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On the complete arcs containing the quadrangles constructing the Fano planes of the left near field plane of order 9

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Abstract: In this paper, we introduce a method to find arcs in the projective plane of order 9 coordinatized by elements of a left nearfield of order 9. We give an algorithm for checking arcs in this projective plane and apply the algorithm (implemented in C#) to determine and classify arcs.

Keywords: Near field, projective plane, arc.

1 Introduction

A projective plane π consists of a set P of points, and a set L of subsets of P, called lines, such that every pair of points is contained in exactly one line, every two distinct lines intersect in exactly one point, and there exist four points in such a position that they pairwise define six distinct lines. A subplane of a projective plane π is a set B of points and lines which is itself a projective plane, relative to the incidence relation given in π.

It is well known that there exist at least four non-isomorphic projective planes of order 9. The known four distinct projective planes of order 9 are extensively studied by Room Kirkpatrick [4]. These are Desarguesian plane, the left nearfield plane, the right nearfield plane and Hughes plane. The last three planes of order 9 are called ”miniquaternion planes” because they can be coordinatized by the miniquaternion near field. O. Veblen and J. M. Wedderburn [6] discovered these miniquaternion planes in 1907.

A $(k,2)-$ arc K is a set of k points no 3 of which are collinear in the plane. A $(k,2)-$arc is called simply an arc of size k or a k arc. A $(k,3)-$ arc K is a set of k points no 4 of which are collinear of this plane.

In [1,5], an algorithm (implemented in C#) to determine and classify Fano subplanes of the projective plane of order 9 coordinatized by elements of a left nearfield of order 9 is given and 18 complete quadrangles which generate Fano plane are determined. In this paper, the projective plane whose algebraic structure is the left nearfield plane of order 9 is constructed by using homogene coordinates and then all complete arcs containing the Fano plane are obtained by using “completion procedure” in this projective plane. We give an algorithm for checking arcs in this projective plane and apply the algorithm (implemented in C#) to determine and classify arcs.

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2 The left nearfield system of order 9

The regular near field of order q^2, for q an odd prime power, are defined taking the elements of $GF(q^2)$, using the field addition and definition a new multiplication on the elements in terms of the field multiplication. This gives an algebraic system in which the non-zero elements form a group under the multiplication and the right or left distributive laws hold. The near field can be used to define and coordinatize the near field plane of order 9.

We give the left near field of order 9 by taking the elements of $GF(3)$ and using the field addition and a new multiplication on the elements in terms of the left near field multiplication, [1,2,3,5].

Definition 1. *A left nearfield is a system* $(S,\oplus,\odot)$, *where* $\oplus$ *and* $\odot$ *are binary operations on the set* S *and*

(i) S *is finite*
(ii) $(S,\oplus)$ *is a group, with identity* 0
(iii) $(S\backslash\{0\},\odot)$ *is a group, with identity* 1
(iv) $0\odot x=0$ *for all* $x\in S$
(v) $\odot$ *is left distributive over* $\oplus$, *that is* $x\odot(y\oplus z)=(x\odot y)\oplus(x\odot z)$ *for all* $x,y,z\in S$
(vi) *Given* $m,n,k\in S$ *with* $m\neq n$, *there exists a unique* $x\in S$ *such that* $-m\odot x\oplus n\odot x=k$.

Let $(F_3,+,.)$ be the Galois field of order 3.We now construct $(S,\oplus,\odot)$, using F_3, a left nearfield of order 9.

The nine elements of S are $a+\lambda b$, $a,b\in F_3$, $\lambda\notin F_3$. Addition in S is defined by the rule

$$(a+\lambda b)\oplus(c+\lambda d)=(a+c)+\lambda(b+d) \tag{1}$$

and multiplication by

$$(a+\lambda b)\odot(c+\lambda d)=\begin{cases} ac+\lambda(ad), & \text{if} \quad b=0\\ ac-b^{-1}d\,f(a)+\lambda(bc-(a-1)d), & \text{if} \quad b\neq 0\end{cases} \tag{2}$$

where, $a,b,c,d\in F_3$, $\lambda\notin F_3$ and $f(t)=t^2+1$ is a irreducible polynom on F_3.

For the sake of sorthness if we use ab instead of $a+\lambda b$ in equation (1) and (2), then addition and multiplication tables are obtained as follows:

Table 1

$\oplus$	00	01	02	10	11	12	20	21	22
00	00	01	02	10	11	12	20	21	22
01	01	02	00	11	12	10	21	22	20
02	02	00	01	12	10	11	22	20	21
10	10	11	12	20	21	22	00	01	02
11	11	12	10	21	22	20	01	02	00
12	12	10	11	22	20	21	02	00	01
20	20	21	22	00	01	02	10	11	12
21	21	22	20	01	02	00	11	12	10
22	22	20	21	02	00	01	12	10	11

Table 2

$\odot$	00	01	02	10	11	12	20	21	22
00	00	00	00	00	00	00	00	00	00
01	00	20	10	01	21	11	02	22	12
02	00	10	20	02	12	22	01	11	21
10	00	01	02	10	11	12	20	21	22
11	00	12	21	11	20	02	22	01	10
12	00	22	11	12	01	20	21	10	02
20	00	02	01	20	22	21	10	12	11
21	00	11	22	21	02	10	12	20	01
22	00	21	12	22	10	01	11	02	20

If we use the following equalities

$$0 = (0,0)$$
$$1 = (1,0)$$
$$2 = (2,0)$$
$$3 = (0,1)$$
$$4 = (1,1)$$
$$5 = (2,1)$$
$$6 = (0,2)$$
$$7 = (1,2)$$
$$8 = (2,2)$$

the addition and multiplication tables in $(S,\oplus,\odot)$ can be arranged as follows :

Table 3

$\oplus$	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	0	4	5	3	7	8	3
2	2	0	1	5	3	4	8	6	7
3	3	4	5	6	7	8	0	1	2
4	4	5	3	7	8	6	1	2	0
5	5	3	4	8	6	7	2	0	1
6	6	7	8	0	1	2	3	4	5
7	7	8	6	1	2	0	4	5	3
8	8	6	7	2	0	1	5	3	4

$\odot$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8
2	0	2	1	6	8	7	3	5	4
3	0	3	6	2	5	8	1	4	7
4	0	4	8	7	2	3	5	6	1
5	0	5	7	4	6	2	8	1	3
6	0	6	3	1	7	4	2	8	5
7	0	7	5	8	3	1	4	2	6
8	0	8	4	5	1	6	7	3	2

The system $(S,\oplus,\odot)$ satisfies the conditions of Definition 2.1 and therefore a left nearfield of order 9.

3 The projective plane P_2S

It is well known that every projective plane has also an algebraic structure obtained by coordinazation. Conversely, certain algebraic structures can be used to construct projective planes.

In this section, we will construct projective plane order 9. The projective plane whose the points and the lines are coordinated by the elements of $(S,\oplus,\odot)$.

The 91 points of P_2S are the elements of the set

$$\{(x,y) : x,y \in S\} \cup \{(m) : m \in S\} \cup \{(\infty)\}.$$

The points of the form (x,y) are called *proper points*, and the unique point (∞) and the points of the form (m) are called*ideal points*. The 91 lines of P_2S are defined to be set of points satisfying one of the three conditions:

$$\begin{aligned}
&[m,k] = \{(x,y) \in S^2,\ y = m \odot x \oplus k\} \cup \{(m)\} \\
&[a] = \{(x,y) \in S^2\ , x = a\} \cup \{(\infty)\} \\
&[\infty] = \{(m) \in S\} \cup \{(\infty)\}.
\end{aligned}$$

The 81 lines having form $y = m \odot x \oplus k$ and 9 lines having equation of the form $x = \lambda$ are called the *proper lines* and the unique line $[\infty]$ is called *the ideal line.*

The system of points, lines and incidence relation given above defines a projective plane of order 9, which is the left nearfiled plane.

Now, we are considering the projective plane of order 9 homogeneous coordinatized by elements of the above left nearfield. We notice that the homogeneous coordinates of a point are not unique. Two triples that are multiples of each other specify the same point. Thus the same point has many sets of homogeneous coordinates: (x,y,z) and (x',y',z') represent the same point if and only if there is some $\lambda \neq 0$, $\lambda \in S$ such that $x' = \lambda \odot x$, $y' = \lambda \odot y$, $z' = \lambda \odot z$. We convert a point expressed in Cartesian coordinates to homogeneous coordinates in left nearfield plane of order 9. We have seen that a point (x,y) in the P_2S has homogeneous coordinates $\lambda \odot (x,y,1) = (\lambda \odot x, \lambda \odot y, \lambda \odot 1)$, $\lambda \neq 0$, $\lambda \in S$. Homogeneous coordinates of the form $\lambda \odot (m,1,0)$ do correspond to all ideal points (m), m, $\lambda \in S^*$ in the P_2S. Homogeneous coordinates of the form $(\lambda,0,0)$ do correspond to the unique point at infinity in the P_2S . We have seen that a line $[m,k]$ in the P_2S has homogeneous coordinates $\mu \odot [m,-1,k] = [\mu \odot m, \mu \odot (-1), \mu \odot k]$, $\mu \neq 0$, $\mu \in S$. Homogeneous coordinates of the form $\mu \odot [x,0,1]$ do correspond to all lines $[a]$, $a \neq 0$, $a \in S$ in the P_2S. Homogeneous coordinates of the form $[0,0,\mu]$ do correspond to the unique line $[\infty]$ at infinity in the P_2S.

A line in the P_2S has general equation $y = m \odot x \oplus k$. Suppose (x_1,x_2,x_3), $x_3 \neq 0$ are the homogeneous coordinates of a point (x,y) on the line; hence $x = x_3^{-1} \odot x_1$ and $y = x_3^{-1} \odot x_2$. Substituting for x and y in the line equation and multiplying through by x_3, yields the conditions for (x_1,x_2,x_3) to be the homogeneous coordinates of a point on the line:

$$m \odot x_1 \oplus (-1) \odot x_2 \oplus k \odot x_3 = 0.$$

The following table lists all homogeneous coordinates of the 91 points and lines in the projective plane of order 9 coordinatized by elements of the above left nearfield.

4 The complete arcs containing the quadrangles constructing the Fano planes in the projective plane P_2S

A $(k,n)-$arc K in a projective plane is a set of k points, such that some n, but no $n+1$ of them are collinear. A $(k,n)-$arc K is complete if it is not contained in a $(k+1,n)-$ arc. A line of the projective plane containing exactly i points of a $(k,n)-$ arc is called an $i-$secant of K. For a $(k,3)-$arc each line of projective plane is a 3-secant, 2-secant, 1-secant, or 0-secant.

It is known that a finite projective plane is coordinatized by any quadrangle O,I,X,P (4 points, no 3 on a line). In [1,5], It

i	P_i	l_i									
1	(1,0,0)	2	11	20	29	38	47	56	65	74	83
2	(0,1,0)	1	11	12	13	14	15	16	17	18	19
3	(1,1,0)	4	11	22	30	44	55	63	68	79	87
4	(2,1,0)	3	11	21	31	41	51	61	71	81	91
5	(3,1,0)	5	11	23	35	40	54	60	66	82	88
6	(4,1,0)	6	11	24	37	43	49	62	72	77	84
7	(5,1,0)	7	11	25	36	46	50	58	69	75	89
8	(6,1,0)	8	11	26	32	39	52	64	67	78	90
9	(7,1,0)	9	11	27	34	42	53	57	73	76	86
10	(8,1,0)	10	11	28	33	45	48	59	70	80	85
11	(0,0,1)	1	2	3	4	5	6	7	8	9	10
12	(1,0,1)	2	13	22	31	40	49	58	67	76	85
13	(2,0,1)	2	12	21	30	39	48	57	66	75	84
14	(3,0,1)	2	14	23	32	41	50	59	68	77	86
15	(4,0,1)	2	15	24	33	42	51	60	69	78	87
16	(5,0,1)	2	16	25	34	43	52	61	70	79	88
17	(6,0,1)	2	17	26	35	44	53	62	71	80	89
18	(7,0,1)	2	18	27	36	45	54	63	72	81	90
19	(8,0,1)	2	19	28	37	46	55	64	73	82	91
20	(0,1,1)	1	29	30	31	32	33	34	35	36	37
21	(1,1,1)	4	13	21	29	46	54	62	70	78	86
22	(2,1,1)	3	12	22	29	42	52	59	72	82	89
23	(3,1,1)	5	14	26	29	45	51	58	73	79	84
24	(4,1,1)	6	15	28	29	40	53	61	68	75	90
25	(5,1,1)	7	16	27	29	41	49	64	66	80	87
26	(6,1,1)	8	17	23	29	43	55	57	69	81	85
27	(7,1,1)	9	18	25	29	44	48	60	67	77	91
28	(8,1,1)	10	19	24	29	39	50	63	71	76	88
29	(0,2,1)	1	20	21	22	23	24	25	26	27	28
30	(1,2,1)	3	13	20	30	43	50	60	73	80	90
31	(2,2,1)	4	12	20	31	45	53	64	69	77	88
32	(3,2,1)	8	14	20	35	46	48	61	72	76	87
33	(4,2,1)	10	15	20	37	41	54	57	67	79	89
34	(5,2,1)	9	16	20	36	39	51	62	68	82	85
35	(6,2,1)	5	17	20	32	42	49	63	70	75	91
36	(7,2,1)	7	18	20	34	40	55	59	71	78	84
37	(8,2,1)	6	19	20	33	44	52	58	66	81	86
38	(0,3,1)	1	38	39	40	41	42	43	44	45	46
39	(1,3,1)	8	13	28	34	38	51	63	66	77	89
40	(2,3,1)	5	12	24	36	38	55	61	67	80	86
41	(3,3,1)	4	14	27	37	38	52	60	71	75	85
42	(4,3,1)	7	15	22	32	38	48	62	73	81	88
43	(5,3,1)	10	16	23	30	38	53	58	72	78	91
44	(6,3,1)	3	17	25	33	38	54	64	68	76	84
45	(7,3,1)	6	18	26	31	38	50	57	70	82	87
46	(8,3,1)	9	19	21	35	38	49	59	69	79	90
47	(0,4,1)	1	47	48	49	50	51	52	53	54	55
48	(1,4,1)	10	13	27	32	44	47	61	69	82	84
49	(2,4,1)	6	12	25	35	41	47	63	73	78	85
50	(3,4,1)	9	14	24	30	40	47	64	70	81	89
51	(4,4,1)	4	15	26	36	43	47	59	66	76	91
52	(5,4,1)	5	16	22	33	46	47	57	71	77	90
53	(6,4,1)	7	17	28	31	39	47	60	72	79	86
54	(7,4,1)	8	18	21	37	42	47	58	68	80	88
55	(8,4,1)	3	19	23	34	45	47	62	67	75	87
56	(0,5,1)	1	56	57	58	59	60	61	62	63	64
57	(1,5,1)	9	13	26	33	41	55	56	72	75	88
58	(2,5,1)	7	12	23	37	44	51	56	70	76	90
59	(3,5,1)	6	14	22	34	39	54	56	69	80	91
60	(4,5,1)	8	15	25	30	45	49	56	71	82	86
61	(5,5,1)	4	16	28	35	42	50	56	67	81	84
62	(6,5,1)	10	17	21	36	40	52	56	73	77	87
63	(7,5,1)	3	18	24	32	46	53	56	66	79	85
64	(8,5,1)	5	19	27	31	43	48	56	68	78	89
65	(0,6,1)	1	65	66	67	68	69	70	71	72	73
66	(1,6,1)	5	13	25	37	39	53	59	65	81	87
67	(2,6,1)	8	12	27	33	40	50	62	65	79	91
68	(3,6,1)	3	14	28	36	44	49	57	65	78	88
69	(4,6,1)	9	15	23	31	46	52	63	65	80	84
70	(5,6,1)	6	16	21	32	45	55	60	65	76	89
71	(6,6,1)	4	17	24	34	41	48	58	65	82	90
72	(7,6,1)	10	18	22	35	43	51	64	65	75	86
73	(8,6,1)	7	19	26	30	42	54	61	65	77	85
74	(0,7,1)	1	74	75	76	77	78	79	80	81	82
75	(1,7,1)	7	13	24	35	45	52	57	68	74	91
76	(2,7,1)	9	12	28	32	43	54	58	71	74	87
77	(3,7,1)	10	14	25	31	42	55	62	66	74	90
78	(4,7,1)	5	15	21	34	44	50	64	72	74	85
79	(5,7,1)	3	16	26	37	40	48	63	69	74	86
80	(6,7,1)	6	17	27	30	46	51	59	67	74	88
81	(7,7,1)	4	18	23	33	39	49	61	73	74	89
82	(8,7,1)	8	19	22	36	41	53	60	70	74	84
83	(0,8,1)	1	83	84	85	86	87	88	89	90	91
84	(1,8,1)	6	13	23	36	42	48	64	71	79	83
85	(2,8,1)	10	12	26	34	46	49	60	68	81	83
86	(3,8,1)	7	14	21	33	43	53	63	67	82	83
87	(4,8,1)	3	15	27	35	39	55	58	70	77	83
88	(5,8,1)	8	16	24	31	44	54	59	73	75	83
89	(6,8,1)	9	17	22	37	45	50	61	66	78	83
90	(7,8,1)	5	18	28	30	41	52	62	69	76	83
91	(8,8,1)	4	19	25	32	40	51	57	72	80	83

is seen that there are 18 Fano subplanes containing $I = (1,1,1)$, $X = (1,0,0)$, $O = (0,0,1)$, $P = (a,b,1)$ with $a,b \in F_3$ in P_2S.

In this section, $(k,2)$-arcs containing Fano planes of P_2S are determined selecting the quadrangle O,I,X,P such that $I = (1,1,1)$, $X = (1,0,0)$, $O = (0,0,1)$, $P = (a,b,1)$ with $a,b \in F_3$ in P_2S and the algorithm used in the classification of $(k,2)$-arcs containing Fano planes of P_2S is described. By applying this algorithm (implemented *C#*) a full listing the complete $(k,2)$-arcs containing Fano planes of P_2S has been achieved.

4.1 Method and algorithm

Our method had been staged into 2 steps. Firstly, we consider a quadrangle from Fano subplanes containing $I = (1,1,1)$, $X = (1,0,0)$, $O = (0,0,1)$, $P = (a,b,1)$ with $a,b \in F_3$ in P_2S. Secondly, a point not on the lines of selected Fano plane is added the set $\{O,I,X,P\}$ and then a new point not in spanned by the previous set. It is iterated this argument. The process ends when a complete arc is obtained. Now, we give an algorithm (implemented *C#*) to find these complete arcs.

Steps of algorithm
$A \leftarrow$Read(Excell File)
$B \leftarrow$Read(Text File)
$C \leftarrow A$
while$s(C) > 0$
$B_i \leftarrow$input(b) , $\{b \mid b \in C, b \notin B, i - s(B) + 1\}$
$j = 1$
while $j \leq s(B)$
for $k = (j+1)$ to $s(B)$
$m \leftarrow$the index of row on B_j, B_k
$D \leftarrow A_{mn}\{A_{mn} \mid A_{mn} \neq B_j, A_{mn} \neq B_k, n = 1,...,10\}$
remove a from A; $\{a \mid a \in A, a \in D\}$
$C \leftarrow c$; $\{c \mid c \in A, c \notin C\}$
end for
$j = j + 1$
end while
end while.

4.2 List of the complete arcs containing the quadrangles constructing the Fano planes

It is determined the complete $(k,2)-$ arcs containing the quadrangles constructing the Fano subplanes in P_2S by using a computer programme in [7].
$(6,2)$- arc is $\{11,6,11,21,38,58\}$
$(7,2)$- arcs are

$$\{1,4,11,21,35,38,88\},\{1,4,11,21,37,38,73\},\{1,4,11,21,37,38,77\}$$
$$\{1,4,11,21,38,53,84\},\{1,4,11,21,38,60,66\},\{1,4,11,21,38,66,88\}$$
$$\{1,4,11,21,38,72,84\},\{1,5,11,21,38,53,89\},\{1,5,11,21,38,54,87\}$$
$$\{1,5,11,21,38,57,70\},\{1,5,11,21,38,57,72\},\{1,6,11,21,38,57,68\}$$
$$\{1,6,11,21,38,57,73\},\{1,8,11,21,30,53,80\},\{1,11,21,33,38,53,64\}$$
$$\{1,8,11,21,38,57,72\},\{1,9,11,21,30,38,70\},\{1,9,11,21,30,38,78\}$$
$$\{1,9,11,21,36,38,66\},\{1,9,11,21,36,38,87\},\{1,9,11,21,38,55,86\}$$
$$\{1,9,11,21,38,62,66\},\{1,9,11,21,38,68,86\},\{1,10,11,21,30,38,73\}$$
$$\{1,10,11,21,30,38,90\},\{1,10,11,21,34,38,84\},\{1,10,11,21,38,57,68\}$$
$$\{1,10,11,21,38,57,73\},\{1,10,11,21,38,59,68\},\{1,10,11,21,38,64,73\}$$
$$\{1,10,11,21,38,68,84\},\{1,11,21,30,38,49,73\},\{1,11,21,30,38,58,70\}$$
$$\{1,11,21,33,38,53,70\},\{1,11,21,33,38,54,68\},\{1,11,21,33,38,64,79\}$$
$$\{1,11,21,33,38,67,89\},\{1,11,21,33,38,79,86\},\{1,11,21,34,38,49,60\}$$
$$\{1,11,21,34,38,49,86\},\{1,11,21,34,38,60,68\},\{1,11,21,34,38,72,78\}$$
$$\{1,11,21,34,38,62,73\},\{1,11,21,34,38,82,86\},\{1,11,21,35,38,49,67\}$$
$$\{1,11,21,35,38,49,72\},\{1,11,21,35,38,55,76\},\{1,11,21,35,38,59,68\}$$
$$\{1,11,21,35,38,59,72\},\{1,11,21,35,38,60,86\},\{1,11,21,36,38,54,59\}$$
$$\{1,11,21,36,38,55,57\},\{1,11,21,36,38,55,76\},\{1,11,21,36,38,55,88\}$$

$$\{1,11,21,36,38,59,82\},\{1,11,21,36,38,60,77\},\{1,11,21,36,38,77,88\}$$
$$\{1,11,21,37,38,50,90\},\{1,11,21,37,38,54,57\},\{1,11,21,37,38,54,89\}$$
$$\{1,11,21,37,38,55,62\},\{1,11,21,37,38,62,76\},\{1,11,21,37,38,62,87\}$$
$$\{1,11,21,37,38,76,90\},\{1,11,21,38,49,58,72\},\{1,11,21,38,49,60,86\}$$
$$\{1,11,21,38,49,62,66\},\{1,11,21,38,49,62,73\},\{1,11,21,38,49,72,78\}$$
$$\{1,11,21,38,49,73,79\},\{1,11,21,38,49,77,87\},\{1,11,21,38,49,77,88\}$$
$$\{1,11,21,38,50,64,88\},\{1,11,21,38,53,64,70\},\{1,11,21,38,53,70,76\}$$
$$\{1,11,21,38,53,70,77\},\{1,11,21,38,53,70,90\},\{1,11,21,38,54,59,89\}$$
$$\{1,11,21,38,54,59,79\},\{1,11,21,38,54,64,79\},\{1,11,21,38,54,68,76\}$$
$$\{1,11,21,38,55,59,72\},\{1,11,21,38,55,62,88\},\{1,11,21,38,55,72,78\}$$
$$\{1,11,21,38,57,77,88\},\{1,11,21,38,57,79,86\},\{1,11,21,38,58,70,76\}$$
$$\{1,11,21,38,58,72,84\},\{1,11,21,38,59,72,78\},\{1,11,21,38,59,78,89\}$$
$$\{1,11,21,38,60,66,76\},\{1,11,21,38,60,68,76\},\{1,11,21,38,60,68,90\}$$
$$\{1,11,21,38,60,77,86\},\{1,11,21,38,62,73,90\},\{1,11,21,38,62,76,90\}$$
$$\{1,11,21,38,64,73,87\},\{1,11,21,38,72,78,89\},\{1,11,21,38,73,79,89\}$$
$$\{1,11,21,38,64,70,77\},\{1,11,21,38,64,79,88\},\{1,11,21,38,64,70,87\}$$
$$\{1,11,21,38,64,70,77\},\{1,11,21,38,64,79,88\},\{1,11,21,38,64,70,87\}$$
$$\{1,11,21,38,67,76,87\},\{1,11,21,38,68,76,84\},\{1,11,21,38,70,76,87\}$$

$(8,2)$- arcs are

$$\{1,4,11,21,34,38,48,64\},\{1,4,11,21,34,38,60,73\},\{1,4,11,21,34,38,64,73\}$$
$$\{1,4,11,21,34,38,64,84\},\{1,4,11,21,34,38,75,88\},\{1,4,11,21,35,38,48,60\}$$
$$\{1,4,11,21,35,38,53,76\},\{1,4,11,21,35,38,72,89\},\{1,4,11,21,36,38,48,59\}$$
$$\{1,4,11,21,36,38,66,77\},\{1,4,11,21,36,38,59,72\},\{1,4,11,21,36,38,72,88\}$$
$$\{1,4,11,21,36,38,59,77\},\{1,4,11,21,37,38,64,67\},\{1,4,11,21,37,38,64,75\}$$
$$\{1,4,11,21,37,38,67,76\},\{1,4,11,21,37,38,67,89\},\{1,4,11,21,38,48,67,76\}$$
$$\{1,4,11,21,38,50,59,72\},\{1,4,11,21,38,50,60,77\},\{1,4,11,21,38,50,64,73\}$$
$$\{1,4,11,21,38,50,64,75\},\{1,4,11,21,38,50,64,84\},\{1,4,11,21,38,50,72,75\}$$
$$\{1,4,11,21,38,50,72,88\},\{1,4,11,21,38,53,67,76\},\{1,4,11,21,38,53,67,89\}$$
$$\{1,4,11,21,38,58,66,76\},\{1,4,11,21,38,58,66,77\},\{1,4,11,21,38,58,67,84\}$$
$$\{1,4,11,21,38,58,67,89\},\{1,4,11,21,38,58,77,89\},\{1,4,11,21,38,59,77,89\}$$
$$\{1,4,11,21,38,64,67,84\},\{1,5,11,21,38,48,70,87\},\{1,6,11,21,33,38,50,78\}$$
$$\{1,4,11,21,38,66,77,89\},\{1,4,11,21,38,72,75,88\},\{1,5,11,21,30,38,49,58\}$$
$$\{1,5,11,21,30,38,49,60\},\{1,5,11,21,30,38,49,77\},\{1,5,11,21,30,38,54,58\}$$
$$\{1,5,11,21,30,38,54,89\},\{1,5,11,21,30,38,58,89\},\{1,5,11,21,30,38,60,77\}$$
$$\{1,5,11,21,34,38,49,58\},\{1,5,11,21,34,38,54,58\},\{1,5,11,21,34,38,54,72\}$$
$$\{1,5,11,21,34,38,58,70\},\{1,5,11,21,34,38,68,75\},\{1,5,11,21,34,38,72,82\}$$
$$\{1,5,11,21,34,38,72,84\},\{1,5,11,21,37,38,54,75\},\{1,5,11,21,37,38,55,57\}$$
$$\{1,5,11,21,37,38,55,86\},\{1,5,11,21,37,38,57,87\},\{1,5,11,21,37,38,82,86\}$$
$$\{1,5,11,21,38,48,82,89\},\{1,5,11,21,38,49,58,77\},\{1,5,11,21,38,49,77,86\}$$
$$\{1,5,11,21,38,53,57,77\},\{1,5,11,21,38,53,68,77\},\{1,5,11,21,38,53,68,84\}$$
$$\{1,5,11,21,38,54,58,72\},\{1,5,11,21,38,54,72,89,\{1,5,11,21,38,55,57,86\}$$
$$\{1,5,11,21,38,55,72,82\},\{1,5,11,21,38,55,82,86\},\{1,5,11,21,38,57,77,86\}$$
$$\{1,5,11,21,38,58,70,77\},\{1,5,11,21,38,58,72,89\},\{1,5,11,21,38,60,68,77\}$$
$$\{1,5,11,21,38,60,68,84\},\{1,5,11,21,38,60,77,87\},\{1,5,11,21,38,68,82,86\}$$

$\{1,5,11,21,38,72,82,84\},\{1,6,11,21,33,38,50,60\},\{1,6,11,21,33,38,50,68\}$
$\{1,6,11,21,33,38,53,68\},\{1,6,11,21,33,38,75,87\},\{1,6,11,21,35,38,48,79\}$
$\{1,6,11,21,35,38,48,89\},\{1,6,11,21,35,38,53,88\},\{1,6,11,21,35,38,57,79\}$
$\{1,6,11,21,35,38,62,75\},\{1,6,11,21,35,38,62,88\},\{1,6,11,21,35,38,79,88\}$
$\{1,6,11,21,36,38,54,57\},\{1,6,11,21,36,38,54,87\},\{1,6,11,21,36,38,57,88\}$
$\{1,6,11,21,36,38,60,66\},\{1,6,11,21,36,38,60,78\},\{1,6,11,21,36,38,60,87\}$
$\{1,6,11,21,36,38,78,87\},\{1,6,11,21,38,48,60,73\},\{1,6,11,21,38,48,60,87\}$
$\{1,6,11,21,38,48,73,79\},\{1,6,11,21,38,50,60,66\},\{1,6,11,21,38,50,60,78\}$
$\{1,6,11,21,38,50,73,78\},\{1,8,11,21,33,38,58,70\},\{1,8,11,21,38,55,62,84\}$
$\{1,6,11,21,38,53,62,68\},\{1,6,11,21,38,53,62,82\},\{1,6,11,21,38,53,62,88\}$
$\{1,6,11,21,38,53,68,82\},\{1,6,11,21,38,53,82,89\},\{1,6,11,21,38,54,57,87\}$
$\{1,6,11,21,38,54,68,75\},\{1,6,11,21,38,54,75,87\},\{1,6,11,21,38,57,79,88\}$
$\{1,6,11,21,38,60,73,78\},\{1,6,11,21,38,62,66,82\},\{1,6,11,21,38,62,66,88\}$
$\{1,6,11,21,38,62,68,75\},\{1,6,11,21,38,62,82,89\},\{1,6,11,21,38,73,78,89\}$
$\{1,6,11,21,38,73,82,89\},\{1,8,11,21,30,38,50,70\},\{1,8,11,21,30,38,62,73\}$
$\{1,8,11,21,30,38,70,90\},\{1,8,11,21,30,38,73,78\},\{1,8,11,21,33,38,50,75\}$
$\{1,8,11,21,33,38,58,84\},\{1,8,11,21,33,38,70,79\},\{1,8,11,21,33,38,79,84\}$
$\{1,8,11,21,35,38,55,62\},\{1,8,11,21,35,38,57,86\},\{1,8,11,21,35,38,72,75\}$
$\{1,8,11,21,37,38,50,73\},\{1,8,11,21,37,38,50,77\},\{1,8,11,21,37,38,50,86\}$
$\{1,8,11,21,37,38,55,57\},\{1,8,11,21,37,38,55,76\},\{1,8,11,21,37,38,57,77\}$
$\{1,8,11,21,37,38,77,86\},\{1,8,11,21,38,48,73,90\},\{1,8,11,21,38,48,79,86\}$
$\{1,8,11,21,38,50,70,77\},\{1,8,11,21,38,50,70,90\},\{1,8,11,21,38,55,57,78\}$
$\{1,8,11,21,38,55,58,72\},\{1,8,11,21,38,55,58,76\},\{1,8,11,21,38,55,62,73\}$
$\{1,8,11,21,38,55,73,78\},\{1,8,11,21,38,55,76,84\},\{1,8,11,21,38,57,77,86\}$
$\{1,8,11,21,38,58,70,79\},\{1,8,11,21,38,58,72,90\},\{1,8,11,21,38,58,76,84\}$
$\{1,8,11,21,38,58,76,90\},\{1,8,11,21,38,58,79,84\},\{1,8,11,21,38,58,79,90\}$
$\{1,8,11,21,38,62,76,84\},\{1,8,11,21,38,70,79,90\},\{1,9,11,21,30,38,58,67\}$
$\{1,9,11,21,30,38,59,67\},\{1,9,11,21,30,38,59,82\},\{1,9,11,21,30,38,59,90\}$
$\{1,9,11,21,33,38,55,58\},\{1,9,11,21,33,38,55,64\},\{1,9,11,21,33,38,55,66\}$
$\{1,9,11,21,33,38,58,67\},\{1,9,11,21,33,38,67,78\},\{1,9,11,21,33,38,67,86\}$
$\{1,9,11,21,33,38,70,87\},\{1,9,11,21,36,38,48,70\},\{1,10,11,21,37,38,73,90\}$
$\{1,9,11,21,36,38,48,90\},\{1,9,11,21,36,38,70,82\},\{1,9,11,21,36,38,70,88\}$
$\{1,9,11,21,38,49,58,66\},\{1,9,11,21,38,49,67,75\},\{1,9,11,21,38,49,67,78\}$
$\{1,9,11,21,38,49,67,86\},\{1,9,11,21,38,49,78,87\},\{1,9,11,21,38,55,58,66\}$
$\{1,9,11,21,38,55,62,82\},\{1,9,11,21,38,55,64,88\},\{1,9,11,21,38,55,66,88\}$
$\{1,9,11,21,38,59,67,86\},\{1,9,11,21,38,59,68,82\},\{1,9,11,21,38,59,68,90\}$
$\{1,9,11,21,38,59,75,90\},\{1,9,11,21,38,62,68,90\},\{1,9,11,21,38,62,75,87\}$
$\{1,9,11,21,38,48,59,82\},\{1,9,11,21,38,48,59,86\},\{1,9,11,21,38,48,70,82\}$
$\{1,9,11,21,38,64,70,82\},\{1,9,11,21,38,64,70,88\},\{1,9,11,21,38,64,75,88\}$
$\{1,10,11,21,30,38,53,88\},\{1,10,11,21,34,38,54,68\},\{1,10,11,21,34,38,54,79\}$
$\{1,10,11,21,34,38,64,88\},\{1,10,11,21,34,38,68,86\},\{1,10,11,21,34,38,79,86\}$
$\{1,10,11,21,36,38,54,57\},\{1,10,11,21,36,38,54,76\},\{1,10,11,21,36,38,57,78\}$
$\{1,10,11,21,36,38,59,90\},\{1,10,11,21,36,38,78,84\},\{1,10,11,21,37,38,54,64\}$
$\{1,10,11,21,37,38,54,66\},\{1,10,11,21,37,38,54,90\},\{1,10,11,21,37,38,66,76\}$
$\{1,10,11,21,30,38,49,67\},\{1,10,11,21,30,38,53,67\},\{1,10,11,21,30,38,53,78\}$
$\{1,10,11,21,38,49,66,76\},\{1,10,11,21,38,49,66,79\},\{1,10,11,21,38,49,66,86\}$

$$\{1,10,11,21,38,49,79,88\},\{1,10,11,21,38,53,57,88\},\{1,10,11,21,38,53,64,67\}$$
$$\{1,10,11,21,38,53,67,78\},\{1,10,11,21,38,53,78,84\},\{1,10,11,21,38,54,57,79\}$$
$$\{1,10,11,21,38,54,66,76\},\{1,10,11,21,38,54,79,90\},\{1,10,11,21,38,57,79,88\}$$
$$\{1,10,11,21,38,59,67,78\},\{1,10,11,21,38,59,79,86\},\{1,10,11,21,38,66,79,86\}$$
$$\{1,10,11,21,38,67,76,84\},\{1,10,11,21,38,67,78,84\},\{1,10,11,21,38,73,79,90\}$$
$$\{1,11,21,30,38,49,60,78\},\{1,11,21,30,38,50,59,78\},\{1,11,21,30,38,53,62,90\}$$
$$\{1,11,21,30,38,53,77,89\},\{1,11,21,35,38,49,62,75\},\{1,11,21,36,38,48,70,90\}$$
$$\{1,11,21,30,38,54,58,90\},\{1,11,21,30,38,59,77,89\},\{1,11,21,33,38,53,78,89\}$$
$$\{1,11,21,33,38,54,64,87\},\{1,11,21,33,38,54,75,87\},\{1,11,21,33,38,55,64,84\}$$
$$\{1,11,21,33,38,55,78,84\},\{1,11,21,33,38,60,68,86\},\{1,11,21,33,38,64,67,75\}$$
$$\{1,11,21,33,38,64,75,87\},\{1,11,21,33,38,67,75,87\},\{1,11,21,34,38,48,73,82\}$$
$$\{1,11,21,34,38,49,58,79\},\{1,11,21,34,38,49,62,75\},\{1,11,21,34,38,49,72,88\}$$
$$\{1,11,21,34,38,49,75,88\},\{1,11,21,34,38,58,79,84\},\{1,11,21,34,38,60,78,84\}$$
$$\{1,11,21,34,38,72,75,88\},\{1,11,21,34,38,62,75,88\},\{1,11,21,35,38,49,60,76\}$$
$$\{1,11,21,35,38,49,62,76\},\{1,11,21,35,38,49,75,88\},\{1,11,21,35,38,53,57,67\}$$
$$\{1,11,21,35,38,53,57,68\},\{1,11,21,35,38,53,57,70\},\{1,11,21,35,38,53,67,76\}$$
$$\{1,11,21,35,38,53,70,88\},\{1,11,21,35,38,55,59,86\},\{1,11,21,35,38,57,68,86\}$$
$$\{1,11,21,35,38,57,70,79\},\{1,11,21,35,38,48,59,86\},\{1,11,21,35,38,48,59,89\}$$
$$\{1,11,21,35,38,59,79,89\},\{1,11,21,35,38,62,68,75\},\{1,11,21,35,38,62,68,76\}$$
$$\{1,11,21,35,38,62,75,88\},\{1,11,21,35,38,48,67,86\},\{1,11,21,35,38,48,67,89\}$$
$$\{1,11,21,35,38,48,79,89\},\{1,11,21,36,38,48,60,87\},\{1,11,21,36,38,48,60,90\}$$
$$\{1,11,21,36,38,48,76,87\},\{1,11,21,36,38,48,76,90\},\{1,11,21,36,38,54,66,76\}$$
$$\{1,11,21,36,38,55,66,82\},\{1,11,21,36,38,57,72,78\},\{1,11,21,36,38,70,76,90\}$$
$$\{1,11,21,36,38,72,78,87\},\{1,11,21,37,38,50,64,75\},\{1,11,21,37,38,50,64,77\}$$
$$\{1,11,21,37,38,55,66,86\},\{1,11,21,37,38,57,73,87\},\{1,11,21,37,38,62,73,89\}$$
$$\{1,11,21,37,38,64,67,75\},\{1,11,21,37,38,64,75,87\},\{1,11,21,37,38,64,77,87\}$$
$$\{1,11,21,37,38,66,77,86\},\{1,11,21,37,38,67,75,87\},\{1,11,21,38,48,59,67,86\}$$
$$\{1,11,21,38,48,59,67,89\},\{1,11,21,38,48,59,82,86\},\{1,11,21,38,48,60,73,87\}$$
$$\{1,11,21,38,48,60,76,87\},\{1,11,21,38,57,67,78,87\},\{1,11,21,38,57,72,78,87\}$$
$$\{1,11,21,38,48,60,76,90\},\{1,11,21,38,48,64,70,79\},\{1,11,21,38,49,60,73,87\}$$
$$\{1,11,21,38,49,67,78,87\},\{1,11,21,38,50,66,77,88\},\{1,11,21,38,50,68,75,90\}$$
$$\{1,11,21,38,53,57,68,77\},\{1,11,21,38,53,57,70,88\},\{1,11,21,38,53,62,76,84\}$$
$$\{1,11,21,38,53,64,82,84\},\{1,11,21,38,54,57,72,87\},\{1,11,21,38,54,58,76,90\}$$
$$\{1,11,21,38,54,59,72,75\},\{1,11,21,38,55,57,73,78\},\{1,11,21,38,55,59,82,86\}$$
$$\{1,11,21,38,57,73,78,87\}$$

$(9,2)$- arcs are

$$\{1,11,21,30,38,50,70,77,88\},\{1,11,21,30,38,62,73,82,89\},\{1,11,21,33,38,50,60,66,86\},$$
$$\{1,11,21,33,38,50,60,68,84\},\{1,11,21,34,38,48,64,70,82\},\{1,11,21,34,38,48,64,73,79\},$$
$$\{1,11,21,36,38,55,72,82,84\},\{1,11,21,37,38,62,66,82,89\},\{1,11,21,38,50,59,72,75,90\},$$
$$\{1,11,21,38,54,59,68,75,90\},\{1,3,11,21,36,38,55,72,84\}.$$

5 Conclusion

We had described procedure for searching all complete $(k,2)-$ arcs with $6 \leq k < 10$ containing the quadrangles O, I, X, $(a,b,1)$, with $a \in F_3$, $b \in S \backslash F_3$ constructing the Fano subplanes in the left nearfield plane of order 9. By applying a method and an algorithm (implemented *C#*) a full listing the complete $(k,2)-$arcs were determined and classified . These can be represented by four groups. There are 1 classe of the complete $(6,2)-$ arcs, 108 classes of the complete $(7,2)-$ arcs, 319 classes of the complete $(8,2)-$arcs, 11 classes of the complete $(9,2)-$ arcs.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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