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Some Inequalities bounding certain ratios of the (p,k)-Gamma function

Kwara Nantomah

Department of Mathematics, Faculty of Mathematical Sciences, University for Development Studies, Navrongo Campus, P. O. Box 24, Navrongo, UE/R, Ghana.

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Abstract: In this paper, we establish some inequalities bounding the ratio $\Gamma_{p,k}(x)/\Gamma_{p,k}(y)$, where $\Gamma_{p,k}(.)$ is the (p,k)-analogue of the Gamma function. Consequently, some previous results are recovered from the obtained results.

Keywords: Gamma function, Polygamma function, (p,k)-analogue, inequality

1 Introduction

Inequalities that provide bounds for the ratio $\Gamma(x)/\Gamma(y)$, where x and y are numbers of some special form, have been studied intensively by several researchers across the globe. A detailed account on inequalities of this nature can be found in the survey article by Qi [10]. In this study, the focus shall be on the type originating from certain problems of traffic flow.

In 1978, Lew, Frauenthal and Keyfitz [5] by studying certain problems of traffic flow established the double-inequality

$$2\Gamma\left(n+\frac{1}{2}\right) \leq \Gamma\left(\frac{1}{2}\right)\Gamma(n+1) \leq 2^n\Gamma\left(n+\frac{1}{2}\right), \quad n\in\mathbb{N} \tag{1}$$

which can be rearranged as

$$\frac{2}{\sqrt{\pi}} \leq \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \leq \frac{2^n}{\sqrt{\pi}}, \quad n\in\mathbb{N}. \tag{2}$$

Then in 2006, Sándor [11] by using the inequality

$$\left(\frac{x}{x+s}\right)^{1-s} \leq \frac{\Gamma(x+s)}{x^s\Gamma(x)} \leq 1, \quad s\in(0,1), x>0 \tag{3}$$

due Wendel [12], extended and refined the inequality (2) by proving the result

$$\sqrt{x} \leq \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \leq \sqrt{x+\frac{1}{2}} \tag{4}$$

for $x>0$.

* Corresponding author e-mail: mykwarasoft@yahoo.com
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Also, in the paper [6], the authors established the q-analogue of (4) as

$$\sqrt{[x]_q} \leq \frac{\Gamma_q(x+1)}{\Gamma_q\left(x+\frac{1}{2}\right)} \leq \sqrt{\left[x+\frac{1}{2}\right]_q}$$

for $q \in (0,1)$ and $x > 0$.

Furthermore, in the paper [7], the authors established the (q,k)-analogue of (4) as

$$[x]_q^{1-\frac{1}{2k}} \leq \frac{\Gamma_{q,k}(x+k)}{\Gamma_{q,k}\left(x+\frac{1}{2}\right)} \leq \left[x+\frac{1}{2}\right]_q^{1-\frac{1}{2k}}$$

for $q \in (0,1)$, $k > 0$ and $x > 0$.

The main objective of this paper is to establish similar inequalities for the (p,k)-analogue of the Gamma function.

2 Preliminaries

The classical Euler's Gamma function, $\Gamma(x)$ is usually defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)(x+2)\dots(x+n)}$$

Closely related to the Gamma function is the Digamma function, $\psi(x)$ which is defined for $x > 0$ as $\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$.

Euler gave another definition of the Gamma function called the p-analogue, which is defined for $p \in \mathbb{N}$ and $x > 0$ as (see [1, p. 270])

$$\Gamma_p(x) = \frac{p!p^x}{x(x+1)\dots(x+p)}$$

with the p-analogue of the Digamma function defined as $\psi_p(x) = \frac{d}{dx}\ln\Gamma_p(x)$.

Also, Díaz and Pariguan [2] defined the k-analogues of the Gamma and Digamma functions as

$$\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}\,dt \quad \text{and} \quad \psi_k(x) = \frac{d}{dx}\ln\Gamma_k(x)$$

for $k > 0$ and $x \in \mathbb{C}\backslash k\mathbb{Z}^-$.

Then in a recent paper [8], the authors introduced a (p,k)-analogue of the Gamma function defined for $p \in \mathbb{N}$, $k > 0$ and $x \in \mathbb{R}^+$ as

$$\begin{aligned}\Gamma_{p,k}(x) &= \int_0^p t^{x-1}\left(1-\frac{t^k}{pk}\right)^p dt \\ &= \frac{(p+1)!k^{p+1}(pk)^{\frac{x}{k}-1}}{x(x+k)(x+2k)\dots(x+pk)}\end{aligned}$$

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satisfying the properties

$$\begin{aligned} \Gamma_{p,k}(x+k) &= \frac{pkx}{x+pk+k}\Gamma_{p,k}(x) \\ \Gamma_{p,k}(ak) &= \frac{p+1}{p}k^{a-1}\Gamma_p(a), \quad a \in \mathbb{R}^+ \\ \Gamma_{p,k}(k) &= 1. \end{aligned} \tag{5}$$

The (p,k)-analogue of the Digamma function is defined for $x > 0$ as

$$\begin{aligned} \psi_{p,k}(x) = \frac{d}{dx}\ln\Gamma_{p,k}(x) &= \frac{1}{k}\ln(pk) - \sum_{n=0}^{p}\frac{1}{nk+x} \\ &= \frac{1}{k}\ln(pk) - \int_0^\infty \frac{1-e^{-k(p+1)t}}{1-e^{-kt}}e^{-xt}\,dt \end{aligned}$$

Also, the (p,k)-analogue of the Polygamma functions are defined as

$$\begin{aligned} \psi_{p,k}^{(m)}(x) = \frac{d^m}{dx^m}\psi_{p,k}(x) &= \sum_{n=0}^{p}\frac{(-1)^{m+1}m!}{(nk+x)^{m+1}} \\ &= (-1)^{m+1}\int_0^\infty \left(\frac{1-e^{-k(p+1)t}}{1-e^{-kt}}\right)t^m e^{-xt}\,dt \end{aligned}$$

where $m \in \mathbb{N}$, and $\psi_{p,k}^{(0)}(x) \equiv \psi_{p,k}(x)$.

The functions $\Gamma_{p,k}(x)$ and $\psi_{p,k}(x)$ satisfy the following commutative diagrams.

$$\begin{array}{ccc} \Gamma_{p,k}(x) & \xrightarrow{p\to\infty} & \Gamma_k(x) \\ {\scriptstyle k\to1}\downarrow & & \downarrow{\scriptstyle k\to1} \\ \Gamma_p(x) & \xrightarrow[p\to\infty]{} & \Gamma(x) \end{array} \qquad \begin{array}{ccc} \psi_{p,k}(x) & \xrightarrow{p\to\infty} & \psi_k(x) \\ {\scriptstyle k\to1}\downarrow & & \downarrow{\scriptstyle k\to1} \\ \psi_p(x) & \xrightarrow[p\to\infty]{} & \psi(x) \end{array}$$

We now present the main findings of the paper in the following section.

3 Main results

Lemma 1. *Let $p \in \mathbb{N}$, $k > 0$ and $s \in (0,1)$. Then the inequality*

$$\frac{\left(\frac{pkx}{x+pk+k}\right)^{1-s}}{\left(\frac{pk(x+sk)}{x+sk+pk+k}\right)^{1-s}} \le \frac{\Gamma_{p,k}(x+sk)}{\left(\frac{pkx}{x+pk+k}\right)^{s}\Gamma_{p,k}(x)} \le 1 \tag{6}$$

holds for $x > 0$.

Proof. We employ the Hölder's inequality for integrals, which is stated for any integrable functions $f,g : (0,a) \to \mathbb{R}$ as

$$\int_0^a |f(t)g(t)|\,dt \le \left[\int_0^a |f(t)|^\alpha\,dt\right]^{\frac{1}{\alpha}}\left[\int_0^a |g(t)|^\beta\,dt\right]^{\frac{1}{\beta}}$$

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where $\alpha > 1$ such that $\frac{1}{\alpha}+\frac{1}{\beta}=1$. We proceed as follows. Let $\alpha=\frac{1}{1-s}$, $\beta=\frac{1}{s}$, $f(t)=t^{(1-s)(x-1)}\left(1-\frac{t^k}{pk}\right)^{p(1-s)}$, $g(t)=t^{s(x+k-1)}\left(1-\frac{t^k}{pk}\right)^{ps}$. Then,

$$\begin{aligned}
\Gamma_{p,k}(x+sk) &= \int_0^p t^{x+sk-1}\left(1-\frac{t^k}{pk}\right)^p dt \\
&= \int_0^p t^{(1-s)(x-1)}\left(1-\frac{t^k}{pk}\right)^{p(1-s)} .t^{s(x+k-1)}\left(1-\frac{t^k}{pk}\right)^{ps} dt \\
&\leq \left[\int_0^p \left(t^{(1-s)(x-1)}\left(1-\frac{t^k}{pk}\right)^{p(1-s)}\right)^{\frac{1}{1-s}} dt\right]^{1-s} \times \\
&\quad \left[\int_0^p \left(t^{s(x+k-1)}\left(1-\frac{t^k}{pk}\right)^{ps}\right)^{\frac{1}{s}} dt\right]^{s} \\
&= \left[\int_0^p t^{x-1}\left(1-\frac{t^k}{pk}\right)^p dt\right]^{1-s}\left[\int_0^p t^{x+k-1}\left(1-\frac{t^k}{pk}\right)^p dt\right]^{s} \\
&= \left[\Gamma_{p,k}(x)\right]^{1-s}\left[\Gamma_{p,k}(x+k)\right]^{s}.
\end{aligned}$$

That is,

$$\Gamma_{p,k}(x+sk) \leq \left[\Gamma_{p,k}(x)\right]^{1-s}\left[\Gamma_{p,k}(x+k)\right]^{s}. \tag{7}$$

Substituting (5) into inequality (7) yields;

$$\Gamma_{p,k}(x+sk) \leq \left(\frac{pkx}{x+pk+k}\right)^{s}\Gamma_{p,k}(x). \tag{8}$$

Replacing s by $1-s$ in inequality (8) gives

$$\Gamma_{p,k}(x+k-sk) \leq \left(\frac{pkx}{x+pk+k}\right)^{1-s}\Gamma_{p,k}(x). \tag{9}$$

Further, upon substituting for x by $x+sk$, we obtain

$$\Gamma_{p,k}(x+k) \leq \left(\frac{pk(x+sk)}{x+sk+pk+k}\right)^{1-s}\Gamma_{p,k}(x+sk). \tag{10}$$

Now combining (8) and (10) gives

$$\frac{\Gamma_{p,k}(x+k)}{\left(\frac{pk(x+sk)}{x+sk+pk+k}\right)^{1-s}} \leq \Gamma_{p,k}(x+sk) \leq \left(\frac{pkx}{x+pk+k}\right)^{s}\Gamma_{p,k}(x)$$

which by (5) can be written as

$$\frac{\left(\frac{pkx}{x+pk+k}\right)}{\left(\frac{pk(x+sk)}{x+sk+pk+k}\right)^{1-s}}\Gamma_{p,k}(x) \leq \Gamma_{p,k}(x+sk) \leq \left(\frac{pkx}{x+pk+k}\right)^{s}\Gamma_{p,k}(x). \tag{11}$$

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Finally, (11) can be rearranged as

$$\frac{\left(\frac{pkx}{x+pk+k}\right)^{1-s}}{\left(\frac{pk(x+sk)}{x+sk+pk+k}\right)^{1-s}} \leq \frac{\Gamma_{p,k}(x+sk)}{\left(\frac{pkx}{x+pk+k}\right)^{s}\Gamma_{p,k}(x)} \leq 1$$

concluding the proof.

Theorem 1. *Let $p\in\mathbb{N}$, $k>0$ and $s\in(0,1)$. Then the inequality*

$$\left(\frac{pkx}{x+pk+k}\right)^{1-s} \leq \frac{\Gamma_{p,k}(x+k)}{\Gamma_{p,k}(x+sk)} \leq \left(\frac{pk(x+sk)}{x+sk+pk+k}\right)^{1-s} \tag{12}$$

holds for $x>0$.

Proof. The inequality (6) implies

$$\frac{\left(\frac{pkx}{x+pk+k}\right)}{\left(\frac{pk(x+sk)}{x+sk+pk+k}\right)^{1-s}} \leq \frac{\Gamma_{p,k}(x+sk)}{\Gamma_{p,k}(x)} \leq \left(\frac{pkx}{x+pk+k}\right)^{s}$$

which by inversion yields

$$\left(\frac{pkx}{x+pk+k}\right)^{-s} \leq \frac{\Gamma_{p,k}(x)}{\Gamma_{p,k}(x+sk)} \leq \frac{\left(\frac{pk(x+sk)}{x+sk+pk+k}\right)^{1-s}}{\left(\frac{pkx}{x+pk+k}\right)}. \tag{13}$$

Then, substituting the identity (5) into (13) completes the proof.

Remark. Let $k=1$ and $p\to\infty$ in (12). Then, we obtain

$$x^{1-s} \leq \frac{\Gamma(x+1)}{\Gamma(x+s)} \leq (x+s)^{1-s} \tag{14}$$

which is an improvement of the Gautschi's inequality [3, eqn. (7)].

Corollary 1. *Let $p\in\mathbb{N}$ and $k>0$. Then the inequality*

$$\left(\frac{pkx}{x+pk+k}\right)^{1-\frac{1}{2k}} \leq \frac{\Gamma_{p,k}(x+k)}{\Gamma_{p,k}(x+\frac{1}{2})} \leq \left(\frac{pk(x+\frac{1}{2})}{x+pk+k+\frac{1}{2}}\right)^{1-\frac{1}{2k}} \tag{15}$$

holds for $x>0$.

Proof. This follows from Theorem 1 by letting $s=\frac{1}{2k}$.

Remark. As a consequence of inequality (6), we obtain

$$\lim_{x\to\infty} \frac{\Gamma_{p,k}(x+sk)}{\left(\frac{pkx}{x+pk+k}\right)^{s}\Gamma_{p,k}(x)} = 1, \quad s\in(0,1). \tag{16}$$

Remark. Let $\alpha,\beta\in(0,1)$. Then by (16), we obtain

$$\lim_{x\to\infty} \left(\frac{pkx}{x+pk+k}\right)^{\beta-\alpha} \frac{\Gamma_{p,k}(x+\alpha k)}{\Gamma_{p,k}(x+\beta k)} = 1. \tag{17}$$

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Remark. We note that the limits (16) and (17) are the (p,k)-analogues of the classical Wendel's asymptotic relation given by [12]

$$\lim_{x\to\infty} \frac{\Gamma(x+s)}{x^s\Gamma(x)} = 1.$$

Remark. By letting $p \to \infty$ as $k \to 1$ in (6), we obtain (3).

Remark. By letting $p \to \infty$ in (15), we obtain

$$x^{1-\frac{1}{2k}} \leq \frac{\Gamma_k(x+k)}{\Gamma_k\left(x+\frac{1}{2}\right)} \leq \left(x+\frac{1}{2}\right)^{1-\frac{1}{2k}} \tag{18}$$

which gives a k-analogue of (4).

Remark. By letting $k \to 1$ in (15), we obtain

$$\sqrt{\left(\frac{px}{x+p+1}\right)} \leq \frac{\Gamma_p(x+1)}{\Gamma_p\left(x+\frac{1}{2}\right)} \leq \sqrt{\left(\frac{p(x+\frac{1}{2})}{x+p+\frac{3}{2}}\right)} \tag{19}$$

which gives a p-analogue of (4).

Remark. By letting $p \to \infty$ as $k \to 1$ in (15), we obtain (4).

Theorem 2. *Let $p \in \mathbb{N}$ and $k > 0$. Then, the inequality*

$$e^{(x-y)\psi_{p,k}(y)} < \frac{\Gamma_{p,k}(x)}{\Gamma_{p,k}(y)} < e^{(x-y)\psi_{p,k}(x)} \tag{20}$$

holds for $x > y > 0$.

Proof. Let H be defined for $p \in \mathbb{N}$, $k > 0$ and $t > 0$ by $H(t) = \ln \Gamma_{p,k}(t)$. Further, let (y,x) be fixed. Then, by the classical mean value theorem, there exists a $\lambda \in (y,x)$ such that

$$H'(\lambda) = \frac{\ln \Gamma_{p,k}(x) - \ln \Gamma_{p,k}(y)}{x-y} = \psi_{p,k}(\lambda).$$

Thus,

$$\psi_{p,k}(\lambda) = \frac{1}{x-y} \ln \frac{\Gamma_{p,k}(x)}{\Gamma_{p,k}(y)}.$$

Recall that $\psi_{p,k}(t)$ is increasing for $t > 0$ (see [8]). Then for $\lambda \in (y,x)$ we have

$$\psi_{p,k}(y) < \frac{1}{x-y} \ln \frac{\Gamma_{p,k}(x)}{\Gamma_{p,k}(y)} < \psi_{p,k}(x).$$

That is

$$(x-y)\psi_{p,k}(y) < \ln \frac{\Gamma_{p,k}(x)}{\Gamma_{p,k}(y)} < (x-y)\psi_{p,k}(x).$$

Then, by taking exponents, we obtain the result (20).

Corollary 2. *Let $p \in \mathbb{N}$ and $k > s > 0$. Then, the inequality*

$$e^{(k-s)\psi_{p,k}(x+s)} < \frac{\Gamma_{p,k}(x+k)}{\Gamma_{p,k}(x+s)} < e^{(k-s)\psi_{p,k}(x+k)} \tag{21}$$

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holds for $x > 0$.

Proof. This follows from Theorem 2 upon replacing x and y respectively by $x+k$ and $x+s$.

Remark. In particular, if $s = \frac{1}{2}$, then inequality (21) becomes

$$e^{(k-\frac{1}{2})\psi_{p,k}(x+\frac{1}{2})} < \frac{\Gamma_{p,k}(x+k)}{\Gamma_{p,k}(x+\frac{1}{2})} < e^{(k-\frac{1}{2})\psi_{p,k}(x+k)} \tag{22}$$

Remark. The inequality (20) provides a (p,k)-analogue of the result

$$e^{(x-y)\psi(y)} < \frac{\Gamma(x)}{\Gamma(y)} < e^{(x-y)\psi(x)} \tag{23}$$

for $x > y > 0$, which was established in [9, Corollary 2] .

Remark. Inequality (21) provides a generalization of [4, Theorem 3.1].

4 Conclusion

We have established some inequalities bounding the ratio $\Gamma_{p,k}(x)/\Gamma_{p,k}(y)$, where $\Gamma_{p,k}(.)$ is the (p,k)-analogue of the Gamma function. From the established results, we recover some known results in the literature.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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