

Remarks on generalized quantum gates

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Abstract

In this paper, we give a characterization of generalized quantum gates. We also show that many important operators are generalized quantum gates, moreover, some of these operators can be represented as the convex combination of only two unitary operators. Our results answer what kinds of operations a duality quantum computing admits. We point out that the set of all generalized quantum gates coincides with the set of all restricted allowable generalized quantum gates. Thus, our results are also valid for restricted allowable generalized quantum gates.

Keywords: Duality quantum computer, generalized quantum gate, isometry.

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1. Introduction

Quantum gate, which is represented by unitary operator mathematically, is a fundamental tool in designing quantum circuits and quantum algorithms in quantum computers. Quantum computations are just executed by a series of quantum gates.

In [1], Professor Long proposed a new type of computing machine, the duality quantum computer, which exploits the wave-particle duality of quantum systems. A quantum wave can be decomposed into parts using slits or beam splitters, for example. The subwaves can move along separate paths and then be combined at which point they interfere. Therefore, one can perform different gate operations at different paths, a property called the duality parallelism. This enables us to perform computation using not only products of unitary operations, but also linear combinations of unitary operations, which was called the duality gates or the generalized quantum gates ([1]). In this sense, the duality computer offers additional

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capability in information processing which is superior to quantum parallelism in ordinary quantum computer.

In duality quantum computer, the quantum wave divider operation and quantum wave combiner operation are allowed ([1]). In such a picture, the duality gate, or generalized quantum gate, can be written as $\sum_{i=0}^{d-1} p_i U_i$, where U_i are unitary operators, d is the number of slits that the duality quantum computer passes, and p_i is the probability that the duality quantum computer passes through the i -th slit, and $\sum_i p_i = 1$. The quantum wave divider operation and quantum wave combiner operation and the properties of generalized quantum gates were discussed in [2]-[6]. In [7], a necessary condition for a contraction to be a generalized quantum gate was given. Furthermore, using spectral resolution, a characterization that a contraction is not a generalized quantum gate was given in [8].

In [9], the allowable generalized quantum gates were introduced which is of the form $\sum_{i=0}^{d-1} c_i U_i$, where U_i are unitary operators, c_i are complex numbers with module less than or equal to 1 and constrained by $|\sum_{i=0}^{d-1} c_i| \leq 1$. In [10], the authors discussed the realization of allowable generalized quantum gates. In [11], the restricted allowable generalized quantum gates were considered satisfying $0 < \sum_{i=0}^{d-1} |c_i| \leq 1$. The connections with other problems and some further generalizations and applications are also investigated in [12]-[13]. For the review of the development of duality quantum computing, duality quantum information processing and duality quantum communication during the past few years and related work, please refer to [14]-[16].

In this paper, we give a characterization of generalized quantum gates. We also show that many important operators are generalized quantum gates, moreover, some of these operators can be represented as the convex combination of only two unitary operators. Our results answer what kinds of operations a duality computing admits. We point out that the set of all generalized quantum gates coincides with the set of all restricted allowable generalized quantum gates. Thus, our results are also valid for restricted allowable generalized quantum gates.

2. Preliminaries

Let H be a separable complex Hilbert space, $B(H)$ the set of all bounded linear operators on H , $G(H)$ the set of all generalized quantum gates on H , $A \in B(H)$. Denote the range and the null space of A by $R(A)$ and $N(A)$, respectively. If $\|A\| \leq 1$, then A is said to be contractive. The set of all contractions in $B(H)$ is denoted by $B(H)_1$. If p is a positive integer and $(A^*A)^p - (AA^*)^p$ is a positive operator, then $A \in B(H)$ is called a p -hyponormal operator. Specifically, when $p = 1$, A is called a hyponormal operator. The set of all normal operators, inverse operators, finite rank operators and compact operators on H is denoted by $N(H)$, $Inv(H)$, $F(H)$ and $K(H)$, respectively.

Let $A \in B(H)$, if for each $x \in N(A)^\perp$, we have $\|Ax\| = \|x\|$, then A is called a partial isometry, where $N(A)^\perp$ is called the initial space of A , and $R(A)$ the final space. If $N(A) = 0$, then A is called an isometry. A surjective isometry is a unitary operator.

The following lemma is a well-known result for partial isometry.

2.1. Lemma ([17]). *For $U \in B(H)$, the following statements are equivalent:*

- (1) U is a partial isometry;
- (2) U^* is a partial isometry;
- (3) U^*U is the orthogonal projection on $N(U)^\perp$;
- (4) UU^* is the orthogonal projection on $R(U)$.

If $A \in B(H)$, the ascent $asc(A)$ of A is defined to be the smallest nonnegative integer k (if it exists) which satisfies that $N(A^k) = N(A^{k+1})$. If such k does not exist, then the ascent of A is defined as infinity. Similarly, the descent $des(A)$ of A is defined as the smallest nonnegative integer k (if it exists) for which $R(A^k) = R(A^{k+1})$ holds. If such k does not exist, then $des(A)$ is defined as infinity, too. If the ascent and the descent of T are finite, then they are equal ([18]).

$A \in B(H)$ is said to be semi-Fredholm if the range $R(A)$ is closed and at least one of $\dim N(A)$ and $\dim N(A^*)$ is finite, and the Fredholm index $\text{ind}(A)$ of A is defined by $\text{ind}(A) = \dim N(A) - \dim N(A^*)$. $A \in B(H)$ is said to be Fredholm if A is semi-Fredholm and $-\infty < \text{ind}(A) < \infty$.

Let $A \in B(H)$. For each positive integer n , define A_n to be the restriction of A to $R(A^n)$. If there is a positive integer n_0 such that $R(A_{n_0})$ is closed and A_{n_0} is Fredholm, then A is called B-Fredholm. It follows from [19] that if A is B-Fredholm and n_0 satisfies the above properties, then A_m is Fredholm and $\text{ind}(A_m) = \text{ind}(A_n)$ for all $m \geq n_0$. Thus, we can define the index of a B-Fredholm operator A as the index of the Fredholm operator A_n , where n is any positive integer such that $R(A_n)$ is closed and A_n is a Fredholm operator.

We first introduce the following important operator classes ([18], [20]-[23]).

2.2. Definition. (i) $T \in B(H)$ is called Browder, if T is Fredholm operator and $asc(T) = des(T) < \infty$.

(ii) $T \in B(H)$ is said to be Drazin invertible, if there exists $T^D \in B(H)$ such that $TT^D = T^D T, T^D T T^D = T^D, T^{k+1} T^D = T^k$ for some nonnegative integer k .

(iii) $T \in B(H)$ is said to be generalized Drazin invertible, if there exist $A, B \in B(H)$ such that B is quasi-nilpotent and $TA = AT, ATA = A, TAT = A + B$.

(iv) $T \in B(H)$ is called Weyl, if T is Fredholm with index 0.

(v) $T \in B(H)$ is called B-Weyl, if T is a B-Fredholm with index 0.

The above operators have the following characterizations ([18],[20]-[23]).

2.3. Lemma. (i) $T \in B(H)$ is Browder iff $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 is nilpotent on some finite dimensional space.

(ii) $T \in B(H)$ is Drazin invertible iff $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 is nilpotent.

(iii) $T \in B(H)$ is generalized Drazin invertible iff $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 is quasi-nilpotent.

(iv) $T \in B(H)$ is B-Fredholm iff $T = T_1 \oplus T_2$, where T_1 is Fredholm and T_2 is nilpotent.

(v) $T \in B(H)$ is B-Weyl iff $T = T_1 \oplus T_2$, where T_1 is Fredholm with index 0 and T_2 is nilpotent.

In [2], Gudder essentially proved the following result.

2.4. Theorem ([2]). *If $\dim H < \infty$, then $G(H) = B(H)_1$.*

In [7], Wang, Du and Dou proved the following theorem.

2.5. Theorem ([7]). *If $A \in B(H)_1$ is a finite-rank perturbation of a semi-Fredholm partial isometry with $\text{ind}(A) \neq 0$, then $A \notin G(H)$.*

It follows from this result that Theorem 2.4 does not hold when $\dim H = \infty$. But we have the following result.

2.6. Lemma ([24]). *If $A \in B(H)$ and $\|A\| < 1 - \frac{2}{n}$ for some $n > 2$, then there exist unitary operators U_1, U_2, \dots, U_n such that*

$$A = \frac{1}{n}(U_1 + U_2 + \dots + U_n).$$

Denote $B(H)_1^\circ = \{A \in B(H) : \|A\| < 1\}$. Then from the above lemma, the following result follows.

2.7. Theorem ([7]). $B(H)_1^\circ \subseteq G(H)$.

By using spectral resolution, Du and Dou in [8] established the necessary and sufficient conditions for $A \notin G(H)$ when $A \in B(H)_1$. As a corollary, they obtained the following result.

2.8. Theorem ([8]). *If $A \in B(H)_1$, then A is not a generalized quantum gate if and only if A is a semi-Fredholm with $\text{ind}(A) \neq 0$ and A is a compact perturbation of a partial isometry.*

3. Main Results

In this section, we give a characterization of generalized quantum gates and show that many important operators are generalized quantum gates.

First, we need the following important lemma.

3.1. Lemma ([8]). *Let $A \in B(H)_1$. Then there exist two unitary operators U_1 and U_2 in $B(H)$ such that*

$$A = \frac{1}{2}(U_1 + U_2)$$

if and only if $\dim N(A) = \dim N(A^)$.*

3.2. Remark. The above lemma shows that each contractive normal operator is a generalized quantum gate. However, the conclusion does not hold for contractive p -hyponormal operator. In fact, if A is the forward shift on H , then it is easy to check that A is a contractive p -hyponormal operator, but not a generalized quantum gate.

3.3. Proposition. *If $A, B \in G(H)$, then $AB \in G(H)$ and $A \oplus B \in G(H \oplus H)$.*

Proof. It follows from the fact $A, B \in G(H)$ that there exist unitary operators $U_1, \dots, U_n, V_1, \dots, V_m$ such that

$$(3.1) \quad A = \sum_{i=1}^n p_i U_i, \quad B = \sum_{j=1}^m q_j V_j,$$

where $p_i > 0, i = 1, 2, \dots, n, q_j > 0, j = 1, 2, \dots, m, \sum_{i=1}^n p_i = \sum_{j=1}^m q_j = 1$. Thus,

$$AB = \sum_{i=1}^n p_i U_i \sum_{j=1}^m q_j V_j = \sum_{i=1}^n \sum_{j=1}^m p_i q_j U_i V_j,$$

$$A \oplus B = \sum_{i=1}^n p_i U_i \oplus \sum_{j=1}^m q_j V_j = \sum_{j=1}^m \sum_{i=1}^n p_i q_j (U_i \oplus V_j).$$

Noting that $\sum_{i=1}^n \sum_{j=1}^m p_i q_j = 1$ and the facts that the product as well as the direct sum of unitary operators are still unitary operators, we obtain $AB \in G(H)$, and $A \oplus B \in G(H \oplus H)$. \square

3.4. Remark. The following two examples show that the converse of Proposition 3.3 does not hold.

3.5. Example. Let A, B be operators on H defined by

$$A : H \rightarrow H, \quad \{z_1, z_2, z_3, \dots\} \mapsto \{z_1, \frac{1}{2}z_2, \frac{1}{2}z_3, \dots\},$$

$$B : H \rightarrow H, \quad \{z_1, z_2, z_3, \dots\} \mapsto \{0, z_1, z_2, \dots\}.$$

Then

$$AB : H \rightarrow H, \quad \{z_1, z_2, z_3, \dots\} \mapsto \{0, \frac{1}{2}z_1, \frac{1}{2}z_2, \frac{1}{2}z_3, \dots\}.$$

Note that $A \in Inv(H)$, $\|A\| = 1, \|AB\| = \frac{1}{2}$. Then it follows from Theorem 2.7 and Lemma 3.1 that $A, AB \in G(H)$, but by Theorem 2.8 we obtain that $B \notin G(H)$.

3.6. Example. Let A be the forward shift on H , that is,

$$A : H \rightarrow H, \quad \{z_1, z_2, z_3, \dots\} \mapsto \{0, z_1, z_2, \dots\},$$

$B = A^*$, and $C = I - AB$. Then

$$A \oplus B = \frac{1}{2} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} + \frac{1}{2} \begin{pmatrix} A & -C \\ 0 & B \end{pmatrix}.$$

Note that $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} A & -C \\ 0 & B \end{pmatrix}$ are both unitary operators. Then it follows from Proposition 3.3 that $A \oplus B \in G(H \oplus H)$. But it is easy to see that neither A nor B is in $G(H)$.

Now, we give the following characterization of generalized quantum gates.

3.7. Theorem. *Let $A \in B(H)_1$. Then $A \in G(H) \iff A^n \in G(H)$ for each positive integer n .*

Proof. The necessity follows from Proposition 3.3 immediately. Sufficiency. Firstly, we show that if $A \notin G(H)$ and $\text{ind}(A) < 0$, then A can be represented as the compact perturbation of an isometry.

In fact, it follows from Theorem 2.8 that $A = U + K$, where U is a partial isometry and K is compact. Noting that $\text{ind}(U) = \text{ind}(U + K) = \text{ind}(A) < 0$, we have $\dim N(U) < \infty$.

If the operator U is written in the following form with respect to the space decomposition $H = N(U)^\perp \oplus N(U)$:

$$U = \begin{pmatrix} U_1 & 0 \\ U_2 & 0 \end{pmatrix},$$

then

$$U^*U = \begin{pmatrix} U_1^* & U_2^* \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_1 & 0 \\ U_2 & 0 \end{pmatrix} = \begin{pmatrix} U_1^*U_1 + U_2^*U_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows from the fact that U is a partial isometry and Lemma 2.1 that U^*U is the orthogonal projection on $N(U)^\perp$. Therefore, $U_1^*U_1 + U_2^*U_2 = I_{N(U)^\perp}$. Moreover, it follows from $\|U^*U\| = 1$ that $\|U_2^*U_2\| \leq 1$. Also, note that $\dim N(U) < \infty$. So $U_2^*U_2$ is a positive contractive finite rank operator. Denote

$$U_2^*U_2 = \sum_{i=1}^k a_i |\alpha_i\rangle\langle\alpha_i|,$$

where $0 < a_i \leq 1$ and $\{|\alpha_i\rangle\}_{i=1}^k$ is the orthogonal set in $N(U)^\perp$. If we extend $\{|\alpha_i\rangle\}$ into an orthonormal basis $\{|\alpha_1\rangle, |\alpha_2\rangle, \dots, |\alpha_k\rangle, |\beta_1\rangle, |\beta_2\rangle, \dots\}$ of $N(U)^\perp$, then we have

$$\begin{aligned} U_1^*U_1 &= I_{N(U)^\perp} - U_2^*U_2 = I_{N(U)^\perp} - \sum_{i=1}^k a_i |\alpha_i\rangle\langle\alpha_i| \\ &= \sum_j |\beta_j\rangle\langle\beta_j| + \sum_{i=1}^k (1 - a_i) |\alpha_i\rangle\langle\alpha_i|. \end{aligned}$$

Moreover, it is easy to see that

$$\begin{aligned} \langle\beta_j|U_1^*U_1|\beta_j\rangle &= \langle\beta_j|\beta_j\rangle = 1, j = 1, 2, \dots, \\ \langle\alpha_i|U_1^*U_1|\alpha_i\rangle &= 1 - a_i, i = 1, 2, \dots, k. \end{aligned}$$

If $0 < a_i < 1, i = 1, 2, \dots, k$, take $b_i = \sqrt{\frac{1}{1-a_i}} - 1, i = 1, 2, \dots, k$, and let

$$\begin{aligned} \tilde{U}_1|\alpha_i\rangle &= b_i U_1|\alpha_i\rangle, i = 1, 2, \dots, k, \\ \tilde{U}_1|\beta_j\rangle &= 0, j = 1, 2, \dots. \end{aligned}$$

Then it is easy to see that

$$(3.2) \quad \begin{aligned} \langle\alpha_i|(U_1 + \tilde{U}_1)^*(U_1 + \tilde{U}_1)|\alpha_i\rangle &= (1 + b_i)^2 \langle\alpha_i|U_1^*U_1|\alpha_i\rangle \\ &= (1 + b_i)^2(1 - a_i) = 1, \quad i = 1, 2, \dots, k, \end{aligned}$$

and

$$(3.3) \quad \langle\beta_j|(U_1 + \tilde{U}_1)^*(U_1 + \tilde{U}_1)|\beta_j\rangle = \langle\beta_j|U_1^*U_1|\beta_j\rangle = 1, j = 1, 2, \dots.$$

If there exists some i , such that $a_i = 1$, for example, assume that $a_m = 1, 0 < a_i < 1, i = 1, 2, \dots, k, i \neq m$. Take $b_i = \sqrt{\frac{1}{1-a_i}} - 1, i = 1, 2, \dots, k, i \neq m$, and let

$$\begin{aligned} \tilde{U}_1|\alpha_m\rangle &= |\alpha_m\rangle, \quad \tilde{U}_1|\alpha_i\rangle = b_i U_1|\alpha_i\rangle, i = 1, 2, \dots, k, i \neq m, \\ \tilde{U}_1|\beta_j\rangle &= 0, j = 1, 2, \dots. \end{aligned}$$

Then we can check that (3.2) and (3.3) still hold. Thus, $U_1 + \tilde{U}_1$ is an isometry on $N(U)^\perp$ and \tilde{U}_1 is a finite rank operator. Hence, A can be written as

$$\begin{aligned} A &= U + K = \begin{pmatrix} U_1 & 0 \\ U_2 & 0 \end{pmatrix} + K \\ &= \begin{pmatrix} U_1 + \tilde{U}_1 & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} -\tilde{U}_1 & 0 \\ U_2 & -I \end{pmatrix} + K = V + F + K, \end{aligned}$$

where V is an isometry, F is a finite rank operator and K is a compact operator. So A can be written in the form $A = V + L$, where V is an isometry and L is compact.

Now, in order to prove the sufficiency, we consider two cases:

Case (I). $\text{ind}(A) < 0$. It is obvious that the conclusion holds for $n = 1$. If the conclusion holds for $n = k$, that is, $A^k \in G(H) \Rightarrow A \in G(H)$, we need to prove it also holds for $n = k + 1$, that is, $A^{k+1} \in G(H) \Rightarrow A \in G(H)$. If $A \notin G(H)$, then $A = V + L$, where V is an isometry and L is compact, by induction assumption we know that $A^k \notin G(H)$, thus $A^k = V_1 + L_1$, where V_1 is an isometry and L_1 is compact. Therefore, $A^{k+1} = A^k A = V_1 V + V_1 L + L_1 V + L_1 L$, where $V_1 V$ is still an isometry and $V_1 L + L_1 V + L_1 L$ is still compact. On the other hand, it follows from the fact that A is a semi-Fredholm operator with $\text{ind}(A) \neq 0$ that A^{k+1} is a semi-Fredholm operator with $\text{ind}(A^{k+1}) \neq 0$. Thus, it follows from Theorem 2.4 that $A^{k+1} \notin G(H)$.

Case (II). $\text{ind}(A) > 0$. In Case (I) we have proved that if $A \notin G(H)$ and $\text{ind}(A) < 0$, then $A^n \notin G(H)$. Note that $A \in G(H) \iff A^* \in G(H)$. Thus, if $A \notin G(H)$ and $\text{ind}(A) > 0$, we have $A^* \notin G(H)$ and $\text{ind}(A^*) < 0$. It follows from the proof of Case (I) that $(A^n)^* = (A^*)^n \notin G(H)$, thus $A^n \notin G(H)$. The sufficiency is proved. \square

3.8. Remark. If we get rid of the assumption $A \in B(H)_1$, then the sufficiency does not hold. In fact, if $A = \begin{pmatrix} \frac{1}{3}I & 2I \\ 0 & 0 \end{pmatrix}$, then $A^2 = \begin{pmatrix} \frac{1}{9}I & \frac{2}{3}I \\ 0 & 0 \end{pmatrix}$. It is easy to see that $\|A\| > 2$, $\|A^2\| < 1$. Thus, it follows from Theorem 2.3 that $A^2 \in G(H)$. But it is obvious that $A \notin G(H)$.

3.9. Remark. The following example shows that $AB \in G(H)$ can not guarantee that $BA \in G(H)$. Let

$$A = \begin{pmatrix} I & 9I \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{3}I & \frac{1}{4}I \\ 0 & 0 \end{pmatrix}.$$

Then $AB = \begin{pmatrix} \frac{1}{3}I & \frac{1}{4}I \\ 0 & 0 \end{pmatrix}$, $BA = \begin{pmatrix} \frac{1}{3}I & 3I \\ 0 & 0 \end{pmatrix}$, $\|AB\| < 1$, $\|BA\| > 1$. Thus, it follows from Theorem 2.7 that $AB \in G(H)$, but $BA \notin G(H)$.

Moreover, the following example shows that $A, B \in B(H)_1$, $AB \in G(H)$ does not imply that $BA \in G(H)$.

3.10. Example. Let A, B be the operators on H defined as follows:

$$A : H \rightarrow H, \quad \{x_1, x_2, x_3, \dots\} \mapsto \{0, 0, 0, x_1, 0, 0, 0, x_2, \dots\},$$

$$B : H \rightarrow H, \quad \{x_1, x_2, x_3, \dots\} \mapsto \{x_2, x_4, x_6, \dots\}.$$

Then we have

$$AB : H \rightarrow H, \quad \{x_1, x_2, x_3, \dots\} \mapsto \{0, 0, 0, x_2, 0, 0, 0, x_4, \dots\},$$

$$BA : H \rightarrow H, \quad \{x_1, x_2, x_3, \dots\} \mapsto \{0, x_1, 0, x_2, 0, x_3, \dots\},$$

and

$$(AB)^* : H \rightarrow H, \quad \{x_1, x_2, x_3, \dots\} \mapsto \{0, x_4, 0, x_8, 0, x_{12}, \dots\},$$

$$(BA)^* : H \rightarrow H, \quad \{x_1, x_2, x_3, \dots\} \mapsto \{x_2, x_4, x_6, \dots\}.$$

Note that $AB \in B(H)_1$ and $\dim N(AB) = \dim N((AB)^*) = \infty$. Then it follows from Lemma 3.1 that $AB \in G(H)$. On the other hand, it is easy to see that $\dim N(BA) = 0$ and $\dim N((BA)^*) = \infty$, so BA is a semi-Fredholm operator with $\text{ind}(BA) \neq 0$. Also, it is easy to verify that BA is an isometry. Thus, it follows from Theorem 2.8 that $BA \notin G(H)$.

Finally, by using Proposition 3.3, we show that many important operators are generalized quantum gates.

First, note that the invertible operators and nilpotent operators are both Fredholm operators with index 0, while the quasi-nilpotent operators are not semi-Fredholm operators. Then it follows from Theorem 2.8 that the contractive invertible operators, contractive nilpotent operators, contractive quasi-nilpotent operators and contractive Weyl operators are all generalized quantum gates. Moreover, it follows from Lemma 2.3 and Proposition 3.3 that the contractive Browder operators, contractive Drazin invertible operators, contractive generalized Drazin operators and contractive B-Weyl operators are all generalized quantum gates, too. Note that the Weyl operators are Fredholm operators with index 0. So it follows from Theorem 2.8 that the contractive Weyl operators are generalized quantum gates. On the other hand, since the index of each B-Fredholm operator is not always 0, the contractive B-Fredholm operator is not always a generalized quantum gate.

Furthermore, we show that many operators are not only generalized quantum gates, but can also be represented as the convex combination of two unitary operators.

3.11. Theorem. *Let $A \in B(H)$ be a Drazin invertible operator (resp. Browder operator, Weyl operator, B-Weyl operator). Then the following statements are equivalent:*

- (1) $A \in G(H)$.
- (2) $A \in B(H)_1$.
- (3) $A = \frac{1}{2}(U_1 + U_2)$, where U_1 and U_2 are unitary operators.

Proof. We only prove the case when A is a Drazin invertible operator. It is obvious that (3) \Rightarrow (1) and (1) \Rightarrow (2) hold. Now, we prove (2) \Rightarrow (3). In fact, since A is Drazin invertible, by Lemma 2.3, we have

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} : H_1 \oplus H_2 \rightarrow H_1 \oplus H_2,$$

where A_1 is invertible and A_2 are nilpotent. It is obvious that both A_1 and A_2 are contractions. We assert that $\dim N(A_2) = \dim N(A_2^*)$. Indeed, if $\dim(H_2) < \infty$,

then $\dim N(A_2) = \dim N(A_2^*) < \infty$; if $\dim(H_2) = \infty$, it follows from A_2 is nilpotent that $\dim N(A_2) = \dim N(A_2^*) = \infty$, thus, $\dim N(A_2) = \dim N(A_2^*)$, and so $\dim N(A) = \dim N(A^*)$. Therefore, it follows from Lemma 3.1 that there exist unitary operators $U_1, U_2 \in B(H)$ such that $A = \frac{1}{2}(U_1 + U_2)$. \square

3.12. Remark. If $A \in B(H)$ is a quasi-nilpotent operator, then $\dim N(A) = \dim N(A^*)$ does not always hold. In fact, if A is the operator on H defined as follows:

$$A : H \rightarrow H, \quad \{x_1, x_2, x_3, x_4, \dots\} \mapsto \{0, x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots\},$$

then it is easy to see that $\dim N(A) = 0 \neq 1 = \dim N(A^*)$. This shows that the generalized Drazin invertible operators can not always be represented as the convex combination of two unitary operators.

3.13. Remark. It is obvious that a generalized quantum gate is a restricted allowable generalized quantum gate. Now, we show the converse is also true. In fact, if $A = \sum_{i=0}^{d-1} c_i U_i$, where U_i are unitary operators and $0 < \sum_{i=0}^{d-1} |c_i| \leq 1$, then $A = \sum_{i=0}^{d-1} |c_i| e^{i\theta_i} U_i$. Let $p_i = |c_i|$, $\tilde{U}_i = e^{i\theta_i} U_i$. Then $A = \sum_{i=0}^{d-1} p_i \tilde{U}_i$, where $\sum p_i \leq 1$, and \tilde{U}_i are unitary operators. If $\sum_{i=0}^{d-1} |c_i| = 1$, then $\sum p_i = 1$ and so $A \in G(H)$. If $\sum_{i=0}^{d-1} |c_i| < 1$, then $\sum p_i < 1$, and we have $\|A\| < 1$, that is, $A \in B(H)_1^o$. Thus, it follows from Theorem 2.7 that $A \in G(H)$. So A is a generalized quantum gate.

It follows from the above fact that our results are not only valid for the generalized quantum gates, but also for the restricted allowable generalized quantum gates.

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