frac{d^2m}{dt^2} - m = \frac{\rho_g'}{\rho_g^3}.$$

Thus the solution of m is as follows:

$$m = c_1 e^{(\int\limits_0^s \rho_g ds)} + c_2 e^{(-\int\limits_0^s \rho_g ds)} + \frac{1}{D^2 - 1}(-\frac{\dot{\rho_g}}{\rho_g^3}), \tag{31}$$

where c_1 and c_2 are real constants. Also from (29)$_3$, it is clear that

$$n(s) = 0. \tag{32}$$

Rewriting (31) and (32) in (11), we obtain the curve α as

$$\alpha(s) = [c_1 e^{(\int\limits_0^s \rho_g ds)} + c_2 e^{(-\int\limits_0^s \rho_g ds)} + \frac{1}{D^2 - 1}(-\frac{\dot{\rho_g}}{\rho_g^3})]\eta_1(s), \tag{33}$$

where $c_1, c_2 \in R$, $\dot{\rho_g} = \frac{d\rho_g}{dt}$, $D = \frac{d}{dt}$.

Thereby, Blaschke vectors of the spacelike ruled surface $X_1(s)$ are determined by the invariants of Darboux trihedron as follows:

$$\begin{aligned} X_1(s) &= \eta_1(s), \\ X_2(s) &= \eta_2(s) + \xi[c_1 e^{(\int\limits_0^s \rho_g ds)} + c_2 e^{(-\int\limits_0^s \rho_g ds)} + \frac{1}{D^2 - 1}(-\frac{\dot{\rho_g}}{\rho_g^3})]\eta_3(s), \\ X_3(s) &= \eta_3(s) + \xi[c_1 e^{(\int\limits_0^s \rho_g ds)} + c_2 e^{(-\int\limits_0^s \rho_g ds)} + \frac{1}{D^2 - 1}(-\frac{\dot{\rho_g}}{\rho_g^3})]\eta_2(s). \end{aligned} \tag{34}$$

Thus we can give the following result.

Corollary 6. *The curve α is determined by (33) and the Blaschke vectors of spacelike ruled surface $X_1(s)$ are determined by the invariants of Darboux trihedron as in (34).*

If $c_1 = c_2 = 0$, the equation (34) becomes

$$\begin{aligned} X_1(s) &= \eta_1(s), \\ X_2(s) &= \eta_2(s) + \xi[\frac{1}{D^2}(\frac{\rho_g'}{\rho_g})]\eta_3(s), \\ X_3(s) &= \eta_3(s) + \xi[\frac{1}{D^2}(\frac{\rho_g'}{\rho_g})]\eta_2(s). \end{aligned} \tag{35}$$

Differentiating $(35)_1$ and $(35)_3$ with respect to s, and then using (8), we have

$$p = \rho_g,\ p^* = 0,\ q = \tau_g,\ q^* = 0. \tag{36}$$

Thus we can give the following result.

Corollary 7. *The real and dual parts of the dual invariants of spacelike ruled surface X_1 are obtained as in (22).*

From (36) and (1), we get the following result.

Corollary 8. *If $c_1 = c_2 = 0$, then the spacelike ruled surface generated by the line X_1 is a developable surface.*

4 The characterizations between timelike osculating strip curves and timelike ruled surfaces in $\mathbb{D}_1^3$

In this condition, we choose both the curve α and the ruled surface X_1 as a timelike curve and a timelike surface, respectively. The vector $\alpha(s)$ is written with respect to Darboux vector as

$$\alpha(s) = m(s)\eta_1(s) + n(s)\eta_2(s) + k(s)\eta_3(s), \tag{37}$$

where $m = m(s), n = n(s), k = k(s)$.

Hence we obtain the following relations

$$\begin{aligned} X_1(s) &= \eta_1(s) + \xi[-k(s)\eta_2(s) + n(s)\eta_3(s)], \\ X_2(s) &= \eta_2(s) + \xi[-k(s)\eta_1(s) + m(s)\eta_3(s)], \\ X_3(s) &= \eta_3(s) + \xi[n(s)\eta_1(s) + m(s)\eta_2(s)]. \end{aligned} \tag{38}$$

If the coefficients $m(s), n(s), k(s)$ are found, then the timelike ruled surface $X_1(s)$ can be determined with respect to the invariants of the curve $\alpha(s)$. Differentiating (37) with respect to s, and then substituting (5) in it gives

$$\begin{cases} m'(s) + \rho_g n(s) + \rho_n k(s) = 1, \\ m(s)\rho_g + n'(s) + k(s)\tau_g = 0, \\ m(s)\rho_n - n(s)\tau_g + k'(s) = 0. \end{cases} \tag{39}$$

Since the condition for the curve α to be a osculating strip curve, that is $\rho_n = 0$, the equation (39) becomes

$$\begin{cases} m'(s) + \rho_g n(s) = 1, \\ m(s)\rho_g + n'(s) + k(s)\tau_g = 0, \\ -n(s)\tau_g + k'(s) = 0. \end{cases} \tag{40}$$

We will study the sytem of differential equations (40) for certain special cases as follows:

Case 1. If $m(s)=0$, the curve α is located in affine subspace combined with the vector space $S_p\{\eta_2(s),\eta_3(s)\}$ at the point $\alpha(s)$, then the system (40) turns into

$$\begin{cases} \rho_g n(s)=1, \\ n'(s)+k(s)\tau_g=0, \\ -n(s)\tau_g+k'(s)=0. \end{cases} \tag{41}$$

From $(41)_2$ and $(41)_3$, we have

$$k''(s)-\frac{\tau_g'}{\tau_g}k'(s)+\tau_g^2 k(s)=0. \tag{42}$$

If we change the parameter as $t=\int_0^s \tau_g ds$, we obtain $\frac{d^2k}{dt^2}+k=0$. Thus the solution of k is as follows:

$$k=c_1\cos(\int_0^s \tau_g ds)+c_2\sin(\int_0^s \tau_g ds), \tag{43}$$

where c_1 and c_2 are real constants. Also from $(41)_1$, it is clear that

$$n(s)=\frac{1}{\rho_g}. \tag{44}$$

Rewriting (43) and (44) in (37), we obtain the curve α as

$$\alpha(s)=\frac{1}{\rho_g}\eta_2(s)+[c_1\cos(\int_0^s \tau_g ds)+c_2\sin(\int_0^s \tau_g ds)]\eta_3(s). \tag{45}$$

Thereby, Blaschke vectors of the timelike ruled surface $X_1(s)$ are determined by the invariants of Darboux trihedron as follows:

$$\begin{aligned} X_1(s)&=\eta_1(s)+\xi[-[c_1\cos(\int_0^s \tau_g ds)+c_2\sin(\int_0^s \tau_g ds)]\eta_2(s)+\frac{1}{\rho_g}\eta_3(s)], \\ X_2(s)&=\eta_2(s)+\xi[-[c_1\cos(\int_0^s \tau_g ds)+c_2\sin(\int_0^s \tau_g ds)]\eta_1(s)], \\ X_3(s)&=\eta_3(s)+\xi[\frac{1}{\rho_g}\eta_1(s)]. \end{aligned} \tag{46}$$

Thus we can give the following result.

Corollary 9. *The curve α is determined by (45) and the Blaschke vectors of the timelike ruled surface $X_1(s)$ are determined by the invariants of Darboux trihedron as in (46).*

If $c_1=c_2=0$, the equation (46) turns

$$\begin{aligned} X_1(s)&=\eta_1(s)+\xi[\frac{1}{\rho_g}\eta_3(s)], \\ X_2(s)&=\eta_2(s), \\ X_3(s)&=\eta_3(s)+\xi[\frac{1}{\rho_g}\eta_1(s)]. \end{aligned} \tag{47}$$

Differentiating $(47)_1$ and $(47)_3$ with respect to s, and then using (9), we have

$$p=\rho_g,\ p^*=\frac{\tau_g}{\rho_g},\ q=-\tau_g,\ q^*=-1. \tag{48}$$

Corollary 10. *The real and dual parts of the dual invariants of the timelike ruled surface X_1 are obtained as in (48).*

Case 2. If $n(s) = 0$, the curve α is located in affine subspace combined with the vector space $S_p\{\eta_1(s), \eta_3(s)\}$ at the point $\alpha(s)$, then the system (40) turns into

$$\begin{cases} m'(s) = 1, \\ m(s)\rho_g + k(s)\tau_g = 0, \\ k'(s) = 0. \end{cases} \tag{49}$$

From $(49)_1$ and $(49)_3$, we have

$$m = s + c_1,\ k = c_2 \tag{50}$$

and also from $(49)_2$, we obtain a ratio between the geodesic torsion and the geodesic curvature as

$$\frac{\rho_g}{\tau_g} = -\frac{c_2}{s + c_1}.$$

Rewriting (50) in (37), we obtain the curve α as

$$\alpha(s) = (s + c_1)\eta_1(s) + c_2\eta_3(s). \tag{51}$$

Thereby, Blaschke vectors of the timelike ruled surface $X_1(s)$ are determined by the invariants of Darboux trihedron as follows:

$$\begin{aligned} X_1(s) &= \eta_1(s) + \xi[-c_2\eta_2(s)], \\ X_2(s) &= \eta_2(s) + \xi[-c_2\eta_1(s) + (s + c_1)\eta_3(s)], \\ X_3(s) &= \eta_3(s) + \xi[(s + c_1)\eta_2(s)]. \end{aligned} \tag{52}$$

Thus we can give the following result.

Corollary 11. *The curve α is determined by (51) and the Blaschke vectors of timelike ruled surface $X_1(s)$ are determined by the invariants of Darboux trihedron as in (52).*

If $c_1 = c_2 = 0$, the equation (52) becomes

$$\begin{aligned} X_1(s) &= \eta_1(s), \\ X_2(s) &= \eta_2(s) + \xi(s\eta_3(s)), \\ X_3(s) &= \eta_3(s) + \xi(s\eta_2(s)). \end{aligned} \tag{53}$$

Differentiating $(53)_1$ and $(53)_3$ with respect to s, and then using (9), we have

$$p = \rho_g,\ p^* = 0,\ q = -\tau_g,\ q^* = -1. \tag{54}$$

Thus we can give the following result.

Corollary 12. *The real and dual parts of the dual invariants of the timelike ruled surface X_1 are obtained as in (54).*

From (54) and (1), we get the following result.

Corollary 13. *If $c_1 = c_2 = 0$, then the timelike ruled surface generated by the line X_1 is a developable surface.*

Case 3. If $k(s) = 0$, the curve α is located in affine subspace combined with the vector space $S_p\{\eta_1(s), \eta_2(s)\}$ at the point $\alpha(s)$, then the system (40) turns into

$$\begin{cases} m'(s) + \rho_g n(s) = 1, \\ m(s)\rho_g + n'(s) = 0, \\ -n(s)\tau_g = 0. \end{cases} \tag{55}$$

From $(55)_1$ and $(55)_3$, we have

$$m''(s)+\frac{\rho_g'}{\rho_g}(1-m'(s))-\rho_g^2 m(s)=0. \tag{56}$$

If we change the parameter as $t=\int_0^s \rho_g ds$ in (56), we obtain

$$\frac{d^2m}{dt^2}-m=\frac{\rho_g'}{\rho_g^3}.$$

Thus the solution of m is as follows:

$$m=c_1e^{(\int_0^s \rho_g ds)}+c_2e^{(-\int_0^s \rho_g ds)}+\frac{1}{D^2-1}(-\frac{\dot{\rho_g}}{\rho_g^3}), \tag{57}$$

where c_1 and c_2 are real constants. Also from $(55)_3$, it is clear that

$$n(s)=0. \tag{58}$$

Rewriting (57) and (58) in (37), we obtain the curve α as

$$\alpha(s)=[c_1e^{(\int_0^s \rho_g ds)}+c_2e^{(-\int_0^s \rho_g ds)}+\frac{1}{D^2-1}(-\frac{\dot{\rho_g}}{\rho_g^3})]\eta_1(s), \tag{59}$$

where $c_1,\, c_2\in R$, $\dot{\rho_g}=\frac{d\rho_g}{dt}$, $D=\frac{d}{dt}$.

Thereby, Blaschke vectors of the timelike ruled surface $X_1(s)$ are determined by the invariants of Darboux trihedron as follows:

$$\begin{aligned}
X_1(s)&=\eta_1(s),\\
X_2(s)&=\eta_2(s)+\xi[c_1e^{(\int_0^s \rho_g ds)}+c_2e^{(-\int_0^s \rho_g ds)}+\frac{1}{D^2-1}(-\frac{\dot{\rho_g}}{\rho_g^3})]\eta_3(s),\\
X_3(s)&=\eta_3(s)+\xi[c_1e^{(\int_0^s \rho_g ds)}+c_2e^{(-\int_0^s \rho_g ds)}+\frac{1}{D^2-1}(-\frac{\dot{\rho_g}}{\rho_g^3})]\eta_2(s).
\end{aligned} \tag{60}$$

Thus we can give the following result.

Corollary 14. *The curve α is determined by (59) and the Blaschke vectors of timelike ruled surface $X_1(s)$ are determined by the invariants of Darboux trihedron as in (60).*

If $c_1=c_2=0$, the equation (60) becomes

$$\begin{aligned}
X_1(s)&=\eta_1(s),\\
X_2(s)&=\eta_2(s)+\xi[\frac{1}{D^2-1}(\frac{\rho_g'}{\rho_g})]\eta_3(s),\\
X_3(s)&=\eta_3(s)+\xi[\frac{1}{D^2-1}(\frac{\rho_g'}{\rho_g})]\eta_2(s).
\end{aligned} \tag{61}$$

Differentiating $(61)_1$ and $(61)_3$ with respect to s, and then using (9), we have

$$p=\rho_g,\; p^*=0,\; q=\tau_g,\; q^*=0. \tag{62}$$

Thus we can give the following result.

Corollary 15. *The real and dual parts of the dual invariants of the timelike ruled surface X_1 are obtained as in (62).*

From (62) and (1), we get the following result.

Corollary 16. *If $c_1 = c_2 = 0$, then the timelike ruled surface generated by the line X_1 is a developable surface.*

5 Conclusion

In this paper, the conditions between osculating strip curves and ruled surfaces were studied in dual Lorentz space $\mathbb{D}_1^3$. For this study, a system of differential equations characterizing both spacelike and timelike ruled surfaces were established in dual Lorentz space $\mathbb{D}_1^3$ by using the invariant quantities of osculating strip curves on the given ruled surfaces. Then the solutions of these systems were obtained for special cases. Regarding to these special solutions, some results of relations between osculating strip curves and ruled surfaces were given in dual Lorentz space $\mathbb{D}_1^3$.

Competing interests

The authors declare that they have no competing interests.

Authors' Contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript

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