Sakarya University Journal of Science

ISSN 1301-4048 | e-ISSN 2147-835X | Period Bimonthly | Founded: 1997 | Publisher Sakarya University | http://www.saujs.sakarya.edu.tr/

Title: Categories internal to crossed modules

Authors: Tunçar Şahan, Jihad Jamil Mohammed

Recieved: 2018-10-20 00:00:00

Accepted: 2018-12-20 00:00:00

Article Type: Research Article
Volume: 23
Issue: 4
Month: August
Year: 2019
Pages: 519-531

How to cite
Tunçar Şahan, Jihad Jamil Mohammed; (2019), Categories internal to crossed modules. Sakarya University Journal of Science, 23(4), 519-531, DOI: 10.16984/saufenbilder.472805
Access link
http://www.saujs.sakarya.edu.tr/issue/43328/472805

Sakarya University Journal of Science 23(4), 519-531, 2019

Categories internal to crossed modules

Tunçar Şahan[*1] and Jihad Jamil Mohammed [2]

Abstract

In this study, internal categories in the category of crossed modules are characterized and it has been shown that there is a natural equivalence between the category of crossed modules over crossed modules, i.e. crossed squares and the category of internal categories within the category of crossed modules. Finally, we obtain examples of crossed squares using this equivalence.

Keywords: Crossed module, internal category, crossed square

1. INTRODUCTION

Crossed modules are first defined in the works of Whitehead [25-27] and has been found important in many areas of mathematics including homotopy theory, group representation theory, homology and cohomology on groups, algebraic K-theory, cyclic homology, combinatorial group theory and differential geometry. See [4-7] for applications of crossed modules. Later, it was shown that the categories of internal categories in the category of groups and the category of crossed modules are equivalent [8-14].

Mucuk et al. [18] interpret the concept of normal subcrossed module and quotient crossed module concepts in the category of internal categories within groups, that is group-groupoids. The equivalences of the categories given in [8, Theorem 1] and [24, Section 3] enable us to generalize some results on group-groupoids to the more general internal groupoids for an arbitrary category of groups with operations (see for example [1], [15], [16] and [17]).

Gerstenhaber [11] and Lichtenbaum, Schlessinger [13] have defined the concept of a crossed module on associative and commutative algebras. In [2] the categories of crossed modules and of 2-crossed modules on commutative algebras are linked with an equivalence.

Crossed squares are first described to be applied to algebraic K-theoretic problems [12]. Crossed squares are two-dimensional analogous of crossed modules and model all connected homotopy 3-types (hence all 3-groups) and correspond in much the same way to pairs of normal subgroups while crossed modules model all connected homotopy 2-types and groups model all connected homotopy 1-types.

Recently, freeness conditions for 2-crossed modules and crossed squares are given in [19] and [20]. See also [3] for commutative algebra case.

Main objective of this study is to characterize internal categories within the category of crossed modules and to prove that the category of internal categories in the category of crossed modules and the category of crossed squares are equivalent. Hence this equivalence allow us to produce more examples of crossed squares.

[*] Corresponding Author

[1] Aksaray University, Department of Mathematics, 68100, Aksaray, Turkey, tuncarsahan@gmail.com

[2] Wan Primary School, Zakho, Iraq, cihad.cemil@gmail.com

2. PRELIMINARIES

In this section we recall some well-known basic definitions and resultss.

2.1. Extensions and crossed modules

Following are detailed descriptions of the ideas given in [24] for the case of groups. An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{\ i\ } E \xrightarrow{\ p\ } B \longrightarrow 0$$

is called *short exact sequence* where 0 is the group with one element. Here i is a monomorphism, p is an epimorphism and $\ker p = A$. In a short exact sequence, group E is called an extension of B by A. An extension is called *split* if there exist a group homomorphism $s : B \to E$ such that $ps = 1_B$.

Let E be a split extension of B by A. Then the function

$$\begin{array}{ccccc} \theta & : & E & \to & A \times B \\ & & e & \mapsto & \left(e - sp(e), p(e)\right) \end{array}$$

is a bijection. The inverse of θ is given by $\theta^{-1}(a,b) = a + s(b)$.

Thus it is possible to define a group structure on $A \times B$ such that θ is an isomorphism of groups. Let $(a,b),(a_1,b_1) \in A \times B$. Then

$$\begin{aligned} (a,b)+(a_1,b_1) &= \theta\left(\theta^{-1}\left((a,b)+(a_1,b_1)\right)\right) \\ &= \theta\left(\theta^{-1}(a,b)+\theta^{-1}(a_1,b_1)\right) \\ &= \theta\left(a + s(b) + a_1 + s(b_1)\right) \\ &= \left(a + s(b) + a_1 + s(b_1) - s(b_1) - s(b), b + b_1\right) \\ &= \left(a + \left(s(b) + a_1 - s(b)\right), b + b_1\right). \end{aligned}$$

Here we note that a split extension of B by A defines an (left) action of B on A with

$$b \cdot a = s(b) + a - s(b)$$

for $a \in A$ and $b \in B$. $A \times B$ is called the semi-direct product group of A and B with the operaton given above and denoted by Aã B.

These kind of actions are called derived actions [23]. Every group A has a split extension by itself in a natural way which gives rise to the conjugation action as

$$0 \longrightarrow A \xrightarrow{\ i\ } A\text{ã } A \underset{s}{\overset{p}{\rightleftarrows}} A \longrightarrow 0$$

where $i(a) = (a,0)$, $p(a,a_1) = a_1$ and $s(a) = (0,a)$ for $a, a_1 \in A$.

Definition 2.1 Let A and B be two groups and let B acts on A on the left. Then a group homomorphism $\alpha : A \to B$ is called a crossed module if $1_A \times \alpha : A\text{ã } A \to A\text{ã } B$ and $\alpha \times 1_B : A\text{ã } B \to B\text{ã } B$ are group homomorphisms [24].

A crossed module is denoted by (A,B,α). It is useful to give the definition of crossed modules in terms of group operations and actions.

Proposition 2.2 Let A and B be two groups, $\alpha : A \to B$ a group homomorphism and B acts on A on the left. Then (A,B,α) is a crossed module if and only if

(CM1) $\alpha(b \cdot a) = b + \alpha(a) - b$ and

(CM2) $\alpha(a) \cdot a_1 = a + a_1 - a$

for all $a, a_1 \in A$ and $b \in B$ [24].

Example 2.3 Following homomorphisms are standart examples of crossed modules.

- **(i)** Let X be a topological space, $A \subset X$ and $x \in A$. Then the boundary map ρ from the second relative homotopy group $\pi_2(X,A,x)$ to the fundamental group $\pi_1(X,x)$ is a crossed module with the natural action given in [27].
- **(ii)** Let G be a group and N a normal subgroup of G. Then the inclusion function $N \xrightarrow{inc} G$ is a crossed module where the action of G on N is conjugation.
- **(iii)** Let G be a group. Then the inner automorphism map $G \to \mathrm{Aut}(G)$ is a crossed module. Here the action is given by $\psi \cdot g = \psi(g)$ for all $\psi \in \mathrm{Aut}(G)$ and $g \in G$.
- **(iv)** Given any G-module, M, the trivial homomorphism $0 : M \to G$ is a crossed G-module with the given action of G on M.

A morphism $f = \langle f_A, f_B \rangle$ of crossed modules from (A,B,α) to (A',B',α') is a pair of group homomorphisms $f_A : A \to A'$ and $f_B : B \to B'$ such that $f_B\alpha = \alpha' f_A$ and $f_A(b \cdot a) = f_B(b) \cdot f_A(a)$ for all $a \in A$ and $b \in B$.

Crossed modules form a category with morphisms defined above. The category of crossed modules is denoted by **XMod**.

Definition 2.4 Let (A,B,α) and (S,T,σ) be two crossed modules. Then (S,T,σ) is called a subcrossed module of (A,B,α) if $S \leq A$, $T \leq B$, σ is the restriction of α to S and the action of T on S is the induced action from that of B on A [21,22].

Definition 2.5 Let (A,B,α) be a crossed module and (S,T,σ) a subcrossed module of (A,B,α). Then (S,T,σ) is called a normal subcrossed module or an ideal of (A,B,α) if

(i) $T \lhd B$,

(ii) $b \cdot s \in S$ for all $b \in B$, $s \in S$ and

(iii) $t \cdot a - a \in S$ for all $t \in T$, $a \in A$ [21,22].

Example 2.6 Let $f:(A,B,\alpha) \to (A',B',\alpha')$ be a morphism of crossed modules. Then the kernel $\ker f = \ker\langle f_A, f_B\rangle = \left(\ker f_A, \ker f_B, \alpha_{|\ker f_A}\right)$ of $f = \langle f_A, f_B\rangle$ is a normal subcrossed module (ideal) of (A,B,α). Moreover, the image $\operatorname{Im} f = \operatorname{Im}\langle f_A, f_B\rangle = \left(\operatorname{Im} f_A, \operatorname{Im} f_B, \alpha'|_{\operatorname{Im} f_A}\right)$ of $f = \langle f_A, f_B\rangle$ is a subcrossed module of (A',B',α').

Definition 2.7 A topological crossed module (A,B,α) is a crossed module where A and B are topological groups such that the boundary homomorphism $\alpha : A \to B$ and the action of B on A are continuous.

Now we give the pullback notion in the category of crossed modules.

Definition 2.8 Let (A,B,α), (M,P,μ) and (C,D,γ) be three crossed modules and $f = \langle f_A, f_B\rangle : (A,B,\alpha) \to (M,P,\mu)$ and $g = \langle g_C, g_D\rangle : (C,D,\gamma) \to (M,P,\mu)$ be two crossed module morphisms. Then the pullback crossed module of f and g is $\left(A_{f_A}\times_{g_C} C, B_{f_B}\times_{g_D} D, \alpha\times\gamma\right)$ where the action of $B_{f_B}\times_{g_D} D$ on $A_{f_A}\times_{g_C} C$ is given by

$$(b,d)\cdot(a,c) = (b\cdot a, d\cdot c)$$

for all $(b,d) \in B_{f_B}\times_{g_D} D$ and $(a,c) \in A_{f_A}\times_{g_C} C$.

2.2. Internal categories and Brown-Spencer Theorem

Definition 2.9 Let □ be a category with pullbacks. Then an internal category C in □ consist of two objects C_1 and C_0 in □ and four structure morphisms $s,t : C_1 \to C_0$, $\varepsilon : C_0 \to C_1$ and $m : C_1\,{}_s\times_t C_1 \to C_1$, where $C_1\,{}_s\times_t C_1$ is the pullback of s and t, such that the following conditions hold:

(i) $s\varepsilon = t\varepsilon = 1_{C_0}$;

(ii) $sm = s\pi_2$, $tm = t\pi_1$;

(iii) $m\left(1_{C_1}\times m\right) = m\left(m\times 1_{C_1}\right)$ and

(iv) $m\left(\varepsilon s, 1_{C_1}\right) = m\left(1_{C_1}, \varepsilon t\right) = 1_{C_1}$.

Morphisms s,t,ε and m are called source, target, identity object maps and composition respectively. An internal category in □ will be denoted by $C = (C_1, C_0, s, t, \varepsilon, m)$ or only by C for short.

If there is a morphism $n : C_1 \to C_1$ in □ such that $m(1,n) = \varepsilon s$ and $m(n,1) = \varepsilon t$, i.e. every morphism in C_1 has an inverse up to the composition, then we say that $C = (C_1, C_0, s, t, \varepsilon, m, n)$ is an internal groupoid in □.

Let C and C' be two internal categories in □. Then a morphism $f = (f_1, f_0)$ from C to C' consist of a pair of morphisms $f_1 : C_1 \to C_1'$ and $f_0 : C_0 \to C_0'$ in □ such that

(i) $sf_1 = f_0 s$, $tf_1 = f_0 t$,

(ii) $\varepsilon f_0 = f_1\varepsilon$ and

(iii) $m(f_1\times f_1) = f_1 m$.

Thus one can construct the category of internal categories in an arbitrary category □ with pullbacks where the morphisms are morphisms of internal categories as given above. This category is denoted by $\mathbf{Cat}(\square)$.

An internal category in the category of groups is called a group-groupoid [8]. Group-groupoids are also the group objects in the category of small categories.

Example 2.10 Let X be a topological group. Then the set πX of all homotopy classes of paths in X defines a groupoid structure on the set of objects X. This groupoid is called the fundamental groupoid of X. Moreover, πX is a group-groupoid [8].

Let G be an internal category in the category of groups, i.e. a group-groupoid. Then the object of morphisms G_1 and object of objects G_0 have group structures and there are four group homomorphisms $s,t: G_1 \to G_0$, $\varepsilon: G_0 \to G_1$ and $m: G_1 {}_s\times_t G_1 \to G_1$ such that the conditions **(i)-(iv)** of Definition 2.9 are satisfied.

Morphisms between group-groupoids are functors which are group homomorphisms. The category of group-groupoids is denoted by **GpGd.**

Since $m: G_1 {}_s\times_t G_1 \to G_1$ is a group homomorphism then we can give the following lemma.

Lemma 2.11 Let G be an internal category in the category of groups. Then

$$m\big((b,a)+(b',a')\big) = m\big((b',a')\big) + m\big((b',a')\big),$$

i.e.

$$(b+b')\circ(a+a') = (b\circ a) + (b'\circ a')$$

whenever one side (hence both sides) make senses, for all $a,a',b,b' \in G_1$ [8].

Equation given in Lemma 2.11 is called the interchange law. Applications of interchange law can be given as in the following.

Let G be a group-groupoid. Then the partial composition in G can be given in terms of group operations [8]. Indeed, let $a \in G(x,y)$ and $b \in G(y,z)$. Then

$$\begin{aligned} b\circ a &= (b+0)\circ\big(1_y + (-1_y + a)\big) \\ &= (b\circ 1_y) + \big(0\circ(-1_y + a)\big) \\ &= b - 1_y + a \end{aligned}$$

and similarly $b\circ a = a - 1_y + b$.

Corollary 2.12 Let G be a group-groupoid. Then the elements of $\ker s$ and $\ker t$ are commute under the group operation [8].

One can give the inverse of a morphism in terms of group operation as another consequence of the interchange law. That is, let $a \in G(x,y)$. Then

$$1_y = a\circ a^{-1} = a - 1_x + a^{-1}.$$

Thus $a^{-1} = 1_x - a + 1_y$. Similarly $a^{-1} = 1_y - a + 1_x$.

A final remark is that if $a, a_1 \in \ker s$ and $t(a) = x$ then $-1_x + a \in \ker t$ so commutes with a_1. This implies that

$$(-1_x + a) + a_1 = a_1 + (-1_x + a)$$

and thus

$$a + a_1 - a = 1_x + a_1 - 1_x.$$

Theorem 2.13 [Brown & Spencer Theorem] The category **GpGd** of group-groupoids and the category **XMod** of crossed modules are equivalent [8].

Proof: We sketch the proof since we need some details in the last section. Define a functor

$$\varphi: \mathbf{GpGd} \to \mathbf{XMod}$$

as follows: Let G be a group-groupoid. Then $\varphi(G) = (A,B,\alpha)$ is a crossed modules where $A = \ker s$, $B = G_0$, α is the restriction of t and the action of B on A is given by $x\cdot a = 1_x + a - 1_x$.

Conversely, define a functor

$$\psi: \mathbf{XMod} \to \mathbf{GpGd}$$

as follows: Let (A,B,α) be a crossed module. Then the semi-direct product group Aã B is a group-groupoid on B where $s(a,b) = b$, $t(a,b) = \alpha(a) + b$, $\varepsilon(b) = (0,b)$ and the composition is

$$(a',b')\circ(a,b) = (a'+a,b)$$

where $b' = \alpha(a) + b$. Other details are straightforward so is omitted.

3. INTERNAL CATEGORIES WITHIN THE CATEGORY OF CROSSED MODULES

In this section we will characterize internal categories in the category **XMod**. Let C be an internal category in the category **XMod** of crossed modules over groups. Then C consist of two crossed modules $C_1=\left(A_1,B_1,\alpha_1\right)$ and $C_0=\left(A_0,B_0,\alpha_0\right)$ and four crossed module morphisms as $s=\left\langle s_A,s_B\right\rangle$, $t=\left\langle t_A,t_B\right\rangle : C_1\to C_0$ which are called the source and the target maps respectively, $\varepsilon=\left\langle \varepsilon_A,\varepsilon_B\right\rangle : C_0\to C_1$ which is called the identity object map and $m=\left\langle m_A,m_B\right\rangle : C_1\,{}_s\!\times_t C_1\to C_1$ which is called the composition map. These are object to the followings:

(i) $s\varepsilon=t\varepsilon=1_{C_0}$;

(ii) $sm=s\pi_2$, $tm=t\pi_1$;

(iii) $m\left(1_{C_1}\times m\right)=m\left(m\times 1_{C_1}\right)$ and

(iv) $m\left(\varepsilon s,1_{C_1}\right)=m\left(1_{C_1},\varepsilon t\right)=1_{C_1}$.

$$\begin{tikzcd}
A_1\,{}_{s_A}\!\times_{t_A} A_1 \arrow[r, "m_A"] \arrow[d, "\alpha_1\times\alpha_1"'] & A_1 \arrow[r, shift left, "s_A"] \arrow[r, shift right, "t_A"'] \arrow[d, "\alpha_1"] & A_0 \arrow[l, bend right, "\varepsilon_A"'] \arrow[d, "\alpha_0"] \\
B_1\,{}_{s_B}\!\times_{t_B} B_1 \arrow[r, "m_B"'] & B_1 \arrow[r, shift left, "s_B"] \arrow[r, shift right, "t_B"'] & B_0 \arrow[l, bend left, "\varepsilon_B"]
\end{tikzcd}$$

An internal category in the category **XMod** will be denoted by $C=\left(C_1,C_0,s,t,\varepsilon,m\right)$ or briefly by C when no confusion arise. Identity objects $\varepsilon_A\left(a_0\right)$ and $\varepsilon_B\left(b_0\right)$ will be denoted by 1_{a_0} and 1_{b_0} for short, respectively. Also the composition of elements will be denoted by

$$m_A\left(a_1,a_1^{'}\right)=a_1\circ a_1^{'}$$

and by

$$m_B\left(b_1,b_1^{'}\right)=b_1\circ b_1^{'}$$

for $a_1,a_1^{'}\in A_1$ and $b_1,b_1^{'}\in B_1$ with $s_A\left(a_1\right)=t_A\left(a_1^{'}\right)$ and $s_B\left(b_1\right)=t_B\left(b_1^{'}\right)$.

Example 3.1 Let (A,B,α) be a crossed module over groups. We know that $(A\times A,B\times B,\alpha\times\alpha)$ is also a crossed module. If we set $C_1=(A\times A,B\times B,\alpha\times\alpha)$, $C_0=(A,B,\alpha)$, $s=\pi_1$, $t=\pi_2$, $\varepsilon=\Delta$ and define m with $(a_1,a_2)\circ(a,a_1)=(a,a_2)$ and $(b_1,b_2)\circ(b,b_1)=(b,b_2)$ for all $a,a_1,a_2\in A$ and $b,b_1,b_2\in B$ then $C=(C_1,C_0,s,t,\varepsilon,m)$ becomes an internal category in **Xmod**.

Example 3.2 Let (A,B,α) be a crossed module over groups. Then $C=((A,B,\alpha),(A,B,\alpha),s,t,\varepsilon,m)$ becomes an internal category in **XMod** where s, t and ε are identity maps.

Example 3.3 Let (A,B,α) be a topological crossed module. Then $(\pi A,\pi B,\pi\alpha)$ is also a crossed module. Moreover,

$$\pi(A,B,\alpha)=((\pi A,\pi B,\pi\alpha),(A,B,\alpha),s,t,\varepsilon,m)$$

is an internal category in **XMod**.

Now we will give the properties of an internal category with a few lemmas individually.

Lemma 3.4 Let C be an internal category in **Xmod**. Then for $i\in\{0,1\}$

(i) $\alpha_i(a_i+a_i^{'})=\alpha_i(a_i)+\alpha_i(a_i^{'})$,

(ii) $\alpha_i\left(b_i\cdot a_i\right)=b_i+\alpha_i\left(a_i\right)-b_i$ and

(iii) $\alpha_i\left(a_i\right)\cdot a_i^{'}=a_i+a_i^{'}-a_i$

for all $a_i,a_i^{'}\in A_i$ and $b_i\in B_i$.

Proof: It follows from the fact that $C_i=\left(A_i,B_i,\alpha_i\right)$ is a crossed module for $i\in\{0,1\}$.

Lemma 3.5 Let C be an internal category in **Xmod**. Then

(i) $s_A\left(a_1+a_1^{'}\right)=s_A\left(a_1\right)+s_A\left(a_1^{'}\right)$,
$s_B\left(b_1+b_1^{'}\right)=s_B\left(b_1\right)+s_B\left(b_1^{'}\right)$,
$t_A\left(a_1+a_1^{'}\right)=t_A\left(a_1\right)+t_A\left(a_1^{'}\right)$,
$t_B\left(b_1+b_1^{'}\right)=t_B\left(b_1\right)+t_B\left(b_1^{'}\right)$,

(ii) $\alpha_0 s_A=s_B\alpha_1$, $\alpha_0 t_A=t_B\alpha_1$,

(iii) $s_A(b_1 \cdot a_1) = s_B(b_1) \cdot s_A(a_1)$,
$t_A(b_1 \cdot a_1) = t_B(b_1) \cdot t_A(a_1)$,

(iv) $\varepsilon_A(a_0 + a_0^{'}) = \varepsilon_A(a_0) + \varepsilon_A(a_0^{'})$,
$\varepsilon_B(b_0 + b_0^{'}) = \varepsilon_B(b_0) + \varepsilon_B(b_0^{'})$,

(v) $\alpha_1 \varepsilon_A = \varepsilon_B \alpha_0$,

(vi) $\varepsilon_A(b_0 \cdot a_0) = \varepsilon_B(b_0) \cdot \varepsilon_A(a_0)$,

(vii) $(a_1 + a_1^{'}) \circ (a_1^{''} + a_1^{'''}) = (a_1 \circ a_1^{''}) + (a_1^{'} \circ a_1^{'''})$ with $s_A(a_1) = t_A(a_1^{''}), s_A(a_1^{'}) = t_A(a_1^{'''})$ and $(b_1 + b_1^{'}) \circ (b_1^{''} + b_1^{'''}) = (b_1 \circ b_1^{''}) + (b_1^{'} \circ b_1^{'''})$ with $s_B(b_1) = t_B(b_1^{''})$, $s_B(b_1^{'}) = t_B(b_1^{'''})$,

(viii) $\alpha_1 m_A = m_B(\alpha_1 \times \alpha_1)$,

(ix) $(b_1 \circ b_1^{'}) \cdot (a_1 \circ a_1^{'}) = (b_1 \cdot a_1) \circ (b_1^{'} \cdot a_1^{'})$

for all $a_1, a_{1'}, a_{1''}, a_{1'''} \in A_1$, $b_1, b_{1'}, b_{1''}, b_{1'''} \in B_1$, $a_0, a_0^{'} \in A_0$, and $b_0, b_0^{'} \in B_0$.

Proof: **(i)-(iii)** follows from the fact that $s = \langle s_A, s_B \rangle$, $t = \langle t_A, t_B \rangle$ being morphisms of crossed modules.

(iv)-(vi) follows from the fact that $\varepsilon = \langle \varepsilon_A, \varepsilon_B \rangle$ being a morphism of crossed modules. In these conditions if we use the symbol $\varepsilon(*) = 1_*$ for identity morphisms then we get

(iv)' $1_{a_0 + a_0^{'}} = 1_{a_0} + 1_{a_0^{'}}$, $1_{b_0 + b_0^{'}} = 1_{b_0} + 1_{b_0^{'}}$,

(v)' $\alpha_1(1_{a_0}) = 1_{\alpha_0(a_0)}$ and

(vi)' $1_{b_0 \cdot a_0} = 1_{b_0} \cdot 1_{a_0}$.

(vii)-(ix) follows from the fact that $m = \langle m_A, m_B \rangle$ being a morphism of crossed modules.

The identities given in condition **(vii)** of Lemma 3.5 are called interchange laws between group operations and compositions. As an application of interchange laws we will give the following corollary.

Corollary 3.6 Let C be an internal category in **XMod**. Then the compositions in A_1 and B_1 can be written in terms of group operations on A_1 and B_1, respectively, as

$$a_1 \circ a_1^{'} = a_1 - 1_{s_A(a_1)} + a_1^{'} = a_1^{'} - 1_{s_A(a_1)} + a_1$$

and

$$b_1 \circ b_1^{'} = b_1 - 1_{s_B(b_1)} + b_1^{'} = b_1^{'} - 1_{s_B(b_1)} + b_1$$

for $a_1, a_1^{'} \in A$, $b_1, b_1^{'} \in B$ with $s_A(a_1) = t_A(a_1^{'})$ and $s_B(b_1) = t_B(b_1^{'})$.

Proof: We will prove the assumption for A_1. If 0 denotes the identity (zero) elements of groups A_1 and A_0 then

$$\begin{aligned} a_1 \circ a_1^{'} &= (a_1 + 0) \circ \left(1_{s_A(a_1)} + \left(-1_{s_A(a_1)} + a_1^{'}\right)\right) \\ &= \left(a_1 \circ 1_{s_A(a_1)}\right) + \left(0 \circ \left(-1_{s_A(a_1)} + a_1^{'}\right)\right) \\ &= a_1 - 1_{s_A(a_1)} + a_1^{'} \end{aligned}$$

and similarly

$$\begin{aligned} a_1 \circ a_1^{'} &= (0 + a_1) \circ \left(\left(a_1^{'} - 1_{s_A(a_1)}\right) + 1_{s_A(a_1)}\right) \\ &= \left(0 \circ \left(a_1^{'} - 1_{s_A(a_1)}\right)\right) + \left(a_1 \circ 1_{s_A(a_1)}\right) \\ &= a_1^{'} - 1_{s_A(a_1)} + a_1. \end{aligned}$$

By this corollary we obtain that if $s_A(a_1) = t_A(a_1^{'}) = 0$, i.e. $a_1 \in \ker s_A$ and $a_1^{'} \in \ker t_A$, then

$$a_1 + a_1^{'} = a_1^{'} + a_1.$$

So the elements of $\ker s_A$ and $\ker t_A$ are commutative. Similarly, the elements of $\ker s_B$ and $\ker t_B$ are commutative too. Moreover, for an element $a_1 \in A_1$, $a_1^{-1} = 1_{s_A(a_1)} - a_1 + 1_{t_A(a_1)} \in A_1$ is the inverse element of a_1 up to the composition m_A. Similarly for an element $b_1 \in B_1$, $b_1^{-1} = 1_{s_B(b_1)} - b_1 + 1_{t_B(b_1)} \in B_1$ is the inverse element of b_1 up to the composition m_B. This means that $C = (C_1, C_0, s, t, \varepsilon, m, n)$ has a groupoid structure where $n = \langle n_A, n_B \rangle : C_1 \to C_1$ is a morphism of crossed modules where

$$\begin{aligned} n_A \;:\; & A_1 \to A_1 \\ & a_1 \mapsto n_A(a_1) = a_1^{-1} = 1_{s_A(a_1)} - a_1 + 1_{t_A(a_1)} \end{aligned}$$

and

$$\begin{array}{cccl} n_B & : & B_1 & \to & B_1 \\ & & b_1 & \mapsto & n_B(b_1) = b_1^{-1} = 1_{s_B(b_1)} - b_1 + 1_{t_B(b_1)}. \end{array}$$

It is easy to see that

$$1_{s_A(a_1)} - a_1 + 1_{t_A(a_1)} = 1_{t_A(a_1)} - a_1 + 1_{s_A(a_1)}$$

for all $a_1 \in A_1$ and similarly

$$1_{s_B(b_1)} - b_1 + 1_{t_B(b_1)} = 1_{t_B(b_1)} - b_1 + 1_{s_B(b_1)}$$

for all $b_1 \in B_1$.

Lemma 3.7 Let $a_1 \in A_1$ and $b_1 \in B_1$. Then $b_1^{-1} \cdot a_1^{-1} = (b_1 \cdot a_1)^{-1}$.

Proof: By the condition **(ix)** of Lemma 3.5

$$\begin{aligned} (b_1 \cdot a_1) \circ (b_1^{-1} \cdot a_1^{-1}) &= (b_1 \circ b_1^{-1}) \cdot (a_1 \circ a_1^{-1}) \\ &= 1_{s_B(b_1)} \cdot 1_{s_A(a_1)} \\ &= 1_{s_B(b_1) \cdot s_A(a_1)} \\ &= 1_{s_A(b_1 \cdot a_1)} \end{aligned}$$

and similarly $(b_1^{-1} \cdot a_1^{-1}) \circ (b_1 \cdot a_1) = 1_{t_A(b_1 \cdot a_1)}$. Thus $b_1^{-1} \cdot a_1^{-1} = (b_1 \cdot a_1)^{-1}$.

It is easy to see that an internal category in the category of crossed modules over groups is indeed a crossed module object in the category of internal categories within groups.

Definition 3.8 Let C and C' be two internal categories in **XMod**. A morphism (internal functor) from C to C' is a pair of crossed module morphisms

$$f = \left(f_1 = \left\langle f_1^A, f_1^B \right\rangle, f_0 = \left\langle f_0^A, f_0^B \right\rangle\right) : C \to C'$$

such that $f_0 s = s f_1$, $f_0 t = t f_1$, $f_1 \varepsilon = \varepsilon f_0$ and $f_1 m = m(f_1 \times f_1)$.

Hence we can construct the category of internal categories (groupoids) within the category of crossed modules over groups where the morphisms are internal functors as defined above. This category will be denoted by **Cat(Xmod)**.

3.1. Crossed squares

Crossed squares are first defined in [12]. In this subsection we recall the definition of a crossed square as given in [7]. Further we prove that the category of crossed squares and that of internal categories within crossed modules are equivalent. Finally we give some examples of crossed squares using this equivalence.

Definition 3.9 A crossed square over groups consists of four morphisms of groups $\lambda : L \to M$, $\lambda' : L \to N$, $\mu : M \to P$ and $\nu : N \to P$ together with actions of the group P on L, M, N on the left, conventionally, (and hence actions of M on L and N via μ and of N on L and M via ν) and a function $h : M \times N \to L$. These are subject to the following axioms:

- **(i)** λ, λ' are P-equivariant and μ, ν and $\kappa = \mu\lambda = \nu\lambda'$ are crossed modules,
- **(ii)** $\lambda h(m,n) = m + n \cdot (-m)$, $\lambda' h(m,n) = m \cdot n - n$,
- **(iii)** $h(\lambda(l), n) = l + n \cdot (-l)$, $h(m, \lambda'(l)) = m \cdot l - l$,
- **(iv)** $h(m + m', n) = m \cdot h(m', n) + h(m,n)$,
  $h(m, n + n') = h(m,n) + n \cdot h(m,n')$,
- **(v)** $h(p \cdot m, p \cdot n) = p \cdot h(m,n)$

for all $l \in L$, $m, m' \in M$, $n, n' \in N$ and $p \in P$ [7].

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

A crossed square will be denoted by $S = (L, M, N, P)$.

Example 3.10. Let (A, B, α) be crossed module and (S, T, σ) a normal subcrossed module of (A, B, α). Then

$$\begin{tikzcd} S \arrow[r, "\sigma"] \arrow[d, hook, "\mathrm{inc}"'] & T \arrow[d, hook, "\mathrm{inc}"] \\ A \arrow[r, "\partial"'] & B \end{tikzcd}$$

forms a crossed square of groups where the action of B on S is induced action from the action of B on A and the action of B on T is conjugation. The h map is defined by $h(t,a) = t \cdot a - a$ for all $t \in T$ and $a \in A$ [21,22].

A topological example of crossed squares is the fundamental crossed square which is defined in [7] as follows: Suppose given a commutative square of spaces

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "a"] \\
B \arrow[r, "b"'] & X
\end{tikzcd}$$

Let $F(f)$ be the homotopy fibre of f and $F(X)$ the homotopy fibre of $F(g) \to F(a)$. Then the commutative square of groups

$$\begin{tikzcd}
\pi_1 F(\boldsymbol{X}) \arrow[r] \arrow[d] & \pi_1 F(g) \arrow[d] \\
\pi_1 F(f) \arrow[r] & \pi_1(C)
\end{tikzcd}$$

is naturally equipped with a structure of crossed square. This crossed square is called the fundamental crossed square [7].

A morphism $f = (f_L, f_M, f_N, f_P)$ of crossed squares from $S_1 = (L_1, M_1, N_1, P_1)$ to $S_2 = (L_2, M_2, N_2, P_2)$ consist of four group homomorphisms $f_L : L_1 \to L_2$, $f_M : M_1 \to M_2$, $f_N : N_1 \to N_2$ and $f_P : P_1 \to P_2$ which are compatible with the actions and the functions h_1 and h_2 such that the following diagram is commutative.

$$\begin{tikzcd}
 & M_1 \arrow[rr, "f_M"] \arrow[dd, dotted, "\mu_1"' near end] & & M_2 \arrow[dd, "\mu_2"] \\
L_1 \arrow[ur, "\lambda_1"] \arrow[rr, "f_L" near end] \arrow[dd, "\lambda'_1"'] & & L_2 \arrow[ur, "\lambda_2"'] \arrow[dd, "\lambda'_2" near start] & \\
 & P_1 \arrow[rr, dotted, "f_P" near start] & & P_2 \\
N_1 \arrow[ur, "\nu_1"] \arrow[rr, "f_N"'] & & N_2 \arrow[ur, "\nu_2"'] &
\end{tikzcd}$$

Category of crossed squares over groups with morphisms between crossed squares defined above is denoted by $\mathbf{X^2Mod}$. Crossed squares are equivalent to crossed modules over crossed modules [22].

Now we prove our main theorem.

Theorem 3.11. The category **Cat(XMod)** of internal categories within the category of crossed modules over groups and the category $\mathbf{X^2Mod}$ of crossed squares over groups are equivalent.

Proof: We first define a functor $\eta : \mathbf{Cat}(\mathbf{XMod}) \to \mathbf{X^2Mod}$ as follows: Let $C = (C_1, C_0, s, t, \varepsilon, m, n)$ be an object in $\mathbf{Cat}(\mathbf{XMod})$. If we set $L = \ker s_A$, $M = \ker s_B$, $N = A_0$, $P = B_0$, $\lambda = \alpha_{1|\ker s_A}$, $\lambda' = t_{A|\ker s_A}$, $\mu = t_{B|\ker s_B}$ and $\nu = \alpha_0$ then $\eta(C) = S = (L, M, N, P)$ becomes a crossed square with the function $h(m,n) = m \cdot 1_n - 1_n$ for all $m \in M$ and $n \in N$.

Here (L, M, λ) is a crossed module since it is the kernel crossed module of

$$s = \langle s_A, s_B \rangle : (A_1, B_1, \alpha_1) \to (A_0, B_0, \alpha_0).$$

Moreover we know that (L, N, λ') and (M, P, μ) are crossed modules by Brown & Spencer Theorem [8, Theorem 1]. Finally (N, P, ν) is already a crossed module since it is (A_0, B_0, α_0). Here the actions of P on N is already given, on M is given by $p \cdot m = 1_p + m - 1_p$ and on L is given by $p \cdot l = 1_p \cdot l$ (where the action on the right side of the equation is the action of B_1 on A_1) for $p \in P$, $m \in M$ and $l \in L$. Now we need to show that the conditions given in the Definition 3.9 is satisfied.

(i) We need to show that λ, λ' are P-equivariant and $\kappa = \mu\lambda$ is a crossed module. Let $l \in L$, and $p \in P$. Then

$$\lambda(p \cdot l) = \alpha_1(1_p \cdot l) = 1_p + \alpha_1(l) - 1_p = p \cdot \lambda(l)$$

and

$$\lambda'(p \cdot l) = t_A(1_p \cdot l) = t_B(1_p) \cdot t_A(l) = p \cdot \lambda'(l)$$

so λ and λ' are P-equivariant. Now we need to show that (L, P, κ) is a crossed module. So

(CM1) Let $l \in L$, and $p \in P$. Then

$$\begin{aligned}
\kappa(p \cdot l) &= \mu\lambda(p \cdot l) \\
&= \mu(p \cdot \lambda(l)) \\
&= p + \mu(\lambda(l)) - p \\
&= p + \kappa(l) - p
\end{aligned}$$

(CM2) Let $l,l' \in L$. Then

$$\begin{aligned}\kappa(l)\cdot l' &= \mu(\lambda(l))\cdot l' \\ &= 1_{\mu(\lambda(l))}\cdot l' \\ &= 1_{\nu(\lambda'(l))}\cdot l' \\ &= \lambda\left(1_{\lambda'(l)}\right)\cdot l' \\ &= 1_{\lambda'(l)} + l' - 1_{\lambda'(l)} \\ &= l + l' - l\end{aligned}$$

(ii) Let $m\in M$ and $n \in N$. Then

$$\begin{aligned}\lambda h(m,n) &= \lambda(m\cdot 1_n - 1_n) \\ &= \lambda(m\cdot 1_n) - \lambda(1_n) \\ &= m + \lambda(1_n) - m - \lambda(1_n) \\ &= m + n\cdot(-m)\end{aligned}$$

and

$$\begin{aligned}\lambda' h(m,n) &= \lambda'(m\cdot 1_n - 1_n) \\ &= \lambda'(m\cdot 1_n) - \lambda'(1_n) \\ &= \mu(m)\cdot\lambda'(1_n) - \lambda'(1_n) \\ &= \mu(m)\cdot n - n \\ &= m\cdot n - n\end{aligned}$$

(iii) Let $l\in L$, $m \in M$ and $n\in N$. Then

$$\begin{aligned}h(\lambda(l),n) &= \lambda(l)\cdot 1_n - 1_n \\ &= (l + 1_n - l) - 1_n \\ &= l + (1_n - l - 1_n) \\ &= l + n\cdot(-l)\end{aligned}$$

and

$$\begin{aligned}h(m,\lambda'(l)) &= m\cdot 1_{\lambda'(l)} - 1_{\lambda'(l)} \\ &= \left(m\cdot 1_{\lambda'(l)} - 1_{\lambda'(l)} + l\right) - l \\ &= \left(m\cdot 1_{\lambda'(l)} \circ l\right) - l \\ &= \left(\left(m\cdot 1_{\lambda'(l)}\right)\circ(1_0\cdot l)\right) - l \\ &= ((m\circ 1_0)\cdot(1_{\lambda'(l)}\circ l)) - l \\ &= m\cdot l - l\end{aligned}$$

(iv) Let $m,m' \in M$ and $n,n' \in N$. Then

$$\begin{aligned}h(m+m',n) &= (m+m')\cdot 1_n - 1_n \\ &= m\cdot(m'\cdot 1_n) - 1_n \\ &= m\cdot(m'\cdot 1_n) + m\cdot(-1_n + 1_n) - 1_n \\ &= m\cdot(m'\cdot 1_n - 1_n) + m\cdot 1_n - 1_n \\ &= m\cdot h(m',n) + h(m,n)\end{aligned}$$

and

$$\begin{aligned}h(m,n+n') &= m\cdot 1_{n+n'} - 1_{n+n'} \\ &= m\cdot(1_n + 1_{n'}) - 1_{n'} - 1_n \\ &= (m\cdot 1_n - 1_n) + 1_n + (m\cdot 1_{n'} - 1_{n'}) - 1_{n'} \\ &= h(m,n) + n\cdot h(m,n')\end{aligned}$$

(v) Let $m \in M$, $n\in N$ and $p \in P$. Then

$$\begin{aligned}h(p\cdot m, p\cdot n) &= h(1_p + m - 1_p, p\cdot n) \\ &= (1_p + m - 1_p)\cdot 1_{p\cdot n} - 1_{p\cdot n} \\ &= (1_p + m - 1_p)\cdot(1_p\cdot 1_n) - (1_p\cdot 1_n) \\ &= 1_p\cdot(m\cdot 1_n) + 1_p\cdot(-1_n) \\ &= 1_p\cdot(m\cdot 1_n - 1_n) \\ &= p\cdot h(m,n)\end{aligned}$$

Now let

$$f = \left(f_1 = \left\langle f_1^A, f_1^B\right\rangle, f_0 = \left\langle f_0^A, f_0^B\right\rangle\right) : C \to C'$$

be a morphism in $\mathbf{Cat}(\mathbf{XMod})$. Then

$$\eta(f) = \left(f^A_{1|\ker s_A}, f^B_{1|\ker s_B}, f_0^A, f_0^B\right) : S \to S'$$

is a morphism of crossed squares.

Conversely define a functor

$$\psi : \mathbf{X^2Mod} \to \mathbf{Cat}(\mathbf{XMod})$$

as follows: Let $S = (L,M,N,P)$ be a crossed square over groups. Then

$$\psi(S) = \left(C_1 = (A_1,B_1,\alpha_1), C_0 = (A_0,B_0,\alpha_0), s,t,\varepsilon,m\right)$$

is an internal category within the category of crossed modules over groups where

$$(A_1,B_1,\alpha_1) = (L \rtimes N, M \rtimes P, \lambda\times\nu),$$

$$(A_0,B_0,\alpha_0) = (N,P,\nu),$$

$$s_A(l,n) = n,\ s_B(m,p) = p,$$

$$t_A(l,n)=\lambda'(l)+n\ ,t_B(m,p)=\mu(m)+p\ ,$$
$$\varepsilon_A(n)=(0,n)\ ,\ \varepsilon_B(p)=(0,p)\ ,$$
$$(l',\lambda'(l)+n)\circ(l,n)=(l'+l,n)$$

and

$$(m',\lambda'(m)+p)\circ(m,p)=(m'+m,p).$$

We know that C_0 is a crossed module over groups. First we need to show that $(L\,\text{ã}\,N,M\,\text{ã}\,P,\lambda\times\nu)$ is a crossed module with the action of $M\,\text{ã}\,P$ on $L\,\text{ã}\,N$ is

$$(m,p)\cdot(l,n)=(m\cdot(p\cdot l)+h(m,p\cdot n),p\cdot n).$$

(CM1) Let $(l,n)\in L\,\text{ã}\,N$ and $(m,p)\in M\,\text{ã}\,P$. Then

$$\begin{aligned}(\lambda\times\nu)((m,p)\cdot(l,n))&=(\lambda\times\nu)(m\cdot(p\cdot l)+h(m,p\cdot n),p\cdot n)\\&=(m,p)+(\lambda\times\nu)(l,n)-(m,p).\end{aligned}$$

(CM2) Let $(l,n),(l',n')\in L\,\text{ã}\,N$. Then

$$\begin{aligned}(\lambda\times\nu)((l,n))\cdot(l',n')&=(\lambda(l),\nu(n))\cdot(l',n')\\&=(l,n)+(l',n')-(l,n).\end{aligned}$$

Thus $C_1=(L\,\text{ã}\,N,M\,\text{ã}\,P,\lambda\times\nu)$ is a crossed module. Now we need to show that $\psi(S)=C$ satisfies the conditions given in Lemma 3.5. We know that $s_A,s_B,t_A,t_B,\varepsilon_A,\varepsilon_B,m_A$ and m_B are group homomorphisms. So the conditions (i), (iv) and (vii) holds.

(ii) Let $(l,n)\in L\,\text{ã}\,N$. Then

$$\begin{aligned}\nu s_A((l,n))&=\nu(n)\\&=s_A(\lambda(l),\nu(n))\\&=s_B((\lambda\times\nu)(l,n))\end{aligned}$$

and

$$\begin{aligned}\nu t_A(l,n)&=\nu(\lambda'(l)+n)\\&=\nu(\lambda'(l))+\nu(n)\\&=\mu(\lambda(l))+\nu(n)\\&=t_B((\lambda(l),\nu(n)))\\&=t_B((\lambda\times\nu)(l,n))\end{aligned}$$

(iii) Let $(l,n)\in L\,\text{ã}\,N$ and $(m,p)\in M\,\text{ã}\,P$. Then

$$s_A((m,p)\cdot(l,n))=p\cdot n=s_B(m,p)\cdot s_A(l,n)$$

and

$$\begin{aligned}t_A((m,p)\cdot(l,n))&=\lambda'(m\cdot(p\cdot l)+h(m,p\cdot n))\\&\quad+p\cdot n\\&=t_B(m,p)\cdot t_A(l,n).\end{aligned}$$

(v) Let $n\in N$. Then

$$\begin{aligned}\alpha_1\varepsilon_A(n)&=\alpha_1(0,n)\\&=(\lambda\times\nu)(0,n)\\&=(\lambda(0),\nu(n))\\&=\varepsilon_B\nu(n)\\&=\varepsilon_B\alpha_0(n).\end{aligned}$$

(vi) Let $n\in N$ and $p\in P$. Then

$$\begin{aligned}\varepsilon_A(p\cdot n)&=(0,p\cdot n)\\&=(0,p)\cdot(0,n)\\&=\varepsilon_B(p)\cdot\varepsilon_A(n).\end{aligned}$$

(viii) Let $(l,n),(l',n')\in L\,\text{ã}\,N$ such that $n'=\lambda'(l)+n$. Then

$$\begin{aligned}\alpha_1 m_A((l',n'),(l,n))&=\alpha_1((l',n')\circ(l,n))\\&=(\lambda\times\nu)((l'+l,n))\\&=(\lambda(l'+l),\nu(n))\\&=(\lambda(l')+\lambda(l),\nu(n))\\&=m_B(\alpha_1\times\alpha_1)((l',n'),(l,n)).\end{aligned}$$

(ix) Let $(l,n),(l',n')\in L\,\text{ã}\,N$ and $(m,p),(m',p')\in M\,\text{ã}\,P$ such that $n'=\lambda'(l)+n$ and $p'=\mu(m)+p$. Firstly,

$$\begin{aligned}h(m'+m,p\cdot n)&=h(m-m+m'+m,p\cdot n)\\&=m\cdot h((-m)\cdot m',p\cdot n)\\&\quad+h(m,p\cdot n)\\&=h(m\cdot((-m)\cdot m'),m\cdot(p\cdot n))\\&\quad+h(m,p\cdot n)\\&=h(m',p'\cdot n)+h(m,p\cdot n).\end{aligned}$$

Then

$$\begin{aligned}&\left((m',p')\circ(m,p)\right)\cdot\left((l',n')\circ(l,n)\right)\\&\qquad=(m'+m,p)\cdot(l'+l,n)\end{aligned}$$

and

$$\begin{aligned}&(m'+m,p)\cdot(l'+l,n)\\&\qquad=\left((m',p')\cdot(l',n')\right)\circ\left((m,p)\cdot(l,n)\right)\end{aligned}$$

Thus $\psi(S)=C$ is an object in **Cat(XMod)**. Now let

$$f:S_1=(L_1,M_1,N_1,P_1)\to S_2=(L_2,M_2,N_2,P_2)$$

be a morphism of crossed squares. Then $\psi(f)=\left(\langle f_L\times f_N,f_M\times f_P\rangle,\langle f_N,f_P\rangle\right):C\to C'$ is a morphism in **Cat(XMod)** where $\psi(S_1)=C$ and $\psi(S_2)=C'$.

Finally we show that composition of these functors are naturally isomorphic to the identity functors on **Cat(XMod)** and **X²Mod** respectively. For any object C in **Cat(XMod)** the natural isomorphism $U:1_{\mathbf{Cat(XMod)}}\Rightarrow\psi\eta$ is given by

$$U_C=\left(f_1=\langle f_1^A,f_1^B\rangle,f_0=\langle f_0^A,f_0^B\rangle\right)$$

where

$$f_1^A(a_1)=\left(a_1-1_{s_A(a_1)},s_A(a_1)\right),$$
$$f_1^B(b_1)=\left(b_1-1_{s_B(b_1)},s_B(b_1)\right),\ f_0^A=1_{A_0}$$

and $f_0^B=1_{B_0}$ for all $a_1\in A_1$ and $b_1\in B_1$.

Conversely, for any object $S=(L,M,N,P)$ in **X²Mod** the natural isomorphism $T:\eta\psi\Rightarrow 1_{\mathbf{X^2Mod}}$ is given by $T_S=\left(f_L=\pi_1,f_M=\pi_1,f_N=1_N,f_P=1_P\right)$. This completes the proof.

Now we can give examples of crossed squares which are obtained from examples of internal categories within the category of crossed modules.

Example 3.12 Let (A,B,α) be a crossed module. Then the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "1"'] & B \arrow[d, "1"] \\ A \arrow[r, "\alpha"'] & B \end{tikzcd}$$

has a structure of a crossed square where $h(b,a)=b\cdot a-a$ for all $a\in A$ and $b\in B$.

Example 3.13 Let (A,B,α) be a crossed module. Then the diagram

$$\begin{tikzcd} 0 \arrow[r, "0"] \arrow[d, "0"'] & A \arrow[d, "\alpha"] \\ 0 \arrow[r, "0"'] & B \end{tikzcd}$$

forms a crossed square where $h(a,0)=0$ for all $a\in A$.

Example 3.14 Let (A,B,α) be a topological crossed module. Then we know that $(\pi A,A,s_A,t_A,\varepsilon_A,m_A)$ and $(\pi B,B,s_B,t_B,\varepsilon_B,m_B)$ are group-groupoids. Then

$$\begin{tikzcd} \ker s_A \arrow[r, "\pi\alpha"] \arrow[d, "t_A"'] & \ker s_B \arrow[d, "t_B"] \\ A \arrow[r, "\alpha"'] & B \end{tikzcd}$$

has a crossed square structure where $h([\beta],a)=[\beta\cdot a-a]$ for all $[\beta]\in\ker s_B$ and $a\in A$. Here the path $(\beta\cdot a-a):[0,1]\to A$ is given by $(\beta\cdot a-a)(r)=\beta(r)\cdot a-a$ for all $r\in[0,1]$.

4. CONCLUSION

We proved that the category **Cat(XMod)** of internal categories within the category of crossed modules over groups and the category **X²Mod** of crossed squares over groups are equivalent. Since crossed squares model all connected homotopy 3-types so are internal categories in within the category of crossed modules.

For further work, in a similar way of thinking one can obtain same results in a more generic algebraic category namely the category of groups with operations or in higher dimensional crossed modules [10]. Also in the light of the results given in [18], notions of normal subcrossed square and of quotient crossed square can be obtained.

REFERENCES

[1] H.F. Akız, N. Alemdar, O. Mucuk and T. Şahan, “Coverings of internal groupoids and crossed modules in the category of groups with operations,” *Georgian Mathematical Journal*, vol. 20, no. 2, pp. 223–238, 2013.

[2] Z. Arvasi, “Crossed Squares and 2-Crossed Modules of Commutative Algebras”, *Theory and Applications of Categories*, vol. 3, no. 7, pp. 160--181, 1997.

[3] Z. Arvasi and T. Porter, “Freeness Conditions for 2-Crossed Modules of Commutative Algebras,” *Applied Categorical Structures*, vol. 6, no. 4, pp. 455--471, 1988.

[4] R. Brown, “Coproducts of crossed P-modules: Applications to second homotopy groups and to the homology of groups,” *Topology*, vol. 23, pp. 337--345, 1984.

[5] R. Brown and P.J. Higgins, “On the connection between the second relative homotopy groups of some related spaces,” *Proceedings of the London Mathematical Society*, vol. 36, pp. 193--212, 1978.

[6] R. Brown, R. and J. Huebschmann, “Identities among relations,” in Low Dimensional Topology, *London Mathematical Society Lecture Note Series*, vol. 48, pp. 153--202, 1982.

[7] R. Brown and J.-L. Loday, “Homotopical excision, and Hurwicz theorems, for n-cubes of spaces,” *Proceedings of the London Mathematical Society*, vol. 54, no. 3, pp. 176--192, 1987.

[8] R. Brown and C.B. Spencer, “G-groupoids, crossed modules and the fundamental groupoid of a topological group,” *Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen series A*, vol. 79, no. 4, pp. 296--302, 1976.

[9] D. Conduché, “Modules croisés généralisés de longueur 2,” *Journal of Pure and Applied Algebra*, vol. 34, pp. 155--178, 1984.

[10] G.J. Ellis, “Higher dimensional crossed modules of algebras,” *Journal of Pure and Applied Algebra*, vol. 52, pp. 277--282, 1988.

[11] M. Gerstenhaber, “On the deformation of rings and algebras,” *Annals of Mathematics*, vol. 84, pp. 1--19, 1966.

[12] D. Guin-Walery, D. and J.-L. Loday, “Obstructions Alexcision en K-theorie Algebrique,” *Springer Lecture Notes in Mathematics*, vol. 854, pp. 179--216, 1981.

[13] Lichtenbaum, S. and Schlessinger, M., {The cotangent complex of a morphism}, *Transactions of the American Mathematical Society*, vol. 128, pp. 41--70, 1967.

[14] J.-L. Loday, “Spaces with finitely many non-trivial homotopy groups,” *Journal of Pure and Applied Algebra*, vol. 24, pp. 179--202, 1982.

[15] O. Mucuk and H.F. Akız, “Monodromy groupoids of an internal groupoid in topological groups with operations,” *Filomat*, vol. 29, no. 10, pp. 2355--2366, 2015.

[16] O. Mucuk, B. Kılıçarslan, T. Şahan and N. Alemdar, “Group-groupoids and monodromy groupoids,” *Topology and its Applications*, vol. 158, no. 15, pp. 2034--2042, 2011.

[17] O. Mucuk and T. Şahan, “Coverings and crossed modules of topological groups with operations,” *Turkish Journal of Mathematics*, vol. 38, no. 5, pp. 833--845, 2014.

[18] O. Mucuk, T. Şahan and N. Alemdar, “Normality and Quotients in Crossed Modules and Group-groupoids,” *Applied Categorical Structures*, vol. 23, pp. 415--428, 2015.

[19] A. Mutlu and T. Porter, “Freeness Conditions for 2-Crossed Modules and Complexes,” *Theory and Applications of Categories,* vol. 4, no. 8, pp. 174--194, 1998.

[20] A. Mutlu and T. Porter, “Freeness Conditions for Crossed Squares and Squared Complexes,” *K-Theory*, vol. 20, no. 4, pp. 345--368, 2000.

[21] K.J. Norrie, “Crossed modules and analogues of group theorems”, Ph. D. Thesis, King's College, England, UK, 1987.

[22] K.J. Norrie, “Actions and automorphisms of crossed modules,” *Bulletin de la Société Mathématique de France*, vol. 118, no. 2, pp. 129--146, 1990.

[23] G. Orzech, “Obstruction theory in categories. I - II,” *Journal of Pure and Applied Algebra*, vol. 2, pp. 287--314, 315--340, 1972.

[24] T. Porter, “Extensions, crossed modules and internal categories in categories of groups with operations,” *Proceedings of the Edinburgh*

Mathematical Society, vol. 30, pp. 373--381, 1987.

[25] J.H.C. Whitehead, “Note on a previous paper entitled "On adding relations to homotopy group",” *Annals of Mathematics*, vol. 47, pp. 806--810, 1946.

[26] J.H.C. Whitehead, “On operators in relative homotopy groups,” *Annals of Mathematics*, vol. 49, pp. 610--640, 1948.

[27] J.H.C. Whitehead, “Combinatorial homotopy II,” *Bulletin of the American Mathematical Society*, vol. 55, pp. 453--496, 1949.