Hacettepe Journal of Mathematics and Statistics
Volume 43 (3) (2014), 435 – 449

Some subclasses of meromorphic multivalent functions involving a generalized differential operator

Yong Sun*†, Yue-Ping Jiang* and Zhi-Hong Liu‡

Abstract

Making use of a generalized differential operator which is defined by means of the Hadamard product, we introduce some new subclasses of meromorphic p-valent functions and investigate their inclusion relationships, integral preserving and convolution properties. The results presented here would provide extensions of those given in earlier works. Several other new results are also obtained.

Keywords: Meromorphic functions, multivalent functions, Hadamard product(or convolution), subordination between analytic functions, generalized differential operator.

2000 AMS Classification: 30C45, 30C80

1. Introduction and Preliminaries

Let Σ_p denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p} z^{n-p} \qquad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are *analytic* in the *punctured* open unit disk

$$\mathbb{U}^* := \{z : \ z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\} =: \mathbb{U} \setminus \{0\}.$$

Let $f, \ g \in \Sigma_p$, where f is given by (1.1) and g is defined by

$$g(z) = z^{-p} + \sum_{n=1}^{\infty} b_{n-p} z^{n-p}.$$

*School of Mathematics and Econometrics, Hunan University, Changsha 410082, Hunan, People's Republic of China.
Email: yongsun2008@foxmail.com; ypjiang731@163.com
†Corresponding Author.
‡Department of Mathematics, Honghe University, Mengzi 661100, Yunnan, People's Republic of China.
Email: liuzhihongmath@163.com

Then the Hadamard product (or convolution) $f * g$ is defined by

$$(f * g)(z) := z^{-p} + \sum_{n=1}^{\infty} a_{n-p} b_{n-p} z^{n-p} =: (g * f)(z).$$

Let $\mathcal{P}$ denote the class of functions of the form:

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

which are analytic and convex in $\mathbb{U}$ and satisfy the condition:

$$\Re(p(z)) > 0 \qquad (z \in \mathbb{U}).$$

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g(\omega(z)) \qquad (z \in \mathbb{U}).$$

Indeed, it is known that (see [12] or [13])

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Analogous to the operator defined recently by Selvaraj and Selvakumaran [20] and Aouf *et al.* [2], we introduce the following integral operator:

$$\mathcal{M}_{\lambda,g}^{\delta} : \Sigma_p \longrightarrow \Sigma_p$$

defined by

$$\mathcal{M}_{\lambda,g}^{0} f(z) = (f * g)(z),$$

$$\mathcal{M}_{\lambda,g}^{1} f(z) = (1+\lambda)(f * g)(z) + \frac{\lambda}{p} z (f * g)'(z),$$

$$\mathcal{M}_{\lambda,g}^{\delta} f(z) = \mathcal{M}_{\lambda,g}^{1} \left(\mathcal{M}_{\lambda,g}^{\delta-1} f(z) \right) \qquad (\delta \in \mathbb{N};\ \lambda \geqq 0). \tag{1.2}$$

If $f \in \Sigma_p$, then we have

$$\mathcal{M}_{\lambda,g}^{\delta} f(z) = z^{-p} + \sum_{n=1}^{\infty} \left(1 + \frac{n\lambda}{p}\right)^{\delta} a_{n-p} b_{n-p} z^{n-p}. \tag{1.3}$$

It easily follows from (1.2) that

$$\frac{\lambda z}{p} \left(\mathcal{M}_{\lambda,g}^{\delta} f \right)'(z) = \mathcal{M}_{\lambda,g}^{\delta+1} f(z) - (1+\lambda) \mathcal{M}_{\lambda,g}^{\delta} f(z). \tag{1.4}$$

Throughout this paper, we assume that

$$p,\ k \in \mathbb{N}, \qquad \varepsilon_k = \exp\left(\frac{2\pi i}{k}\right),$$

and

$$f_{p,k}^{\delta}(\lambda; g; z) = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{jp} \mathcal{M}_{\lambda,g}^{\delta} f(\varepsilon_k^j z) = z^{-p} + \cdots \qquad (f \in \Sigma_p). \tag{1.5}$$

Clearly, for $k = 1$, we have

$$f_{p,1}^{\delta}(\lambda; g; z) = \mathcal{M}_{\lambda,g}^{\delta} f(z).$$

Making use of the integral operator $\mathcal{M}_{\lambda,g}^{\delta}$ and the above-mentioned principle of subordination between analytic functions, we now introduce and investigate the following subclasses of the class Σ_p of mermorphically p-valent functions.

1.1. Definition. A function $f \in \Sigma_p$ is said to be in the class $\mathcal{S}_{p,k}^{\delta}(\lambda; g; h)$ if it satisfies the following subordination condition:

$$-\frac{z\left(\mathcal{M}_{\lambda,g}^{\delta} f\right)'(z)}{p f_{p,k}^{\delta}(\lambda; g; z)} \prec h(z) \qquad (z \in \mathbb{U}), \tag{1.6}$$

where

$$h \in \mathcal{P} \quad \text{and} \quad f_{p,k}^{\delta}(\lambda; g; z) \neq 0 \quad (z \in \mathbb{U}^*).$$

1.2. Remark. In a recent paper, Srivastava *et al.* [21] introduced an investigated a subclass $\Sigma_{p,k}(a, c; h)$ of Σ_p consisting of functions which are satisfy the following subordination condition:

$$-\frac{z\left(\mathcal{L}_p(a, c) f\right)'(z)}{p f_{p,k}(a, c; z)} \prec h(z) \qquad (z \in \mathbb{U};\ c \neq 0, -1, -2, \ldots),$$

where $h \in \mathcal{P}$,

$$\mathcal{L}_p(a, c) f(z) = \varphi_p(a, c; z) * f(z) = \left(z^{-p} + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{n-p}\right) * f(z) \qquad (z \in \mathbb{U}^*),$$

and

$$f_{p,k}(a, c; z) = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{jp} (\mathcal{L}_p(a, c) f)(\varepsilon_k^j z) \neq 0 \qquad (z \in \mathbb{U}^*).$$

The above $(\mu)_n$ is the Pochhammer symbol defined by

$$(\mu)_0 = 1 \quad \text{and} \quad (\mu)_n = \mu(\mu + 1) \cdots (\mu + n - 1) \qquad (n \in \mathbb{N}).$$

It is also easy to see that, if we set

$$\lambda = 0,\ \delta = 1, \quad \text{and} \quad g(z) = \varphi_p(a, c; z)$$

in the class $\mathcal{S}_{p,k}^{\delta}(\lambda; g; h)$, then it reduces to the class $\Sigma_{p,k}(a, c; h)$.

More recently, Wang *et al.* [22] studied a subclass $\mathcal{F}_{p,k}^{q,s}(\alpha; \alpha_1; h)$ of Σ_p consisting of functions which are satisfy the following subordination condition:

$$-\frac{z\left[(1+\alpha)\left(H_p^{q,s}(\alpha_1) f\right)'(z) + \alpha\left(H_p^{q,s}(\alpha_1 + 1) f\right)'(z)\right]}{p\left[(1+\alpha) f_{p,k}^{q,s}(\alpha_1; z) + \alpha f_{p,k}^{q,s}(\alpha_1 + 1; z)\right]} \prec h(z) \qquad (z \in \mathbb{U}),$$

where $h \in \mathcal{P}$,

$$H_p^{q,s}(\alpha_1)f(z) = h_p^{q,s}(\alpha_1;z)*f(z) = \left(z^{-p} + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^{n-p}}{n!}\right)*f(z) \qquad (z \in \mathbb{U}^*),$$

and

$$f_{p,k}^{q,s}(\alpha_1;z) = \frac{1}{k}\sum_{n=1}^{k-1} \varepsilon_k^{jp}(H_p^{q,s}(\alpha_1)f)(\varepsilon_k^j z) \neq 0 \qquad (z \in \mathbb{U}^*).$$

It is also easy to see that, if we set

$$\lambda = 0,\ \delta = 1, \quad \text{and} \quad g(z) = (1+\alpha)h_p^{q,s}(\alpha_1;z) + \alpha h_p^{q,s}(\alpha_1+1;z)$$

in the class $\mathcal{S}_{p,k}^{\delta}(\lambda;g;h)$, then it reduces to the class $\mathcal{F}_{p,k}^{q,s}(\alpha;\alpha_1;h)$.

1.3. Definition. A function $f \in \Sigma_p$ is said to be in the class $\mathcal{K}_{p,k}^{\delta}(\lambda;g;h)$ if it satisfies the following subordination condition:

$$-\frac{z\left(\mathcal{M}_{\lambda,g}^{\delta}f\right)'(z)}{p\varphi_{p,k}^{\delta}(\lambda;g;z)} \prec h(z) \qquad (z \in \mathbb{U}) \tag{1.7}$$

for some $\varphi \in \mathcal{S}_{p,k}^{\delta}(\lambda;g;h)$, where

$$h \in \mathcal{P} \quad \text{and} \quad \varphi_{p,k}^{\delta}(\lambda;g;z) \neq 0 \quad (z \in \mathbb{U}^*).$$

1.4. Remark. If we set

$$\lambda = 0,\ \delta = 1, \quad \text{and} \quad g(z) = \varphi_p(a,c;z)$$

in the class $\mathcal{K}_{p,k}^{\delta}(\lambda;g;h)$, then it reduces to the class $\mathcal{K}_{p,k}(a,c;h)$, which was also introduced and studied recently by Srivastava *et al.* [21].

If we set

$$\lambda = 0,\ \delta = 1, \quad \text{and} \quad g(z) = (1+\alpha)h_p^{q,s}(\alpha_1;z) + \alpha h_p^{q,s}(\alpha_1+1;z)$$

in the class $\mathcal{K}_{p,k}^{\delta}(\lambda;g;h)$, then it reduces to the class $\mathcal{G}_{p,k}^{q,s}(\alpha;\alpha_1;h)$, which was also introduced and studied recently by Wang *et al.* [22].

1.5. Definition. A function $f \in \Sigma_p$ is said to be in the class $\mathcal{H}_{p,k}^{\delta}(\alpha,\lambda;g;h)$ if it satisfies the following subordination condition:

$$-(1-\alpha)\frac{z\left(\mathcal{M}_{\lambda,g}^{\delta}f\right)'(z)}{p\varphi_{p,k}^{\delta}(\lambda;g;z)} - \alpha\frac{z\left(\mathcal{M}_{\lambda,g}^{\delta+1}f\right)'(z)}{p\varphi_{p,k}^{\delta+1}(\lambda;g;z)} \prec h(z) \qquad (z \in \mathbb{U}), \tag{1.8}$$

for some $\alpha \geqq 0$ and $\varphi \in \mathcal{S}_{p,k}^{\delta}(\lambda;g;h)$, where

$$h \in \mathcal{P} \quad \text{and} \quad \varphi_{p,k}^{\delta+1}(\lambda;g;z) \neq 0 \quad (z \in \mathbb{U}^*).$$

1.6. Remark. If we set

$$\lambda = 0,\ \delta = 1, \quad \text{and} \quad g(z) = \varphi_p(a,c;z)$$

in the class $\mathcal{H}_{p,k}^{\delta}(\lambda;g;h)$, then it reduces to the class $\mathcal{K}_{p,k}(\alpha;a,c;h)$, which was also introduced and studied recently by Srivastava *et al.* [21].

1.7. Remark. By suitably specifying the values of $p,\ k,\ \delta,\ \lambda,\ \alpha,\ g$ and h, the classes

$$\mathcal{S}_{p,k}^{\delta}(\lambda; g; h),\ \mathcal{K}_{p,k}^{\delta}(\lambda; g; h) \quad \text{and} \quad \mathcal{H}_{p,k}^{\delta}(\alpha, \lambda; g; h)$$

reduce to the various subclasses introduced and studied in [9, 10, 26, 27]. For some recent investigations on meromorphic functions, see (for example) the earlier works [1, 3, 4, 5, 6, 7, 8, 11, 14, 15, 17, 18, 23, 24, 25] and the references cited therein.

In order to establish our main results, we shall also make use of the following lemmas.

1.8. Lemma. (See [12]) *Let $\vartheta,\ \gamma \in \mathbb{C}$ with $\vartheta \neq 0$. Suppose that φ is convex and univalent in $\mathbb{U}$ with*

$$\varphi(0) = 1 \quad \textit{and} \quad \Re(\vartheta\varphi(z) + \gamma) > 0 \quad (z \in \mathbb{U}).$$

If $\mathfrak{p}$ is analytic in $\mathbb{U}$ with $\mathfrak{p}(0) = 1$, then the following subordination

$$\mathfrak{p}(z) + \frac{z\mathfrak{p}'(z)}{\vartheta\mathfrak{p}(z) + \gamma} \prec \varphi(z) \qquad (z \in \mathbb{U})$$

implies that

$$\mathfrak{p}(z) \prec \varphi(z) \qquad (z \in \mathbb{U}).$$

1.9. Lemma. (See [13]) *Let η be analytic and convex univalent in $\mathbb{U}$ and let ζ be analytic in $\mathbb{U}$ with*

$$\Re(\zeta(z)) \geqq 0 \quad (z \in \mathbb{U}).$$

If $\mathfrak{q}$ is analytic in $\mathbb{U}$ with $\mathfrak{q}(0) = \eta(0)$, then the following subordination

$$\mathfrak{q}(z) + \zeta(z)z\mathfrak{q}'(z) \prec \eta(z) \qquad (z \in \mathbb{U})$$

implies that

$$\mathfrak{q}(z) \prec \eta(z) \qquad (z \in \mathbb{U}).$$

1.10. Lemma. *Let $f \in \mathcal{S}_{p,k}^{\delta}(\lambda; g; h)$. Then*

$$-\frac{z\left(f_{p,k}^{\delta}(\lambda; g; z)\right)'}{pf_{p,k}^{\delta}(\lambda; g; z)} \prec h(z) \qquad (z \in \mathbb{U}). \tag{1.9}$$

Proof. Making use of (1.5), we have

$$\begin{aligned} f_{p,k}^{\delta}(\lambda; g; \varepsilon_k^j z) &= \frac{1}{k}\sum_{n=0}^{k-1} \varepsilon_k^{np}\mathcal{M}_{\lambda,g}^{\delta} f(\varepsilon_k^{n+j} z) \\ &= \varepsilon_k^{-jp} \cdot \frac{1}{k}\sum_{n=0}^{k-1} \varepsilon_k^{(n+j)p}\mathcal{M}_{\lambda,g}^{\delta} f(\varepsilon_k^{n+j} z) \\ &= \varepsilon_k^{-jp} f_{p,k}^{\delta}(\lambda; g; z) \qquad (j \in \{0, 1, \ldots, k-1\}), \end{aligned} \tag{1.10}$$

and

$$\left(f_{p,k}^{\delta}(\lambda; g; z)\right)' = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{j(p+1)}(\mathcal{M}_{\lambda,g}^{\delta} f)'(\varepsilon_k^j z).$$

Hence

$$
\begin{aligned}
(1.11)\qquad -\frac{z\left(f_{p,k}^{\delta}(\lambda;g;z)\right)'}{pf_{p,k}^{\delta}(\lambda;g;z)} &= \frac{1}{k}\sum_{j=0}^{k-1}\frac{\varepsilon_k^{j(p+1)}z(\mathcal{M}_{\lambda,g}^{\delta}f)'(\varepsilon_k^j z)}{-pf_{p,k}^{\delta}(\lambda;g;z)} \\
&= \frac{1}{k}\sum_{j=0}^{k-1}\frac{\varepsilon_k^{j}z(\mathcal{M}_{\lambda,g}^{\delta}f)'(\varepsilon_k^j z)}{-pf_{p,k}^{\delta}(\lambda;g;\varepsilon_k^j z)} \qquad (z\in\mathbb{U}).
\end{aligned}
$$

Moreover, since $f\in\mathcal{S}_{p,k}^{\delta}(\lambda;g;h)$, we have

$$
(1.12)\qquad -\frac{\varepsilon_k^{j}z(\mathcal{M}_{\lambda,g}^{\delta}f)'(\varepsilon_k^j z)}{pf_{p,k}^{\delta}(\lambda;g;\varepsilon_k^j z)} \prec h(z) \qquad (z\in\mathbb{U};\ j\in\{0,1,\ldots,k-1\}).
$$

Noting that h is convex and univalent in $\mathbb{U}$, from (1.11) and (1.12), we conclude that the assertion (1.9) of Lemma 1.10 holds true. $\square$

Let $\mathcal{A}$ be the class of functions of the form:

$$
f(z)=z+\sum_{n=2}^{\infty}a_nz^n \qquad (z\in\mathbb{U}).
$$

A function $f\in\mathcal{A}$ is said to be starlike of order α in $\mathbb{U}$ if it satisfies the following inequality:

$$
\Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha \qquad (z\in\mathbb{U};\ \alpha<1).
$$

We denote this class by $\mathcal{S}^*(\alpha)$. A function $f\in\mathcal{A}$ is said to be prestarlike of order α in $\mathbb{U}$ if

$$
\frac{z}{(1-\alpha)^{2-2\alpha}}*f(z)\in\mathcal{S}^*(\alpha) \qquad (z\in\mathbb{U};\ \alpha<1).
$$

We denote this class by $\mathcal{R}(\alpha)$. It is clear that a $f\in\mathcal{A}$ is in the class $\mathcal{R}(0)$ if and only if f is convex univalent in $\mathbb{U}$ and that

$$
\mathcal{R}(\frac{1}{2})=\mathcal{S}^*(\frac{1}{2}).
$$

1.11. Lemma. (See [13]) *Let $\alpha<1$, $f\in\mathcal{R}(\alpha)$ and $\phi\in\mathcal{S}^*(\alpha)$. Then, for any analytic function H in $\mathbb{U}$,*

$$
\frac{f*(\phi H)}{f*\phi}(\mathbb{U})\subset\overline{co}(H(\mathbb{U})),
$$

where $\overline{co}(H(\mathbb{U}))$ denotes the close convex hull of $H(\mathbb{U})$.

In the present paper, we aim at proving such results inclusion relationships, integral preserving and convolution properties for each of the function classes. The results presented here would provide extensions of those given in a number of earlier works. Several other new results are also obtained.

2. A Set of Inclusion Relationships

We first provide some inclusion relationships for the function classes

$$\mathcal{S}^{\delta}_{p,k}(\lambda; g; h),\ \mathcal{K}^{\delta}_{p,k}(\lambda; g; h) \quad \text{and} \quad \mathcal{H}^{\delta}_{p,k}(\alpha, \lambda; g; h)$$

which were defined in preceding section.

2.1. Theorem. *Let $h \in \mathcal{P}$ with*

(2.1) $\quad \Re(h(z)) < 1 + \dfrac{1}{\lambda} \qquad (\lambda > 0;\ z \in \mathbb{U}).$

Then

$$\mathcal{S}^{\delta+1}_{p,k}(\lambda; g; h) \subset \mathcal{S}^{\delta}_{p,k}(\lambda; g; h).$$

Proof. By using (1.4) and (1.5), we have

(2.2)

$$(1+\lambda)f^{\delta}_{p,k}(\lambda; g; z)+\frac{\lambda z}{p}\left(f^{\delta}_{p,k}(\lambda; g; z)\right)' = \frac{1}{k}\sum_{j=0}^{k-1}\varepsilon_k^{jp}\mathcal{M}^{\delta+1}_{\lambda,g}f(\varepsilon_k^{j}z) = f^{\delta+1}_{p,k}(\lambda; g; z).$$

Let $f \in \mathcal{S}^{\delta+1}_{p,k}(\lambda; g; h)$ and suppose that

(2.3) $\quad \varpi(z) = -\dfrac{z\left(f^{\delta}_{p,k}(\lambda; g; z)\right)'}{pf^{\delta}_{p,k}(\lambda; g; z)} \qquad (z \in \mathbb{U}),$

then ϖ is analytic in $\mathbb{U}$ with $\varpi(0) = 1$. It follows from (2.2) and (2.3) that

(2.4) $\quad (1-\lambda) - \lambda\varpi(z) = \dfrac{f^{\delta+1}_{p,k}(\lambda; g; z)}{f^{\delta}_{p,k}(\lambda; g; z)}.$

Differentiating both sides of (2.4) with respect to z and using (2.3), we have

(2.5) $\quad \varpi(z) + \dfrac{z\varpi'(z)}{p(1+\frac{1}{\lambda}) - p\varpi(z)} = -\dfrac{z\left(f^{\delta+1}_{p,k}(\lambda; g; z)\right)'}{pf^{\delta+1}_{p,k}(\lambda; g; z)}.$

From (2.5) and Lemma 1.10, we find that

(2.6) $\quad \varpi(z) + \dfrac{z\varpi'(z)}{p(1+\frac{1}{\lambda}) - p\varpi(z)} \prec h(z) \qquad (z \in \mathbb{U}).$

Now, in view of (2.1) and (2.6), an application of Lemma 1.8 yields

(2.7) $\quad \varpi(z) \prec h(z) \qquad (z \in \mathbb{U}).$

Set

(2.8) $\quad q(z) = -\dfrac{z(\mathcal{M}^{\delta}_{\lambda,g}f)'(z)}{pf^{\delta}_{p,k}(\lambda; g; z)},$

then q is analytic in $\mathbb{U}$ with $q(0) = 1$. We obtain from (1.4) that

(2.9) $\quad f^{\delta}_{p,k}(\lambda; g; z)q(z) = (1+\dfrac{1}{\lambda})\mathcal{M}^{\delta}_{\lambda,g}f(z) - \dfrac{1}{\lambda}\mathcal{M}^{\delta+1}_{\lambda,g}f(z).$

Differentiating both sides of (2.9) and using (2.8), we get

$$zq'(z)+\left(p(1+\frac{1}{\lambda})+\frac{z\left(f^{\delta}_{p,k}(\lambda;g;z)\right)'}{f^{\delta}_{p,k}(\lambda;g;z)}\right)q(z)=-\frac{z(\mathcal{M}^{\delta+1}_{\lambda,g}f)'(z)}{\lambda f^{\delta}_{p,k}(\lambda;g;z)}. \tag{2.10}$$

Since $f\in\mathcal{S}^{\delta+1}_{p,k}(\lambda;g;h)$, we find from (2.2), (2.3) and (2.10) that

$$q(z)+\frac{zq'(z)}{p(1+\frac{1}{\lambda})-p\varpi(z)}=-\frac{z(\mathcal{M}^{\delta+1}_{\lambda,g}f)'(z)}{pf^{\delta+1}_{p,k}(\lambda;g;z)}\prec h(z)\qquad(z\in\mathbb{U}). \tag{2.11}$$

From (2.1) and (2.7), we observe that

$$\Re\left(p(1+\frac{1}{\lambda})-p\varpi(z)\right)>0.$$

Therefore, from (2.11) and Lemma 1.9, we conclude that

$$q(z)\prec h(z)\qquad(z\in\mathbb{U}),$$

which implies $f\in\mathcal{S}^{\delta}_{p,k}(\lambda;g;h)$. The proof of Theorem 2.1 is thus completed. □

2.2. Theorem. *Let $h\in\mathcal{P}$ with*

$$\Re(h(z))<1+\frac{1}{\lambda}\qquad(\lambda>0;\ z\in\mathbb{U}). \tag{2.12}$$

Then

$$\mathcal{K}^{\delta+1}_{p,k}(\lambda;g;h)\subset\mathcal{K}^{\delta}_{p,k}(\lambda;g;h).$$

Proof. Let $f\in\mathcal{K}^{\delta+1}_{p,k}(\lambda;g;h)$, then there exists a function $\varphi\in\mathcal{S}^{\delta+1}_{p,k}(\lambda;g;h)$ such that

$$-\frac{z(\mathcal{M}^{\delta+1}_{\lambda,g}f)'(z)}{p\varphi^{\delta+1}_{p,k}(\lambda;g;z)}\prec h(z)\qquad(z\in\mathbb{U}). \tag{2.13}$$

An application of Theorem 2.1 yields $\varphi\in\mathcal{S}^{\delta}_{p,k}(\lambda;g;h)$ and Lemma 1.10 leads to

$$\psi(z)=-\frac{z\left(\varphi^{\delta}_{p,k}(\lambda;g;z)\right)'}{p\varphi^{\delta}_{p,k}(\lambda;g;z)}\prec h(z)\qquad(z\in\mathbb{U}). \tag{2.14}$$

Let

$$q(z)=-\frac{z(\mathcal{M}^{\delta}_{\lambda,g}f)'(z)}{p\varphi^{\delta}_{p,k}(\lambda;g;z)}\qquad(z\in\mathbb{U}). \tag{2.15}$$

By using (1.4), (2.15) can be written as follows:

$$\varphi^{\delta}_{p,k}(\lambda;g;z)q(z)=(1+\frac{1}{\lambda})\mathcal{M}^{\delta}_{\lambda,g}f(z)-\frac{1}{\lambda}\mathcal{M}^{\delta+1}_{\lambda,g}f(z). \tag{2.16}$$

Differentiating both sides of (2.16) and using (2.2)(with f replaced by φ), we find that

$$q(z)+\frac{zq'(z)}{p(1+\frac{1}{\lambda})-p\psi(z)}=-\frac{z(\mathcal{M}^{\delta+1}_{\lambda,g}f)'(z)}{p\varphi^{\delta+1}_{p,k}(\lambda;g;z)}\qquad(z\in\mathbb{U}). \tag{2.17}$$

Combining (2.13) and (2.17), we obtain

$$q(z)+\frac{zq'(z)}{p(1+\frac{1}{\lambda})-p\psi(z)}\prec h(z)\qquad(z\in\mathbb{U}). \tag{2.18}$$

Combining (2.12), (2.14) and (2.18), we deduce from Lemma 1.9 that

$$q(z)\prec h(z)\qquad(z\in\mathbb{U}),$$

which shows that $f\in\mathcal{K}^{\delta}_{p,k}(\lambda;g;h)$. □

By carefully selecting the function h involved in Theorem 2.1 and Theorem 2.2, we can obtain a number of useful corollaries.

2.3. Corollary. *Let* $0<\alpha\leqq 1$, $-1\leqq B<A\leqq 1$ *and*

$$h(z)=\left(\frac{1+Az}{1+Bz}\right)^{\alpha}\qquad(z\in\mathbb{U}). \tag{2.19}$$

If $\lambda>\left[\left(\frac{1+A}{1+B}\right)^{\alpha}-1\right]^{-1}$, *then*

$$\mathcal{S}^{\delta+1}_{p,k}(\lambda;g;h)\subset\mathcal{S}^{\delta}_{p,k}(\lambda;g;h)\quad\text{and}\quad\mathcal{K}^{\delta+1}_{p,k}(\lambda;g;h)\subset\mathcal{K}^{\delta}_{p,k}(\lambda;g;h).$$

Proof. The analytic function h defined by (2.19) is convex univalent in $\mathbb{U}$ (see [21]), $h(0)=1$ and $h(\mathbb{U})$ is symmetric with respect to real axis. Thus $h\in\mathcal{P}$ and

$$0<\left(\frac{1-A}{1-B}\right)^{\alpha}<\Re(h(z))<\left(\frac{1+A}{1+B}\right)^{\alpha}\qquad(z\in\mathbb{U};\ 0<\alpha\leqq 1;\ -1\leqq B<A\leqq 1).$$

Hence, by using Theorem 2.1 and 2.2, we have the corollaryllary. □

2.4. Corollary. *Let* $0<\alpha<1$ *and*

$$h(z)=1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{\alpha z}}{1-\sqrt{\alpha z}}\right)\right)^2\qquad(z\in\mathbb{U}). \tag{2.20}$$

If $\lambda>\frac{\pi^2}{2}\left(\log\left(\frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}\right)\right)^{-2}$, *then*

$$\mathcal{S}^{\delta+1}_{p,k}(\lambda;g;h)\subset\mathcal{S}^{\delta}_{p,k}(\lambda;g;h)\quad\text{and}\quad\mathcal{K}^{\delta+1}_{p,k}(\lambda;g;h)\subset\mathcal{K}^{\delta}_{p,k}(\lambda;g;h).$$

Proof. The function h defined by (2.20) is in the class $\mathcal{P}$ (see [19]) and $h(\bar{z})=\overline{h(z)}$. Therefore

$$\frac{1}{2}<h(-1)<\Re(h(z))<h(1)=1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}\right)\right)^2\qquad(z\in\mathbb{U};\ 0<\alpha<1).$$

Hence, by using Theorem 2.1 and 2.2, we have the corollary. □

2.5. Theorem. *Let* $h\in\mathcal{P}$ *with*

$$\Re(h(z))<1+\frac{1}{\lambda}\qquad(\lambda>0;\ z\in\mathbb{U}). \tag{2.21}$$

Then

$$\mathcal{H}^{\delta}_{p,k}(\alpha_2,\lambda;g;h)\subset\mathcal{H}^{\delta}_{p,k}(\alpha_1,\lambda;g;h)\qquad(0\leqq\alpha_1<\alpha_2).$$

Proof. For $f \in \mathcal{H}^{\delta}_{p,k}(\alpha_2, \lambda; g; h)$, there exists a function $\varphi \in \mathcal{S}^{\delta}_{p,k}(\lambda; g; h)$ satisfying $\varphi^{\delta+1}_{p,k}(\lambda; g; z) \neq 0$ such that

$$-(1-\alpha_2)\frac{z(\mathcal{M}^{\delta}_{\lambda,g}f)'(z)}{p\varphi^{\delta}_{p,k}(\lambda; g; z)} - \alpha_2\frac{z(\mathcal{M}^{\delta+1}_{\lambda,g}f)'(z)}{p\varphi^{\delta+1}_{p,k}(\lambda; g; z)} \prec h(z) \qquad (z \in \mathbb{U}). \tag{2.22}$$

Put

$$q(z) = -\frac{z(\mathcal{M}^{\delta}_{\lambda,g}f)'(z)}{p\varphi^{\delta}_{p,k}(\lambda; g; z)} \qquad (z \in \mathbb{U}).$$

Since $\varphi \in \mathcal{S}^{\delta}_{p,k}(\lambda; g; h)$, it follows from (2.14) to (2.17)(using in the proof of Theorem 2.2) and (2.22) that

(2.23)

$$q(z)+\frac{\alpha_2 zq'(z)}{p(1+\frac{1}{\lambda}) - p\psi(z)} = -(1-\alpha_2)\frac{z(\mathcal{M}^{\delta}_{\lambda,g}f)'(z)}{p\varphi^{\delta}_{p,k}(\lambda; g; z)} - \alpha_2\frac{z(\mathcal{M}^{\delta+1}_{\lambda,g}f)'(z)}{p\varphi^{\delta+1}_{p,k}(\lambda; g; z)} \prec h(z) \qquad (z \in \mathbb{U}).$$

In light of (2.14) and (2.21), we thus observe that

$$\frac{1}{\alpha_2}\Re\left(p(1+\frac{1}{\lambda} - p\psi(z)\right) > 0 \qquad (z \in \mathbb{U}).$$

Hence, by (2.23) and Lemma 1.9, we have

$$q(z) \prec h(z) \qquad (z \in \mathbb{U}). \tag{2.24}$$

Since $0 \leqq \frac{\alpha_1}{\alpha_2} < 1$ and h is convex univalent in $\mathbb{U}$, we deduce from (2.22) and (2.24) that

$$\begin{aligned} &-(1-\alpha_1)\frac{z(\mathcal{M}^{\delta}_{\lambda,g}f)'(z)}{p\varphi^{\delta}_{p,k}(\lambda; g; z)} - \alpha_1\frac{z(\mathcal{M}^{\delta+1}_{\lambda,g}f)'(z)}{p\varphi^{\delta+1}_{p,k}(\lambda; g; z)} = \left(1 - \frac{\alpha_1}{\alpha_2}\right)q(z) \\ &+ \frac{\alpha_1}{\alpha_2}\left(-(1-\alpha_2)\frac{z(\mathcal{M}^{\delta}_{\lambda,g}f)'(z)}{p\varphi^{\delta}_{p,k}(\lambda; g; z)} - \alpha_2\frac{z(\mathcal{M}^{\delta+1}_{\lambda,g}f)'(z)}{p\varphi^{\delta+1}_{p,k}(\lambda; g; z)}\right) \prec h(z) \qquad (z \in \mathbb{U}). \end{aligned} \tag{2.25}$$

Thus $f \in \mathcal{H}^{\delta}_{p,k}(\alpha_1, \lambda; g; h)$. The proof of Theorem 2.5 is evidently completed. $\square$

3. Integral Preserving Properties

In this section, we prove some integral preserving properties of the subclasses

$$\mathcal{S}^{\delta}_{p,k}(\lambda; g; h) \quad \text{and} \quad \mathcal{K}^{\delta}_{p,k}(\lambda; g; h).$$

3.1. Theorem. *Let $h \in \mathcal{P}$ with*

$$\Re(h(z)) < \frac{\Re(c)}{p} \qquad (z \in \mathbb{U};\ \Re(c) > p). \tag{3.1}$$

If $f \in \mathcal{S}^{\delta}_{p,k}(\lambda; g; h)$, then the function defined by

$$F(z) = \frac{c-p}{z^c}\int_0^z t^{c-1}f(t)dt \tag{3.2}$$

is also in the class $\mathcal{S}^{\delta}_{p,k}(\lambda; g; h)$, provided that

$$F^{\delta}_{p,k}(\lambda; g; z) \neq 0 \qquad (z \in \mathbb{U}^*).$$

Proof. Let $f \in \mathcal{S}_{p,k}^{\delta}(\lambda; g; h))$, we then find from (3.2) that

$$c\mathcal{M}_{\lambda,g}^{\delta} F(z) + z(\mathcal{M}_{\lambda,g}^{\delta} F)'(z) = (c-p)\mathcal{M}_{\lambda,g}^{\delta} f(z). \tag{3.3}$$

By using (3.3), we get

$$cF_{p,k}^{\delta}(\lambda; g; z) + z\left(F_{p,k}^{\delta}(\lambda; g; z)\right)' = (c-p) f_{p,k}^{\delta}(\lambda; g; z). \tag{3.4}$$

Let

$$\chi(z) = -\frac{z\left(F_{p,k}^{\delta}(\lambda; g; z)\right)'}{pF_{p,k}^{\delta}(\lambda; g; z)}.$$

Then χ is analytic in $\mathbb{U}$, with $\chi(0) = 1$, and from (3.4) we observe that

$$c - p\chi(z) = (c-p)\frac{f_{p,k}^{\delta}(\lambda; g; z)}{F_{p,k}^{\delta}(\lambda; g; z)}. \tag{3.5}$$

Differentiating both sides of (3.5) with respect to z and using Lemma 1.10, we obtain

$$\chi(z) + \frac{z\chi'(z)}{c - p\chi(z)} = -\frac{z\left(f_{p,k}^{\delta}(\lambda; g; z)\right)'}{pf_{p,k}^{\delta}(\lambda; g; z)} \prec h(z). \tag{3.6}$$

In view of (3.6), Lemma 1.9 leads to $\chi(z) \prec h(z)$. If we let

$$\kappa(z) = -\frac{z(\mathcal{M}_{\lambda,g}^{\delta} F)'(z)}{pF_{p,k}^{\delta}(\lambda; g; z)}, \tag{3.7}$$

then κ is analytic in $\mathbb{U}$ with $\kappa(0) = 1$. It follows from (3.3) that

$$F_{p,k}^{\delta}(\lambda; g; z)\kappa(z) = -\frac{c-p}{p}\mathcal{M}_{\lambda,g}^{\delta} f(z) + \frac{c}{p}\mathcal{M}_{\lambda,g}^{\delta} F(z). \tag{3.8}$$

Differentiating both sides of (3.8) and using (3.7), we get

$$z\kappa'(z) + (c - p\chi(z))\kappa(z) = (c-p)\frac{z(\mathcal{M}_{\lambda,g}^{\delta} f)'(z)}{-pF_{p,k}^{\delta}(\lambda; g; z)}. \tag{3.9}$$

Since $f \in \mathcal{S}_{p,k}^{\delta}(\lambda; g; h))$, from (3.5) and (3.9), we deduce that

$$\kappa(z) + \frac{z\kappa'(z)}{c - p\chi(z)} = \frac{c-p}{c - p\chi(z)}\frac{z(\mathcal{M}_{\lambda,g}^{\delta} f)'(z)}{-pF_{p,k}^{\delta}(\lambda; g; z)} = -\frac{z(\mathcal{M}_{\lambda,g}^{\delta} f)'(z)}{pf_{p,k}^{\delta}(\lambda; g; z)} \prec h(z). \tag{3.10}$$

Combining $\Re(h(z)) < \frac{\Re(c)}{p}$ and $\chi(z) \prec h(z)$, we find that

$$\Re(c - p\chi(z)) > 0.$$

Therefore, from (3.10) and Lemma 1.9, we have $\kappa(z) \prec h(z)$, which implies that $F \in \mathcal{S}_{p,k}^{\delta}(\lambda; g; h))$. □

By arguments similar to those used in the proofs of Theorems 2.2 and 3.1, the following result can be proved. We omit the details involved.

3.2. Corollary. *Let $h \in \mathcal{P}$ with*

$$\Re(h(z)) < \frac{\Re(c)}{p} \qquad (z \in \mathbb{U}).$$

If $f \in \mathcal{K}_{p,k}^{\delta}(\lambda; g; h)$ with $\varphi \in \mathcal{S}_{p,k}^{\delta}(\lambda; g; h)$, then the function

$$F(z) = \frac{c-p}{z^c} \int_0^z t^{c-1} f(t) dt$$

belongs to the class $\mathcal{K}_{p,k}^{\delta}(\lambda; g; h)$ with

$$G(z) = \frac{c-p}{z^c} \int_0^z t^{c-1} \varphi(t) dt$$

provided that $G_{p,k}^{\delta}(\lambda; g; z) \neq 0$ $(z \in \mathbb{U}^*)$.

4. Convolution Properties

At last, we prove the convolution properties associated with the function classes

$$\mathcal{S}_{p,k}^{\delta}(\lambda; g; h) \quad \text{and} \quad \mathcal{K}_{p,k}^{\delta}(\lambda; g; h).$$

4.1. Theorem. *Let $h \in \mathcal{P}$ with*

(4.1) $$\Re(h(z)) < 1 + \frac{1-\alpha}{p} \qquad (z \in \mathbb{U};\ \alpha < 1).$$

If $f \in \mathcal{S}_{p,k}^{\delta}(\lambda; g; h)$,

(4.2) $$\phi \in \Sigma_p \quad \text{and} \quad z^{p+1}\phi(z) \in \mathcal{R}(\alpha).$$

Then

$$f * \phi \in \mathcal{S}_{p,k}^{\delta}(\lambda; g; h).$$

Proof. Let $f \in \mathcal{S}_{p,k}^{\delta}(\lambda; g; h))$ and suppose that

(4.3) $$\rho(z) = z^{p+1} f_{p,k}^{\delta}(\lambda; g; z) \qquad (z \in \mathbb{U}).$$

Then $\rho \in \mathcal{A}$ and

(4.4) $$H(z) := -\frac{z(\mathcal{M}_{\lambda,g}^{\delta} f)'(z)}{p f_{p,k}^{\delta}(\lambda; g; z)} \prec h(z) \qquad (z \in \mathbb{U}).$$

By using Lemma 1.10, we find that

(4.5) $$\frac{z\rho'(z)}{\rho(z)} = p + 1 + \frac{z\left(f_{p,k}^{\delta}(\lambda; g; z)\right)'}{f_{p,k}^{\delta}(\lambda; g; z)} \prec p + 1 - ph(z) \qquad (z \in \mathbb{U}).$$

In view of (4.1) and (4.5), we have

(4.6) $$\Re\left(\frac{z\rho'(z)}{\rho(z)}\right) > \alpha,$$

that is, that

$$\rho \in \mathcal{S}^*(\alpha).$$

For $\phi \in \Sigma_p$, it is easy to verify that

(4.7)

$$z^{p+1}\left(\mathcal{M}_{\lambda,g}^{\delta}(f*\phi)(\varepsilon_k^j z)\right)=(z^{p+1}\phi(z))*\left(z^{p+1}\mathcal{M}_{\lambda,g}^{\delta}f(\varepsilon_k^j z)\right)\qquad (j\in\{0,1,\dots,k-1\})$$

and

$$z^{p+2}\mathcal{M}_{\lambda,g}^{\delta}(f*\phi)'(z)=(z^{p+1}\phi(z))*\left(z^{p+2}(\mathcal{M}_{\lambda,g}^{\delta}f)'(z)\right). \tag{4.8}$$

Making use of (4.3), (4.4), (4.7) and (4.8), we find that

$$\begin{aligned}\Phi(z):=\frac{z\mathcal{M}_{\lambda,g}^{\delta}(f*\phi)'(z)}{\frac{p}{k}\sum_{j=0}^{k-1}\varepsilon_k^{jp}\mathcal{M}_{\lambda,g}^{\delta}(f*\phi)(\varepsilon_k^j z)}&=-\frac{(z^{p+1}\phi(z))*\left(z^{p+2}(\mathcal{M}_{\lambda,g}^{\delta}f)'(z)\right)}{p(z^{p+1}\phi(z))*\left(z^{p+1}f_{p,k}^{\delta}(\lambda;g;z)\right)}\\ &=\frac{(z^{p+1}\phi(z))*(\rho(z)H(z))}{(z^{p+1}\phi(z))*(\rho(z))}\qquad (z\in\mathbb{U}).\end{aligned} \tag{4.9}$$

Since h is convex univalent in $\mathbb{U}$, it follows from (4.2), (4.4), (4.6), (4.9) and Lemma 1.11 that

$$\Phi(z)\prec h(z)\qquad (z\in\mathbb{U}).$$

Hence

$$f*\phi\in\mathcal{S}_{p,k}^{\delta}(\lambda;g;h).$$

□

By similarly applying the method of proof of Theorem 4.1, we can get the following result.

4.2. Corollary. *Let $h\in\mathcal{P}$ with*

$$\Re(h(z))<1+\frac{1-\alpha}{p}\qquad (z\in\mathbb{U};\ \alpha<1).$$

If $f\in\mathcal{K}_{p,k}^{\delta}(\lambda;g;h)$ with $\varphi\in\mathcal{S}_{p,k}^{\delta}(\lambda;g;h)$,

$$\phi\in\Sigma_p\quad\text{and}\quad z^{p+1}\phi(z)\in\mathcal{R}(\alpha).$$

Then

$$f*\phi\in\mathcal{K}_{p,k}^{\delta}(\lambda;g;h).$$

Acknowledgements. The present investigation was supported by the *National Natural Science Foundation* under Grant 11371126 of the People's Republic of China.

References

[1] Aouf, M.K. *Certain subclasses of meromorphically multivalent functions associated with generalized hypergeometric function*, Computers and Mathematics with Applications **55**, 494–509, 2008.

[2] Aouf, M.K. Shamandy, A. Mostafa, A.O. and Madian, S.M. *Properties of some families of meromorphic p-valent functions involving certain differetial operator*, Acta Universitatis Aplulensis **20**, 7–16, 2009.

[3] Cho, N.E. Kwon, O.S. Owa, S. and Srivastava, H.M. *A class of integral operators preserving subordination and superordination for meromorphic functions*, Applied Mathematics and Computation **193**, 463–474, 2007.

[4] Cho, N.E. Kwon, O.S. and Srivastava, H.M. *Inclusion and argument properties for certain subclasses of meromorphic functions associated with a family of multiplier transformations*, Journal of Mathematical Analysis and Applications **300**, 505–520, 2004.

[5] Cho, N.E. and Noor, K.I. *Inclusion properties for certain classes of meromorphic functions associated with Cho-Saigo-Srivastava operator*, Journal of Mathematical Analysis and Applications **320**, 779–786, 2006.

[6] El-Ashwah, R.M. and Aouf, M.K. *Hadamard product of certain meromorphic starlike and convex functions*, Computers and Mathematics with Applications **57**, 1102–1106, 2009.

[7] Jin-Lin, L. and Owa, S. *On a class of meromorphic p-valent starlike functions involving certain linear operators*, International Journal of Mathematics and Mathematical Sciences **32**, 271–280, 2002.

[8] Jin-Lin, L. and Owa, S. *Some families of meromorphic multivalent functions involving certain linear operators*, Indian Journal of Mathematics **46**, 47–62, 2004.

[9] Jin-Lin, L. and Srivastava, H.M. *A linear operator and associated families of meromorphically multivalent functions*, Journal of Mathematical Analysis and Applications **259**, 566–581, 2001.

[10] Jin-Lin, L. and Srivastava, H.M. *Classes of meromorphically multivalent functions associated with the generalized hypergeometric function*, Mathematical and Computer Modelling **39**, 21–34, 2004.

[11] Mohd, M.H. Ali, R.M. Keong, L.S. and Ravichandran, V. *Subclasses of meromorphic functions associated with convolution*, Journal of Inequalities and Applications, Article ID 190291, pp. 1–10, 2009.

[12] Miller,S.S. and Mocanou, P.T. *On some classes of first order differential subordinations*, Michigan Mathematical Journal **32**, 185–195, 1985.

[13] Miller,S.S. and Mocanou, P.T. *Differential subordinations and inequalities in the complex plane*, Journal of Differential Equations **67**, 199–211, 1987.

[14] Nunokawa, M. and Ahuja, O.P. *On meromorphic starlike and convex functions*, Indian Journal of Pure Applied Mathematics **32**, 1027–1032, 2001.

[15] Piejko, K. and Sokól, J. *Subclasses of meromorphic functions associated with the Cho-Kwon-Srivastava operator*, Journal of Mathematical Analysis and Applications **337**, 1261–1266, 2008.

[16] Roning, F. *Uniformly convex functions and a corresponding class of starlike functions*, Proceedings of American Mathematical Society **118**, 189–196, 1993.

[17] Raina, R.K. and Srivastava, H.M. *A new class of meromorphically multivalent functions with applications to generalized hypergeometric functions*, Mathematical and Computer Modelling **43**, 350–356, 2006.

[18] Srivastava, H.M. and Patel, J. *Certain subclasses of meromorphically multivalent functions involving a family of linear operators*, Southeast Asian Bulletin of Mathematics **30**, 123–140, 2006.

[19] Ruscheweyh, S. *Convolutions in Geometric Function Theory: Fundamental Theories of Physics*, vol. 83 of Seminaire de Mathematiques Superieures, Presses de Universite de Montreal, Montreal, Canada, 1982.

[20] Selvaraj, C. and Selvakumaran, K.A. *On certain classes of multivalent functions involving a generalized differential operator*, Bulletin of the Korean Mathematical Society **46**, 905–915, 2009.

[21] Srivastava, H.M. Ding-Gong, Y. and Neng, X. *Some subclasses of meromorphically multivalent functions associated with a linear operator*, Applied Mathematics and Computation **195**, 11–23, 2008.

[22] Zhi-Gang, W. Yue-Ping, J. and Srivastava, H.M. *Some subclasses of meromorphically multivalent functions associated with the generalized hypergeometric function*, Computers and Mathematics with Applications **57**, 571–586, 2009.

[23] Zhi-Gang, W. Yong, S. and Zhi-Hua, Z. *Certain classes of meromorphic multivalent functions*, Computers and Mathematics with Applications **57**, 1408–1417, 2009.

[24] Ding-Gong, Y. *Certain convolution operators for meromorphic functions*, Southeast Asian Bulletin of Mathematics **25**, 175–186, 2001.

[25] Shao-Mou, Y. Zai-Ming L. and Srivastava, H.M. *Some inclusion relationships and integral-preserving properties of certain subclasses of meromorphic functions associated with a family of integral operators*, Journal of Mathematical Analysis and Applications **337**, 505–515, 2008.

[26] Zhong-Zhu Z. and Zhuo-Ren W. *On meromorphically starlike functins and functions meromorphically starlike with respect to symmetric conjugate points*, Journal of Mathematical Analysis and Applications **261**, 17–27, 2001.

[27] Zhong-Zhu Z. and Zhuo-Ren W. *On functions meromorphically starlike with respect to symmetric points*, Journal of Mathematical Research and Exposition **23**, 71–76, 2003.