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The Kummer beta Birnbaum-Saunders: An alternative fatigue life distribution

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Abstract

Birnbaum and Saunders [11] introduced a positive continuous distribution commonly used in reliability studies. Based on this distribution, we propose and study the called Kummer beta Birnbaum-Saunders distribution for modeling fatigue life data. Various properties of the new distribution including explicit expressions for the moments, generating function, mean deviations, density function of the order statistics and their moments are derived. We investigate maximum likelihood estimation of the parameters. The superiority of the new distribution is illustrated by means of two failure real data sets.

Keywords: Birnbaum-Saunders distribution, Fatigue life distribution, Kummer beta distribution, Lifetime data, Maximum likelihood estimation.

2000 AMS Classification: 60E05, 62N05

1. Introduction

Fatigue is a structural damage which occurs when a material is exposed to stress and tension fluctuations. When the effect of vibrations on material specimens and structures is studied, the first point to be considered is the mechanism that could cause fatigue of these materials. To understand the fatigue process and the genesis of the fatigue life and cumulative damage distributions, we recall concepts related to crack, cycle, fatigue, and load.

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In summary, the fatigue process (*fatigue life*) begins with an imperceptible fissure, the initiation, growth, and propagation of which produces a dominant crack in the specimen due to cyclic patterns of stress, whose ultimate extension causes the rupture or failure of this specimen. The failure occurs when the total extension of the crack exceeds a critical threshold for the first time. The partial extension of a crack produced by fatigue in each cycle is modeled by a random variable which depends on the type of material, the magnitude of the stress, and the number of previous cycles, among other factors. More details about the fatigue process can be found, for example, in Valluri [71], Birnbaum and Saunders [11], Murthy [53], Saunders [68], and Volodin [75].

The most popular model used to describe the lifetime process under fatigue is the Birnbaum-Saunders (BS) distribution. However, it allows for unimodal hazard rates only, hence cannot provide reasonable fits for modeling phenomenon with bathtub hazard rates, which are common in reliability studies. The distributions allowing for unimodal and bathtub hazard rates are sufficiently complex (Nelson, [55]) and usually require five or more parameters.

Motivated by problems of vibration in commercial aircraft that caused fatigue in the materials, Birnbaum and Saunders [11], [12] proposed the two-parameter BS distribution, also known as the fatigue life distribution, with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$, say BS(α, β). This distribution can be used to model lifetime data and it is widely applicable to represent failure times of fatiguing materials. If Z is a standard normal random variable, the random variable X defined by

$$X = \beta \left[\frac{\alpha Z}{2} + \left\{ \left(\frac{\alpha Z}{2} \right)^2 + 1 \right\}^{1/2} \right]^2$$

has a BS(α, β) distribution whose cumulative distribution function (cdf) is given by

$$G(x) = \Phi(\nu) \tag{1.1}$$

for $x > 0$, where $\nu = (1/\alpha)\rho(x/\beta)$, $\rho(z) = z^{1/2} - z^{-1/2}$, and $\Phi(\cdot)$ is the standard cdf. The parameter β is the median of the distribution, i.e., $G(\beta) = \Phi(0) = 1/2$. For any $k > 0$, $kX \sim$ BS$(\alpha, k\beta)$. Kundu et al. [36] investigated the shape of the BS hazard rate function. Results on improved statistical inference for this distribution are discussed by Wu and Wong [78] and Lemonte et al. [46], [48]. Díaz-Garcia and Leiva [21] proposed a new family of generalized BS distributions based on contoured elliptical distributions, whereas Guiraud et al. [29] introduced a non-central version of the BS distribution. The probability density function (pdf) corresponding to (1.1) is

$$g(x) = r(\alpha, \beta) x^{-3/2} (x + \beta) \exp \left[-\frac{\tau(x/\beta)}{2\alpha^2} \right] \tag{1.2}$$

for $x > 0$, where $r(\alpha, \beta) = \exp\left(\alpha^{-2}\right) \left(2\alpha\sqrt{2\pi\beta}\right)^{-1}$ and $\tau(z) = z - z^{-1}$. The fractional moments of (1.2) (Rieck, [65]) are

$$E\left(X^p\right) = \beta^p I(p, \alpha),$$

where

$$I(p, \alpha) = \frac{K_{p+1/2}\left(\alpha^{-2}\right) + K_{p-1/2}\left(\alpha^{-2}\right)}{2K_{1/2}\left(\alpha^{-2}\right)} \tag{1.3}$$

and $K_p(z)$ denotes the modified Bessel function of the third kind with p representing its order and z the argument. Its integral representation is $K_p(z) = 0.5\int_{-\infty}^{\infty} \exp\left\{-z\cosh(t) - pt\right\} dt$. A discussion of this function can be found in Watson [77].

The Kummer beta (KB) distribution may be characterized by the pdf (Ng and Kotz, [57])

$$F_{\mathrm{KB}}(x) = Kx^{a-1}(1-x)^{b-1}e^{-cx} \tag{1.4}$$

for $0 < x < 1$, $a > 0$, $b > 0$ and $-\infty < c < \infty$, where

$$K^{-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} {}_1F_1(a; a+b; -c),$$

where

$${}_1F_1(a; a+b; -c) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 t^{a-1}(1-t)^{b-1}e^{-ct}dt = \sum_{k=0}^{\infty}\frac{(a)_k(-c)^k}{(a+b)_k k!}$$

is the confluent hypergeometric function (Abramowitz and Stegun, [1]), $\Gamma(\cdot)$ is the gamma function and $(d)_k = d(d+1)\cdots(d+k-1)$ denotes the ascending factorial. An important special case of (1.4) for $c = 0$ is the beta pdf.

There has been much theoretical developments with respect to the BS distribution. The developments have covered many aspects of the distribution. Some of these are: acceptance sampling (Balakrishnan et al., [6]; Aslam et al., [3]), Bayes estimation (Xu and Tang, [80]), bivariate generalizations (Kundu et al., [35]), bootstrap estimation (Lemonte et al., [48]), censored estimation (Barreto et al., [8]), confidence intervals (Leiva et al., [40]), discrimination (Butler-Mccullough, [13]), EM estimation (Balakrishnan et al., [7]), graphical estimation (Chang and Tang, [15]), hazard rate (Kundu et al., [36]), influence diagnostics (Li et al., [49]), interval estimation (Wang, [76]), log linear models (Rieck and Nedelman, [66]), matrix-variate generalizations (Caro-Lopera et al., [14]), maximum likelihood estimation (Engelhardt et al., [23]), mixture models (Patriota, [61]), moment estimation (Ng et al., [56]), moment generating function (Rieck, [65]), percentiles estimation (Vilca et al., [74]), random number generation (Leiva et al., [39]), reference analysis (Xu and Tang, [79]), regression models (Lemonte and Cordeiro, [44]), reliability models (Upadhyay et al., [70]), robust estimation (Paula et al., [62]), shape and change point analyses (Azevedo et al., [5]), statistical software (Barros et al., [9]), testing hypotheses (Lemonte and Ferrari, [47]), time series models (Bhatti, [10]), truncated versions (Ahmed et al., [2]), and univariate generalizations (Owen, [60]; Vilca and Leiva, [72]; Gómes et al., [26]; Leiva et al., [38]; Leiva et al., [41]; Athayde et al., [4]; Ferreira et al., [25]; Santos-Neto et al., [67]; Lemonte, [42]).

The BS distribution has also received wide ranging applications. Some recent applications include: modeling of hourly SO_2 concentrations at ten monitoring stations located in different zones in Santiago (Leiva et al., [37]); modeling of diameter at breast height distributions of near-natural complex structure silver fir-European beech forests (Podlaski, [64]); modeling of hourly dissolved oxygen (DO) concentrations observed at four monitoring stations located at different areas of Santiago (Leiva et al., [38]; Vilca et al., [73]); statistical analysis of redundant systems with one warm stand-by unit (Nikulin and Tahir, [59]).

Because of the widespread study and applications of the BS distribution, there is a need for new generalizations. This aim of this paper is to introduce a new generalization of the BS distribution.

For an arbitrary baseline cdf $G(x)$ with parameter vector γ and pdf $g(x)$, the Kummer beta generalized (denoted by the prefix "KB-G" for short) cdf defined in Pescim et al. [63] is

$$F_{\mathrm{KBG}}(x) = K \int_0^{G(x)} t^{a-1}(1-t)^{b-1} e^{-c\,t} dt, \tag{1.5}$$

where $a > 0$ and $b > 0$ are shape parameters which induce skewness, and thereby promote weight variation of the tails, whereas the parameter $-\infty < c < \infty$ "squeezes" the pdf to the left or right, i.e., it gives weights to the extremes of the pdfs. For more details, see Pescim et al. [63].

The pdf corresponding to (1.5) can be expressed as:

$$f_{\mathrm{KBG}}(x) = K g(x) G^{a-1}(x) \left\{1 - G(x)\right\}^{b-1} \exp\left\{-c\,G(x)\right\}. \tag{1.6}$$

Clearly, the KB pdf (1.4) is a basic exemplar of equation (1.6) for $G(x) = x$, where $x \in (0, 1)$. Additionally, we obtain the classical beta distribution for $c = 0$. Equation (1.6) will be most tractable when both $G(x)$ and $g(x)$ have simple analytic expressions. Its major benefit is to offer more flexibility to extremes (right and/or left) of the pdfs and therefore it becomes suitable for analyzing data with high degree of asymmetry.

The shape parameters a, b and c have the following effects on $f(x)$: increasing values of a make the lower and upper tails of f lighter; increasing values of b make the upper tails of f lighter but they do not change the lower tails of f; increasing values of c make the lower and upper tails of f lighter. So, each of the shape parameters adds more flexibility.

The class of distributions (1.6) includes two important special cases: the beta-generalized (BG) and exponentiated generalized (EG) distributions defined by Eugene et al. [24] and Mudholkar et al. [52] when $c = 0$ and $c = 0$ and $b = 1$, respectively. We can note that the BG distributions can be limited in one aspect. They have only two additional shape parameters and so they can add only a limited structure to the generated distribution. For instance, a BG distribution may have problems to capture the behavior of random variables with symmetric but highly leptokurtic distributions. While the beta parameters offer explicit control over skewness when the parent is symmetric, they have less control over higher moments such as kurtosis. Further, the EG distribution still introduces only one extra shape parameter, whereas three may be required to control both tail weights and the distribution of weight in the center. Hence, the generated distribution (1.6) is a more flexible since it has one more shape parameter than the classical beta or exponentiated generators.

In this paper, we introduce a new five-parameter distribution called the Kummer beta Birnbaum-Saunders (KBBS) distribution which contains as sub-models the BS and beta Birnbaum-Saunders (BBS) (Cordeiro and Lemonte, [18]) distributions. The main motivation for this extension is that the new distribution is a highly flexible life distribution which admits different degrees of kurtosis and asymmetry. Moreover, the new distribution due to its flexibility in accommodating bathtub shaped and unimodal forms of the hazard rate function could be an important distribution in a variety of problems in survival analysis and reliability studies. The KBBS distribution is not only convenient for modeling comfortable bathtub shaped and unimodal hazard rates but it is also suitable for testing goodness of fit of its sub-models.

The KBBS distribution comes from (1.6) by taking $G(x)$ and $g(x)$ as the cdf and the pdf of the BS(α, β) distribution, respectively. We also provide a comprehensive description of some of its mathematical properties with the hope that it will attract wider applications in reliability, engineering and in other areas of research.

The article is outlined as follows. In Section 2, we define the KBBS distribution and plot its pdfs and hazard rate functions. Section 3 provides useful expansions for the pdf and the cdf. We obtain explicit expressions for the moments and generating function (Section 4), incomplete moments (Section 5), mean deviations, Bonferroni and Lorenz curves and reliability (Section 6) and order statistics (Section 7). Several expressions in Sections 3 to 7 involve infinite series. The computational issues relating to these infinite series are discussed in Section 8. In Section 9, we discuss maximum likelihood estimation and statistical inference. Also discussed in Section 9 is a simulation study assessing the performance of the maximum likelihood estimators (MLEs). Two applications presented in Section 10 reveal the usefulness of the new distribution for fatigue life data. Concluding remarks are noted in Section 11.

2. The KBBS distribution

By taking the cdf (1.1) and the pdf (1.2) of the BS distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$, the cdf and the pdf of the KBBS distribution are obtained from equations (1.5) and (1.6) as

$$F(x) = K \int_0^{\Phi(\nu)} t^{a-1}(1-t)^{b-1} e^{-ct} dt \tag{2.1}$$

and

$$\begin{aligned} f(x) &= K r(\alpha,\beta) x^{-3/2}(x+\beta)\Phi(\nu)\left[1-\Phi(\nu)\right]^{b-1} \\ &\times \exp\left\{-\left[\frac{\tau(x/\beta)}{2\alpha^2} + c\,\Phi(\nu)\right]\right\} \end{aligned} \tag{2.2}$$

for $x > 0$. Hereafter, we denote by X the random variable following (2.2), say $X \sim$ KBBS(a, b, c, α, β). This pdf has four shape parameters a, b, c and α, which allow for a high degree of flexibility. The parameter c controls tail weights to the extremes of the distribution. The associated hazard rate function becomes

$$h(x) = \frac{K r(\alpha,\beta) x^{-3/2}(x+\beta)\Phi(\nu)}{\left[1-F(x)\right]\left[1-\Phi(\nu)\right]^{1-b}} \exp\left\{-\left[\frac{\tau(x/\beta)}{2\alpha^2} + c\,\Phi(\nu)\right]\right\}.$$

The study of the new distribution is important since it extends some distributions previously considered in the literature. In fact, the BS distribution (with parameters α and β) is clearly a basic exemplar for $a = b = 1$ and $c = 0$, with a continuous crossover towards distributions with different shapes (e.g., a specified combination of skewness and kurtosis). The KBBS distribution contains as sub-models the beta-BS (BBS) and the exponentiated Birnbaum-Saunders (EBS) (Cordeiro et al., [19]) distributions when $c = 0$ and $b = 1$ in addition to $c = 0$, respectively. Plots of the KBBS pdf and hazard rate functions for selected parameter values are displayed in Figures 1 and 2. It is evident that the shapes of the new pdf are much more flexible than the BS distribution. Further, it allows four major hazard shapes: increasing, decreasing, bathtub and unimodal hazard rates.

3. Expansions for cdf and pdf

Expansions for equations (2.1) and (2.2) can be derived using the concept of exponentiated distributions. Cordeiro et al. [19] defined a random variable Y following the EBS distribution with parameters α, β and $\gamma > 0$, say $Y \sim \mathrm{EBS}(\alpha, \beta, \gamma)$. The cdf and the pdf of Y are denoted by $H(y; \alpha, \beta, \gamma) = \Phi^{\gamma}(\nu)$ and $h(y; \alpha, \beta, \gamma) = \gamma g_{\alpha,\beta}(y)\Phi^{\gamma-1}(\nu)$, respectively, where ν is defined in (1.1). The properties of some exponentiated distributions have been studied by several authors, see Mudholkar and Srivastava [51] and Mudholkar et al. [52] for the exponentiated Weibull distribution, Gupta et al. [30] for the exponentiated Pareto distribution, Gupta and Kundu [31] for the exponentiated exponential distribution, Nadarajah and Gupta [54] for the exponentiated gamma distribution, Cordeiro et al. [20] for the exponentiated generalized gamma distribution, Lemonte and Cordeiro [45] for the exponentiated generalized inverse Gaussian distribution, and Lemonte et al. [43] for the exponentiated Kumaraswamy distribution.

By expanding the term $\exp[-c\,\Phi(\nu)]$ and the binomial in equation (2.2), we obtain the linear combination (for $a > 0$ integer)

$$f(x) = \sum_{j,k=0}^{\infty} w_{j,k} h(x; \alpha, \beta, a + j + k), \tag{3.1}$$

where $h(x; \alpha, \beta, a + j + k)$ denotes the $\mathrm{EBS}(\alpha, \beta, a + j + k)$ pdf and the coefficient $w_{j,k}$ is given by

$$w_{j,k} = \frac{K(-1)^{j+k} c^j}{j!(a + j + k)} \binom{b-1}{k}.$$

By integrating (3.1), we obtain

$$F(x) = \sum_{j,k=0}^{\infty} w_{j,k} \Phi^{a+j+k}(\nu). \tag{3.2}$$

If a is a positive non-integer, we can expand $\Phi^{a+j+k}(\nu)$ as

$$\Phi^{a+j+k}(\nu) = \sum_{r=0}^{\infty} s_r(a + j + k)\Phi^{r}(\nu), \tag{3.3}$$

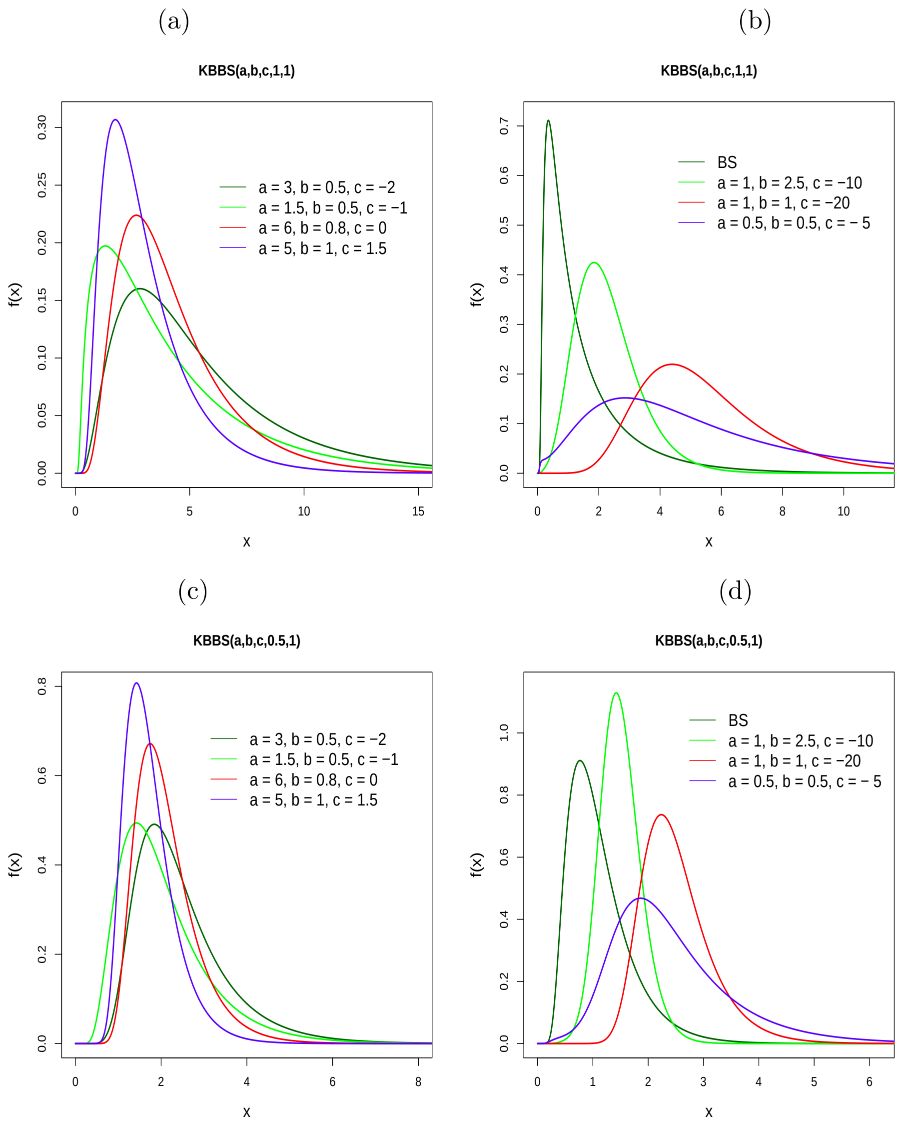

Figure 1. Plots of the pdf (2.2) for some parameter values.

where

$$s_r(m) = \sum_{k=r}^{\infty} (-1)^{k+r} \binom{m}{k} \binom{k}{r}.$$

Thus, from equations (1.2), (3.2) and (3.3), the KBBS cdf can be expressed as

$$F(x) = \sum_{r=0}^{\infty} b_r \Phi^r(\nu), \tag{3.4}$$

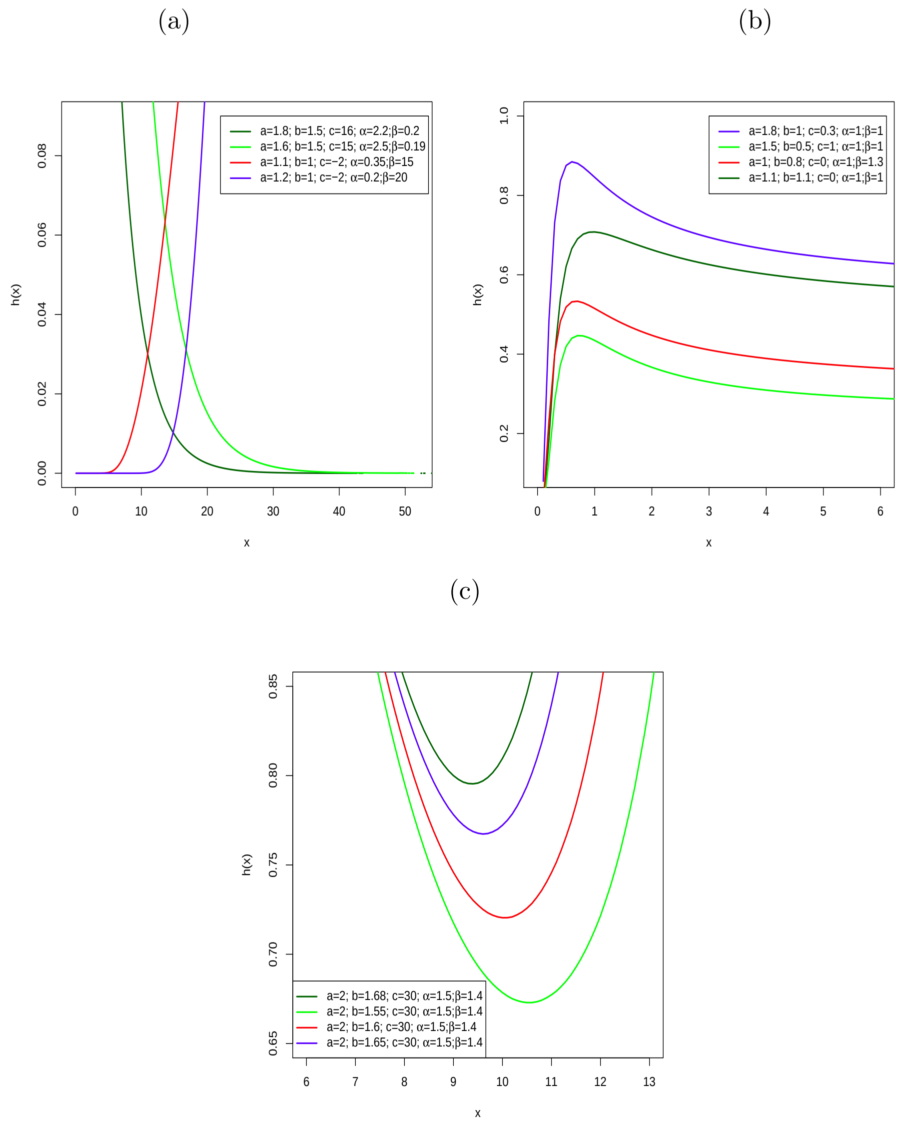

Figure 2. The KBBS hazard rate function. (a) Increasing and decreasing hazard rate function. (b) Unimodal hazard rate function. (c) Bathtub hazard rate function.

where

$$b_r = \sum_{j,k=0}^{\infty} w_{j,k} s_r(a+j+k).$$

For $a > 0$ real non-integer, the KBBS pdf expansion corresponding to (3.4) is obtained by simple differentiation

$$f(x) = \sum_{r=0}^{\infty} b_r h(x; \alpha, \beta, r). \tag{3.5}$$

Equation (3.5) reveals that the KBBS pdf is a linear combination of EBS pdfs. This result is important to derive some properties of the KBBS distribution from those of the EBS distribution.

4. Moments and generating function

4.1. Moments. The ordinary moments of X can be determined from the probability weighted moments (Greenwood et al., [28]) of the BS distribution formally defined for p and r non-negative integers by

$$\tau_{p,r-1} = \int_0^{\infty} x^p g(x) \Phi^{r-1}(\nu) dx. \tag{4.1}$$

The integral (4.1) can be easily computed numerically in software such as MAPLE, MATLAB, MATHEMATICA, Ox and R. Cordeiro and Lemonte [18] proposed an alternative representation to compute $\tau_{p,r-1}$ given by

$$\begin{aligned} \tau_{p,r-1} &= \frac{\beta^p}{2^{r-1}} \sum_{j=0}^{r-1} \binom{r-1}{j} \sum_{k_1,\ldots,k_j=0}^{\infty} A(k_1,\ldots,k_j) \\ &\times \sum_{m=0}^{2s_j+j} (-1)^m \binom{2s_j+j}{m} I\left(p + s_j - m + \frac{j}{2}, \alpha\right), \end{aligned} \tag{4.2}$$

where $s_j = k_1 + \cdots + k_j$, $A(k_1,\ldots,k_j) = \alpha^{-2s_j - j} a_{k_1} \cdots a_{k_j}$,
$a_k = (-1)^k 2^{(1-2k)/2} \left[\sqrt{\pi}(2k+1)\right]^{-1}$ and $I(p + (2s_j + j - 2m)/2, \alpha)$ is determined from (1.3).

The sth moment of X can be expressed from equation (3.5) as

$$\mu_s^{'} = \sum_{r=0}^{\infty} b_r \tau_{s,r-1}, \tag{4.3}$$

where $\tau_{s,r-1}$ is obtained from (4.1) and b_r is defined in (3.4).

he four first moments of the KBBS distribution were calculated by numerical integration and through infinite weighted sums in equation (4.3) using the statistical software package R. The values from both techniques are usually close when ∞ is replaced by a large number as 500 in (4.3). For selected values $a = 2$, $b = 1.5$, $c = 4$, $\alpha = 0.5$ and $\beta = 1$, Table (1) gives some numerical analysis for those moments and for variance, skewness and kurtosis.

The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. Plots of the skewness and kurtosis of the KBBS distribution as a function of c for selected values of a and b for $\alpha = 0.5$ and $\beta = 1.0$ are displayed in Figures 3 and 4. Figures 3a and 3b immediately indicate that the additional parameter c promotes high levels of asymmetry.

Table 1. Values of the four first moments, variance, skewness and kurtosis of the KBBS distribution for $a = 2$, $b = 1.5$, $c = 4$, $\alpha = 0.5$ and $\beta = 1$ obtained by numerical integration and through infinite weighted sums, where $j, k, r = 0, \ldots, p$.

Moments	Infinite weighted sums				Numerical integration
	p=50	p=100	p=250	p=500	
μ_1'	0.85967	0.85920	0.85898	0.85893	0.85890
μ_2'	0.83508	0.83355	0.83278	0.83258	0.83242
μ_3'	0.93435	0.92920	0.92633	0.92550	0.92479
μ_4'	1.23327	1.21506	1.20395	1.20042	1.19703
Variance	0.09604	0.0953	0.09492	0.09481	0.09471
Skewness	1.72439	1.6716	1.63790	1.62691	1.61629
Kurtosis	9.18582	8.6644	8.28549	8.14676	7.99257

(a) (b)

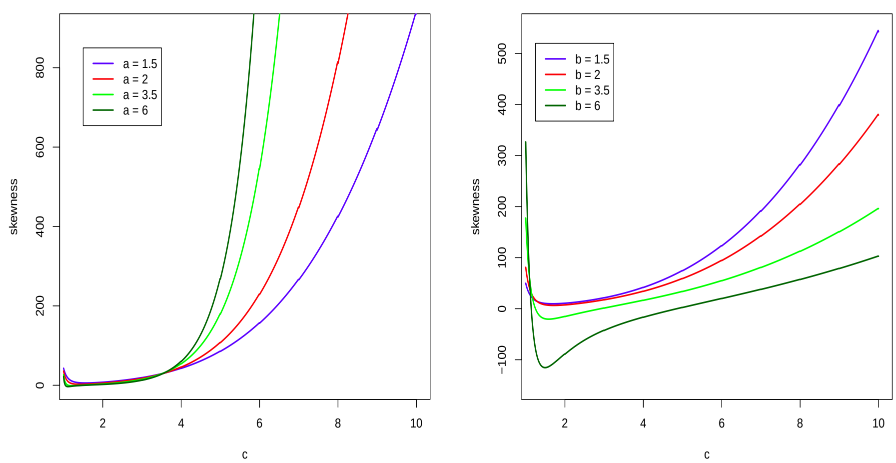

Figure 3. Skewness of the KBBS distribution as a function of c for some values of a and b for $\alpha = 0.5$ and $\beta = 1.0$. (a) $b = 1.5$ and (b) $a = 1.2$.

4.2. Generating function. Here, we provide a representation for the moment generating function (mgf) of X, say $M(t) = E\left[\exp(tX)\right]$, which is obtained as a linear combination of the mgf's of the EBS distributions. From expansion (3.5), we obtain

$$M(t) = \sum_{r=0}^{\infty} b_r M_r(t), \tag{4.4}$$

where $M_r(t)$ is the mgf of the EBS(α, β, r) distribution and b_r is defined by (3.4).

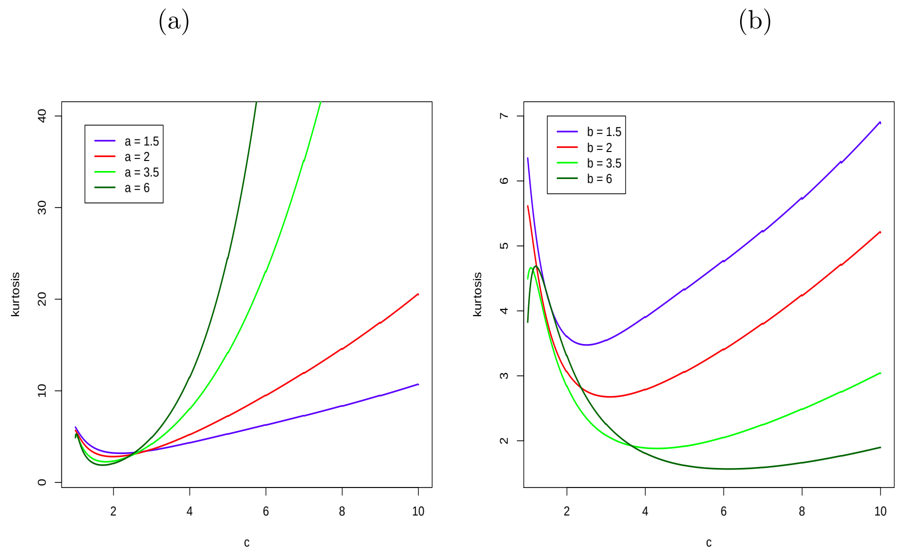

Figure 4. Kurtosis of the KBBS distribution as a function of c for some values of a and b for $\alpha = 0.5$ and $\beta = 1.0$. (a) $b = 1.5$ and (b) $a = 1.2$.

Thus, $M_r(t)$ can be expressed as

$$M_r(t) = r \int_0^\infty \exp(tx) g_{\alpha,\beta}(x) \Phi^{r-1}(\nu) dx, \tag{4.5}$$

where $g_{\alpha,\beta}(x)$ is the BS(α, β) pdf. Setting $u = \Phi(\nu)$ in (4.5), we have

$$M_r(t) = r \int_0^1 u^{r-1} \exp\left[tQ(u)\right] du, \tag{4.6}$$

where $x = Q(u)$ is the quantile function of the BS distribution and $u = \Phi(\nu)$ is given by (1.1).

Now, we derive a power series expansion for the quantile function of the EBS distribution that can be useful to calculate the mgf of the KBBS distribution. We use throughout an equation in Section 0.314 of Gradshteyn and Ryzhik [27] for a power series raised to a positive integer j given by

$$\left(\sum_{i=0}^{\infty} a_i x^i\right)^j = \sum_{i=0}^{\infty} c_{j,i} x^i, \tag{4.7}$$

where the coefficients $c_{j,i}$ (for $i = 1, 2, \ldots$) are easily computed from the recurrence equation

$$c_{j,i} = (i a_0)^{-1} \sum_{m=1}^{i} \left[m(j+1) - i\right] a_m c_{j,i-m} \tag{4.8}$$

and $c_{j,0} = a_0^j$. The coefficient $c_{j,i}$ can be determined from $c_{j,0}, \ldots, c_{j,i-1}$ and hence from the quantities $a_0, \ldots, a_i$. In fact, $c_{j,i}$ can be given explicitly in terms of the coefficients a_i, although it is not necessary for programming numerically our expansions in any algebraic or numerical software.

Following Cordeiro and Lemonte [18], we can invert $u = \Phi(\nu)$ if the condition $-2 < (x/\beta)^{1/2} - (\beta/x)^{1/2} < 2$ holds, to express x as a power series expansion of u

$$x = Q(u) = \sum_{i=0}^{\infty} \rho_i (u - 1/2)^i, \tag{4.9}$$

where the coefficients are $\rho_0 = \beta$, $\rho_{2q+1} = \beta\alpha^{2q+1}\binom{1/2}{q}4^{-q}$ for $q \geq 0$, $\rho_2 = \beta\alpha^2/2$ and $\rho_{2q} = 0$ for $q \geq 2$ and the quantities $e_{q,i}$ follow recursively from equations (4.7) and (4.8) by $e_{q,0} = d_0^q$ and

$$e_{q,i} = (i d_0)^{-1} \sum_{m=1}^{q} \left[m(q+1) - i\right] d_m e_{q,i-m}.$$

Here, the quantities d_m are defined by $d_m = 0$ (for $m = 0, 2, 4, \ldots$) and $d_m = j_{(m-1)/2}$ (for $m = 1, 3, 5, \ldots$), where the j_m's are calculated recursively from

$$j_{m+1} = \frac{1}{2(2m+3)} \sum_{v=0}^{m} \frac{(2v+1)(2m-2v+1) j_v j_{m-v}}{(v+1)(2v+1)}.$$

We have $j_0 = 1$, $j_1 = 1/6$, $j_2 = 7/120$, $j_3 = 127/7560$, and so on.

Substituting equation (4.9) into (4.6) and using the exponential expansion, we obtain

$$M_r(t) = r \sum_{p=0}^{\infty} \frac{t^p}{p!} \int_0^1 u^{r-1} \left(\sum_{i=0}^{\infty} \rho_i w^i \right)^p du, \tag{4.10}$$

where $w = u - 1/2$. From equations (4.7) and (4.8), we have

$$\left(\sum_{i=0}^{\infty} \rho_i w^i \right)^p = \sum_{i=0}^{\infty} \delta_{p,i} w^i = \sum_{i=0}^{\infty} \delta_{p,i} (u - 1/2)^i,$$

where $\delta_{p,0} = \rho_0^p$ and

$$\delta_{p,i} = (i \rho_0)^{-1} \sum_{m=1}^{i} \left[m(p+1) - i\right] \rho_m \delta_{p,i-m}.$$

Then, equation (4.10) becomes

$$M_r(t) = r \sum_{p,i=0}^{\infty} \frac{t^p}{p!} \delta_{p,i} \int_0^1 u^{r-1} (u - 1/2)^i \, du. \tag{4.11}$$

Using the binomial expansion in (4.11), the mgf of the EBS distribution can be expressed as

$$M_r(t) = \sum_{p=0}^{\infty} \delta_{p,i}^{*} t^p, \tag{4.12}$$

where

$$\delta_{p,i}^{*}=r\sum_{i=0}^{\infty}\sum_{q=0}^{i}\binom{i}{q}\frac{(-1)^{i-q}\delta_{p-i}}{p!(q+r)2^{i-q}}.$$

Finally, substituting (4.12) into (4.4), the mgf of the KBBS distribution reduces to

$$M(t)=\sum_{p=0}^{\infty}\eta_p t^p, \tag{4.13}$$

where

$$\eta_p=\sum_{r=0}^{\infty}b_r\delta_{p,r}^{*}.$$

5. Incomplete moments

Many important questions in econometrics require more than just knowing the mean of a distribution, but its shape as well. This is also obvious not only in the study of econometrics and income distributions but in many other areas of research. For empirical purposes, the shape of many distributions can be usefully described by what we call the incomplete moments. These types of moments play an important role for measuring inequality, for example, income quantiles and Lorenz and Bonferroni curves, which depend upon the incomplete moments of a distribution. The nth incomplete moment of X is given by $T_n(y)=\int_0^y x^n f(x)dx$. By inserting (3.5) in $T_n(y)$, we obtain

$$T_n(y)=r(\alpha,\beta)\sum_{r=0}^{\infty}b_r\int_0^y x^{n-3/2}(x+\beta)\Phi^{r-1}(\nu)\exp\left\{-\frac{\tau(x/\beta)}{2\alpha^2}\right\}dx.$$

From Cordeiro and Lemonte [18], we have

$$\begin{aligned}\Phi^{r-1}(\nu) &= 2^{1-r}\sum_{j=0}^{r-1}\binom{r-1}{j}\sum_{k_1,\ldots,k_j=0}^{\infty}\beta^{-(2s_j+j)/2}A\left(k_1,\ldots,k_j\right)\\ &\quad\times\sum_{m=0}^{2s_j+j}(-\beta)^m\binom{2s_j+j}{m}x^{(2s_j+j-2m)/2},\end{aligned}$$

where s_j and $A\left(k_1,\ldots,k_j\right)$ are defined in (4.2). Thus,

$$\begin{aligned}T_n(y) &= r(\alpha,\beta)\sum_{r=0}^{\infty}b_r 2^{1-r}\sum_{j=0}^{r-1}\binom{r-1}{j}\sum_{k_1,\ldots,k_j=0}^{\infty}\beta^{-(2s_j+j)/2}A\left(k_1,\ldots,k_j\right)\\ &\quad\times\sum_{m=0}^{2s_j+j}(-\beta)^m\binom{2s_j+j}{m}\int_0^y x^{n+(2s_j+j-2m-3)/2}(x+\beta)\exp\left\{-\frac{\tau(x/\beta)}{2\alpha^2}\right\}dx.\end{aligned}$$

Let

$$D(p,q)=\int_0^q x^q\exp\left\{-\frac{x/\beta+\beta/x}{2\alpha^2}\right\}dx=\int_0^{q/\beta}u^q\exp\left\{-\frac{u+u^{-1}}{2\alpha^2}\right\}du.$$

From Terras [69], we can write

$$D(p,q)=\beta^{p+1}K_{p+1}\left(\alpha^{-2}\right)-q^{p+1}K_{p+1}\left(\frac{q}{2\alpha^2\beta},\frac{\beta}{2\alpha^2 q}\right),$$

where $K_p\left(x_1,x_2\right)$ denotes the incomplete Bessel function with arguments x_1 and x_2 and order p. For further details, see Jones [33], [34] and Harris [32].

Hence, the nth incomplete moment of X can be expressed as

$$\begin{aligned}
T_n(y) \;=\;& r(\alpha,\beta)\sum_{r=0}^{\infty}b_r 2^{1-r}\sum_{j=0}^{r-1}\binom{r-1}{j}\sum_{k_1,\ldots,k_j=0}^{\infty}\beta^{-(2s_j+j)/2}A\left(k_1,\ldots,k_j\right)\\
&\times\sum_{m=0}^{2s_j+j}(-\beta)^m\binom{2s_j+j}{m}\\
&\times\left\{D\left(n+\frac{2s_j+j-2m-1}{2},y\right)+\beta D\left(n+\frac{2s_j+j-2m-3}{2},y\right)\right\}. \qquad (5.1)
\end{aligned}$$

Equation (5.1) is the main result of this section.

6. Other measures

Here, we derive the means deviations, Lorenz and Bonferroni curves and the reliability of the KBBS distribution.

6.1. Mean deviations. We can derive the mean deviations about the mean $\mu_1^{'}\left(\delta_1\right)$ and about the median $M\left(\delta_2\right)$ in terms of the first incomplete moment. The median is obtained by inverting $F(M)=K\int_0^{\Phi(\nu)}t^{a-1}(1-t)^{b-1}e^{-ct}dt=1/2$ numerically. They can be expressed as

$$\delta_1=2\left[\mu_1^{'}F\left(\mu_1^{'}\right)-T_1\left(\mu_1^{'}\right)\right],\qquad \delta_2=\mu_1^{'}-2T_1\left(M\right),$$

where $T_1(\cdot)$ is the first incomplete moment of X given by (5.1) with $n=1$. We have

$$\begin{aligned}
T_1(y) \;=\;& r(\alpha,\beta)\sum_{r=0}^{\infty}b_r 2^{1-r}\sum_{j=0}^{r-1}\binom{r-1}{j}\sum_{k_1,\ldots,k_j=0}^{\infty}\beta^{-(2s_j+j)/2}A\left(k_1,\ldots,k_j\right)\\
&\times\sum_{m=0}^{2s_j+j}(-\beta)^m\binom{2s_j+j}{m}\\
&\times\left\{D\left(\frac{2s_j+j-2m+1}{2},y\right)+\beta D\left(\frac{2s_j+j-2m-1}{2},y\right)\right\}. \qquad (6.1)
\end{aligned}$$

The measures δ_1 and δ_2 are immediately calculated from (6.1) by setting $y=\mu_1^{'}$ and $y=M$, respectively.

An application of the mean deviations refer to the Lorenz and Bonferroni curves defined by $L(\pi)=T_1(q)/\mu_1^{'}$ and $B(\pi)=T_1(q)/\pi\mu_1^{'}$, respectively, where $q=F^{-1}(\pi)$ can be computed for a given probability π by inverting (2.1) numerically. These curves have applications in several fields. They measures are immediately calculated from equation (6.1).

6.2. Reliability. In the context of reliability, the stress-strength model describes the life of a component which has a random strength X_1 that is subjected to a random stress X_2. The component fails at the instant that the stress applied to it exceeds the strength, and the component will function satisfactorily whenever $X_1 > X_2$. Hence, $R = \Pr(X_1 < X_2)$ is a measure of component reliability which has many applications in engineering. We derive the reliability R when X_1 and X_2 have independent KBBS$(\alpha, \beta, a_1, b_1, c_1)$ and KBBS$(\alpha, \beta, a_2, b_2, c_2)$ distributions with the same shape parameters α and β.

The pdf of X_1 and the cdf of X_2 can be written from equations (3.1) and (3.2) as

$$f_1(x) = g(x) \sum_{i,j=0}^{\infty} w_{1,i,j} \, (a_1 + i + j) \, \Phi^{a_1+i+j}(\nu), \qquad F_2(x) = \sum_{k,p=0}^{\infty} w_{2,k,p} \Phi^{a_2+k+p}(\nu),$$

respectively, where

$$w_{1,i,j} = \frac{K_1(-1)^{i+j} c_1^i}{i! \, (a_1 + i + j)} \binom{b_1 - 1}{j}, \qquad w_{2,k,p} = \frac{K_2(-1)^{k+p} c_2^k}{k! \, (a_2 + k + p)} \binom{b_2 - 1}{p}.$$

The reliability, R, is given by

$$R = \int_0^\infty f_1(x) F_2(x) dx \tag{6.2}$$

and then

$$R = \sum_{i,j,k,p=0}^{\infty} w_{1,i,j} w_{2,k,p} \int_0^\infty g(x) \Phi^{a_1+a_2+i+j+k+p-1}(\nu) dx.$$

From equation (3.3), we can write

$$\Phi^{a_1+a_2+i+j+k+p-1}(\nu) = \sum_{r=0}^{\infty} s_r \, (a_1 + a_2 + i + j + k + p - 1) \, \Phi^r(\nu),$$

and then R reduces to

$$R = \sum_{i,j,k,p=0}^{\infty} w_{1,i,j} w_{2,k,p} \sum_{r=0}^{\infty} s_r \, (a_1 + a_2 + i + j + k + p - 1) \, \tau_{0,r-1}, \tag{6.3}$$

where $\tau_{0,r-1}$ can be computed from (4.2).

7. Order statistics

Order statistics have been used in a wide range of problems, including robust statistical estimation and detection of outliers, characterization of probability distributions and goodness-of-fit tests, entropy estimation, analysis of censored samples, reliability analysis, quality control and strength of materials.

Suppose $X_1, \ldots, X_n$ is a random sample from the KBBS distribution and let $X_{1:n} < \cdots < X_{n:n}$ denote the corresponding order statistics. Using (3.4) and (3.5), the pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \left[g(x) \sum_{r=0}^{\infty} b_r \Phi^r(\nu) \right] \left[\sum_{r=0}^{\infty} b_r \Phi^r(\nu) \right]^{i+j-1}.$$

From equations (4.7) and (4.8), we obtain

$$\left[\sum_{r=0}^{\infty} b_r \Phi^r(\nu)\right]^{i+j-1} = \sum_{r=0}^{\infty} c_{i+j-1,r} \Phi^r(\nu),$$

where $c_{i+j-1,0} = b_0^{i+j-1}$ and

$$c_{i+j-1,r} = (r b_0)^{-1} \sum_{m=1}^{r} \left[m(i+j) - r\right] b_m c_{i+j-1,r-m}.$$

Hence, the pdf of the ith order statistic for the KBBS distribution can be expressed as

$$f_{i:n}(x) = \sum_{r=0}^{\infty} m_r \, h(x; \alpha, \beta, 2r), \tag{7.1}$$

where

$$m_r = \frac{n! b_r}{(2r+1)(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} c_{i+j-1,r}.$$

Equation (7.1) is the main result of this section. It gives the pdf of the KBBS order statistics as a linear combination of EBS pdfs with parameters α, β and $2r$. So, several mathematical quantities of the KBBS order statistics such as ordinary and incomplete moments, generating function, mean deviations (and several others) can come immediately from those quantities of the EBS distribution.

8. Computational issues

Here, we show the practical values of (3.4), (3.5), (4.3), (4.13), (5.1), (6.3) and (7.1). These formulas with the infinite series truncated provide a simple way to compute the cdf, pdf, moments, mgf, incomplete moments, reliability and the pdf of order statistics. The question is: how large should the truncation limit be?

We now show evidence that each infinite summation can be truncated at twenty to yield sufficient accuracy. Let $D1$ denote the absolute difference between the integrated version, (2.1), and the truncated version of (3.4) averaged over $x = 0.01, 0.02, \ldots, 5$, $a = 0.01, 0.02, \ldots, 10$, $b = 0.01, 0.02, \ldots, 10$, $c = -10, -9.99, \ldots, 10$, $\alpha = 0.01, 0.02, \ldots, 10$ and $\beta = 0.01, 0.02, \ldots, 10$. Let $D2$ denote the absolute difference between (1.6) and the truncated version of (3.5) averaged over $x = 0.01, 0.02, \ldots, 5$, $a = 0.01, 0.02, \ldots, 10$, $b = 0.01, 0.02, \ldots, 10$, $c = -10, -9.99, \ldots, 10$, $\alpha = 0.01, 0.02, \ldots, 10$ and $\beta = 0.01, 0.02, \ldots, 10$. Let $D3$ denote the absolute difference between the truncated version of (4.3) and the integrated version,

$$\mu_s^{'} = K r(\alpha, \beta) \int_0^{\infty} x^{s-3/2} (x+\beta) \Phi(\nu) \left[1 - \Phi(\nu)\right]^{b-1} \exp\left\{-\left[\frac{\tau(x/\beta)}{2\alpha^2} + c\Phi(\nu)\right]\right\} dx,$$

averaged over $s = 1, 2, \ldots, 50$, $a = 0.01, 0.02, \ldots, 10$, $b = 0.01, 0.02, \ldots, 10$, $c = -10, -9.99, \ldots, 10$, $\alpha = 0.01, 0.02, \ldots, 10$ and $\beta = 0.01, 0.02, \ldots, 10$. Let $D4$ denote the absolute difference between the truncated version of (4.13) and the integrated version,

$$
\begin{aligned}
M(t) &= Kr(\alpha, \beta) \int_0^\infty \exp(tx) x^{-3/2} (x+\beta) \Phi(\nu) \left[1 - \Phi(\nu)\right]^{b-1} \\
&\quad \times \exp\left\{ - \left[\frac{\tau(x/\beta)}{2\alpha^2} + c\Phi(\nu) \right] \right\} dx,
\end{aligned}
$$

averaged over $t = 0.01, 0.02, \ldots, 0.99$, $a = 0.01, 0.02, \ldots, 10$, $b = 0.01, 0.02, \ldots, 10$, $c = -10, -9.99, \ldots, 10$, $\alpha = 0.01, 0.02, \ldots, 10$ and $\beta = 0.01, 0.02, \ldots, 10$. Let $D5$ denote the absolute difference between the truncated version of (5.1) and the integrated version,

$$
T_n(y) = Kr(\alpha, \beta) \int_0^y x^{n-3/2} (x+\beta) \Phi(\nu) \left[1 - \Phi(\nu)\right]^{b-1} \exp\left\{ - \left[\frac{\tau(x/\beta)}{2\alpha^2} + c\Phi(\nu) \right] \right\} dx,
$$

averaged over $n = 1, 2, \ldots, 50$, $y = 0.01, 0.02, \ldots, 5$, $a = 0.01, 0.02, \ldots, 10$, $b = 0.01, 0.02, \ldots, 10$, $c = -10, -9.99, \ldots, 10$, $\alpha = 0.01, 0.02, \ldots, 10$ and $\beta = 0.01, 0.02, \ldots, 10$. Let $D6$ denote the absolute difference between the integrated version, (6.2), and the truncated version of (6.3) averaged over $a_1 = 0.01, 0.02, \ldots, 10$, $b_1 = 0.01, 0.02, \ldots, 10$, $c_1 = -10, -9.99, \ldots, 10$, $a_2 = 0.01, 0.02, \ldots, 10$, $b_2 = 0.01, 0.02, \ldots, 10$, $c_2 = -10, -9.99, \ldots, 10$, $\alpha = 0.01, 0.02, \ldots, 10$ and $\beta = 0.01, 0.02, \ldots, 10$. Let $D7$ denote the absolute difference between (7.1) and the truncated version of (7.1) averaged over $i = 1, 2, \ldots, n$, $n = 1, 2, \ldots, 50$, $a = 0.01, 0.02, \ldots, 10$, $b = 0.01, 0.02, \ldots, 10$, $c = -10, -9.99, \ldots, 10$, $\alpha = 0.01, 0.02, \ldots, 10$ and $\beta = 0.01, 0.02, \ldots, 10$.

We obtained the following estimates after extensive computations: $D1 = 1.21 \times 10^{-21}$, $D2 = 9.43 \times 10^{-20}$, $D3 = 2.39 \times 10^{-33}$, $D4 = 3.54 \times 10^{-21}$, $D5 = 7.6 \times 10^{-25}$, $D6 = 1.68 \times 10^{-20}$ and $D7 = 2.78 \times 10^{-22}$. These estimates are small enough to suggest that the truncated versions of (3.4), (3.5), (4.3), (4.13), (5.1), (6.3) and (7.1) are reasonable for practical use.

It would ideal to show that each (untruncated) infinite series (like (3.4), (3.5), (4.3), (4.13), (5.1), (6.3) and (7.1)) is convergent and gives valid values for all values of its arguments. This will be a difficult mathematical problem and a possible future work.

9. Inference

Section 9.1 gives procedures for maximum likelihood estimation of the KBBS distribution. Section 9.2 assesses the performance of the MLEs in terms of biases, mean squared errors, coverage probabilities and coverage lengths by means of a simulation study.

9.1. Estimation. The estimation of the parameters of the KBBS distribution will be investigated by maximum likelihood. Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random sample of this distribution with unknown parameter vector $\boldsymbol{\theta} = (\alpha, \beta, a, b, c)^T$.

The total log-likelihood function for $\boldsymbol{\theta}$ is

$$\begin{aligned}\ell(\boldsymbol{\theta}) \quad = \quad & n\log K + n\log r(\alpha,\beta) - \frac{3}{2}\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log\left(x_i+\beta\right) - \frac{1}{2\alpha^2}\sum_{i=1}^{n}\tau\left(x_i/\beta\right) \\ & -c\sum_{i=1}^{n}\Phi\left(\nu_i\right) + (a-1)\sum_{i=1}^{n}\log\Phi\left(\nu_i\right) + (b-1)\sum_{i=1}^{n}\log\left[1-\Phi\left(\nu_i\right)\right]. \qquad (9.1)\end{aligned}$$

The elements of score vector are given by

$$\begin{aligned}U_\alpha(\boldsymbol{\theta}) &= -\frac{n}{\alpha}\left(1+\frac{2}{\alpha^2}\right) + \frac{1}{\alpha^3}\sum_{i=1}^{n}\left(\frac{x_i}{\beta}+\frac{\beta}{x_i}\right) \\ &\quad -\frac{1}{\alpha}\sum_{i=1}^{n}\nu_i\phi\left(\nu_i\right)\left\{\frac{a-1}{\Phi\left(\nu_i\right)} - \frac{b-1}{1-\Phi\left(\nu_i\right)} - 2c\right\}, \\ U_\beta(\boldsymbol{\theta}) &= -\frac{n}{2\beta} + \sum_{i=1}^{n}\frac{1}{x_i+\beta} + \frac{1}{2\alpha^2\beta}\sum_{i=1}^{n}\left(\frac{x_i}{\beta}-\frac{\beta}{x_i}\right) \\ &\quad -\frac{1}{2\alpha\beta}\sum_{i=1}^{n}\tau\left(\sqrt{x_i/\beta}\right)\phi\left(\nu_i\right)\left\{\frac{a-1}{\Phi\left(\nu_i\right)} - \frac{b-1}{1-\Phi\left(\nu_i\right)} - c\right\}, \\ U_a(\boldsymbol{\theta}) &= \frac{n}{K}\frac{\partial K}{\partial a} + \sum_{i=1}^{n}\log\Phi\left(\nu_i\right), \\ U_b(\boldsymbol{\theta}) &= \frac{n}{K}\frac{\partial K}{\partial b} + \sum_{i=1}^{n}\log\left[1-\Phi\left(\nu_i\right)\right], \\ U_c(\boldsymbol{\theta}) &= \frac{n}{K}\frac{\partial K}{\partial c} + \sum_{i=1}^{n}\Phi\left(\nu_i\right),\end{aligned}$$

where $\phi(\cdot)$ is the standard normal pdf, $\nu_i = \alpha^{-1}\left\{\sqrt{x_i/\beta} - \sqrt{\beta/x_i}\right\}$ and $\tau\left(\sqrt{x_i/\beta}\right) = \sqrt{x_i/\beta} + \sqrt{\beta/x_i}$ for $i = 1, 2, \ldots, n$. The partial derivatives of K with respect to a, b and c are

$$\begin{aligned}\frac{\partial K}{\partial a} &= -\frac{\left[\psi(a)-\psi(a+b)\right]{}_1F_1(a,a+b,-c) + \dfrac{\partial_1 F_1(a,a+b,-c)}{\partial a}}{B(a,b)\left[{}_1F_1(a,a+b,-c)\right]^2}, \\ \frac{\partial K}{\partial b} &= -\frac{\left[\psi(b)-\psi(a+b)\right]{}_1F_1(a,a+b,-c) + \dfrac{\partial_1 F_1(a,a+b,-c)}{\partial b}}{B(a,b)\left[{}_1F_1(a,a+b,-c)\right]^2}, \\ \frac{\partial K}{\partial c} &= \frac{a_1F_1(a+1,a+b+1,-c)}{(a+b)B(a,b)_1F_1(a,a+b,-c)},\end{aligned}$$

where

$$\begin{aligned}\frac{\partial_1 F_1(a,a+b,-c)}{\partial a} \quad = \quad & -\left[\psi(a)-\psi(a+b)\right]{}_1F_1(a,a+b,-c) \\ & -\sum_{k=0}^{\infty}\frac{(a)_k(-c)^k}{k!(a+b)_k}\left[\psi(a+b+k)-\psi(a+k)\right]\end{aligned}$$

and

$$\frac{\partial_1 F_1(a,a+b,-c)}{\partial c} = \psi(a+b)_1F_1(a,a+b,-c) + \sum_{k=0}^{\infty} \frac{(a)_k(-c)^k}{k!(a+b)_k}\psi(a+b+k).$$

Maximization of (9.1) can be performed by using well established routines such as the nlm routine or optimize in the R statistical package. Setting these equations to zero, $\mathbf{U}(\boldsymbol{\theta}) = \mathbf{0}$, and solving them simultaneously yields the MLE $\widehat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$. These equations cannot be solved analytically and statistical software can be used to solve them numerically by means of iterative techniques such as the Newton-Raphson algorithm.

For interval estimation and hypothesis tests on the parameters in $\boldsymbol{\theta}$, we require the 5×5 total observed information matrix $\mathbf{J}(\boldsymbol{\theta}) = -\{U_{r,s}\}$, where the elements $U_{r,s}$ for $r, s = \alpha, \beta, a, b, c$ are given in the Appendix. The estimated asymptotic multivariate normal $N_5\left(\mathbf{0}, \mathbf{J}\left(\widehat{\boldsymbol{\theta}}\right)^{-1}\right)$ distribution of $\widehat{\boldsymbol{\theta}}$ can be used to construct approximate confidence regions for the parameters and for the hazard rate and survival functions. An asymptotic confidence interval with significance level γ for each parameter θ_r is given by

$$\mathrm{ACI}\left(\theta_r, 100(1-\gamma)\%\right) = \left(\widehat{\theta}_r - z_{1-\gamma/2}\sqrt{\widehat{\kappa}^{\theta_r,\theta_r}}, \widehat{\theta}_r + z_{1-\gamma/2}\sqrt{\widehat{\kappa}^{\theta_r,\theta_r}}\right),$$

where $\widehat{\kappa}^{\theta_r,\theta_r}$ is the rth diagonal element of $\mathbf{J}\left(\boldsymbol{\theta}\right)^{-1}$ estimated at $\widehat{\boldsymbol{\theta}}$ for $r = 1, \ldots, 4$, and $z_{1-\gamma/2}$ is the $100(1-\gamma/2)$ percentile of the standard normal distribution.

The likelihood ratio (LR) statistic is useful for comparing the new distribution with some of its sub-models. For example, we may adopt the LR statistic to check if the fit using the KBBS distribution is statistically "superior" to a fit using the BS distribution for a given data set. In any case, considering the partition $\boldsymbol{\theta} = \left(\boldsymbol{\theta}_1^T, \boldsymbol{\theta}_2^T\right)^T$, tests of hypotheses of the type $H_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_1^{(0)}$ versus $H_A : \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_1^{(0)}$ can be performed using the LR statistic $w = 2\left\{\ell\left(\widehat{\boldsymbol{\theta}}\right) - \ell\left(\widetilde{\boldsymbol{\theta}}\right)\right\}$, where $\widehat{\boldsymbol{\theta}}$ and $\widetilde{\boldsymbol{\theta}}$ are the estimates of $\boldsymbol{\theta}$ under H_A and H_0, respectively. Under the null hypothesis H_0, w approaches χ_q^2 as $n \to \infty$, where q is the dimension of the vector $\boldsymbol{\theta}_1$ of interest. The LR test rejects H_0 if $w > \xi_\gamma$, where ξ_γ denotes the upper 100γ percentile of the χ_q^2 distribution.

9.2. Simulation study. Here, we assess the performance of the MLEs with respect to sample size n. The assessment is based on a simulation study:

(1) generate ten thousand samples of size n from (2.1). The inversion method was used to generate samples, i.e., variates of the KBBS distribution were generated by solving

$$K\int_0^{\Phi(\rho(X/\beta)/\alpha)} t^{a-1}(1-t)^{b-1}\exp(-ct)dt = U,$$

where $U \sim U(0,1)$ is a uniform variate on the unit interval.

(2) compute the MLEs for the ten thousand samples, say $\left(\widehat{a}_i, \widehat{b}_i, \widehat{c}_i, \widehat{\alpha}_i, \widehat{\beta}_i\right)$ for $i = 1, 2, \ldots, 10000$.

(3) compute the standard errors of the MLEs for the ten thousand samples, say $\left(s_{\widehat{a}_i}, s_{\widehat{b}_i}, s_{\widehat{c}_i}, s_{\widehat{\alpha}_i}, s_{\widehat{\beta}_i} \right)$ for $i = 1, 2, \ldots, 10000$. The standard errors were computed by inverting the observed information matrices.

(4) compute the biases and mean squared errors given by

$$\mathrm{bias}_h(n) = \frac{1}{10000} \sum_{i=1}^{10000} \left(\widehat{h}_i - h \right),$$

$$\mathrm{MSE}_h(n) = \frac{1}{10000} \sum_{i=1}^{10000} \left(\widehat{h}_i - h \right)^2$$

for $h = a, b, c, \alpha, \beta$.

(5) compute the coverage probabilities and coverage lengths given by

$$\mathrm{CP}_h(n) = \frac{1}{10000} \sum_{i=1}^{10000} I \left\{ \widehat{h}_i - 1.959964 s_{\widehat{h}_i} < h < \widehat{h}_i + 1.959964 s_{\widehat{h}_i} \right\},$$

$$\mathrm{CL}_h(n) = \frac{3.919928}{10000} \sum_{i=1}^{10000} s_{\widehat{h}_i}$$

for $h = a, b, c, \alpha, \beta$, where $I\{\cdot\}$ denotes the indicator function.

We repeated these steps for $n = 10, 11, \ldots, 100$ with $a = 1$, $b = 1$, $c = 1$, $\alpha = 1$ and $\beta = 1$, so computing $\mathrm{bias}_h(n)$, $\mathrm{MSE}_h(n)$, $\mathrm{CP}_h(n)$ and $\mathrm{CL}_h(n)$ for $h = a, b, c, \alpha, \beta$ and $n = 10, 11, \ldots, 100$.

Figure 5 shows how the five biases vary with respect to n. Figure 6 shows how the five mean squared errors vary with respect to n. Figure 7 shows how the five coverage probabilities vary with respect to n. Figure 8 shows how the five coverage lengths vary with respect to n. The broken line in Figure 5 corresponds to the biases being zero. The broken line in Figure 6 corresponds to the mean squared errors being zero. The broken line in Figure 7 corresponds to the nominal coverage probability of 0.95.

The following observations can be drawn from the figures: the biases for a, c and α are generally positive; the biases for b and β are generally negative; the biases appear smallest for the parameter, b; the biases appear largest for the parameter, c; the biases for each parameter either decrease or increase to zero as $n \to \infty$; the mean squared errors appear smallest for the parameters, b and β; the mean squared errors appear largest for the parameter, c; the mean squared errors for each parameter decrease to zero as $n \to \infty$; the coverage probabilities for each parameter are reasonably close enough to the nominal level for n greater than or equal to sixty; the coverage lengths appear smallest for the parameter, b; the coverage lengths appear largest for the parameters, a and c; the coverage lengths for each parameter decrease to zero as $n \to \infty$. These observations are for only one choice for (a, b, c, α, β), namely that $(a, b, c, \alpha, \beta) = (1, 1, 1, 1, 1)$. But the results were similar for other choices.

Section 10 presents two real data applications. The sample size for the first data set is sixty six. The sample size for the second data set is one hundred and one.

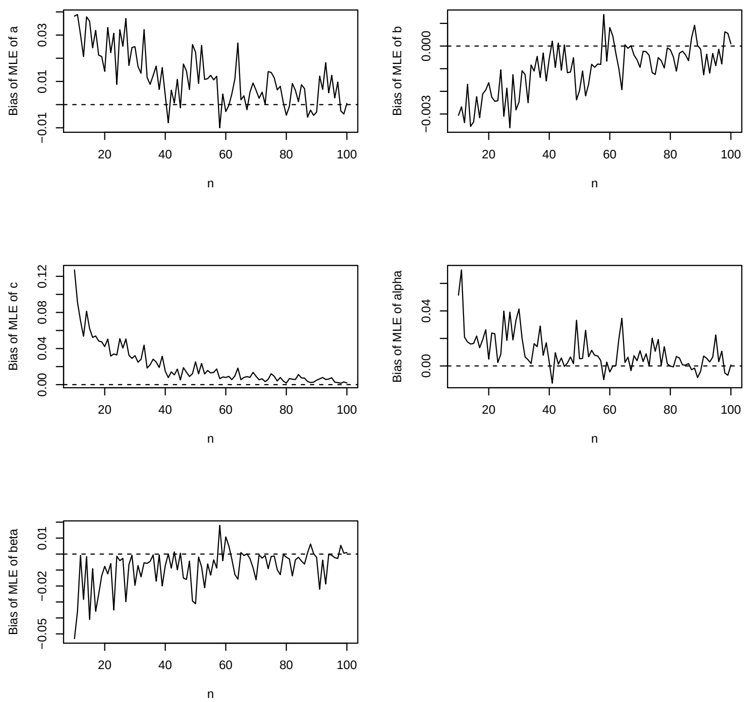

Figure 5. Biases of the MLEs of (a, b, c, α, β) versus $n = 10, 11, \ldots, 100$.

Hence, the biases for $\widehat{a}$, $\widehat{b}$, $\widehat{c}$, $\widehat{\alpha}$ and $\widehat{\beta}$ can be expected to be less than 0.01, 0.001, 0.02, 0.02 and 0.01, respectively, for both data sets. The mean squared errors for $\widehat{a}$, $\widehat{b}$, $\widehat{c}$, $\widehat{\alpha}$ and $\widehat{\beta}$ can be expected to be less than 0.005, 0.003, 0.02, 0.002 and 0.002, respectively, for both data sets. The coverage probabilities can be expected to be accurate for $\widehat{a}$, $\widehat{b}$, $\widehat{c}$, $\widehat{\alpha}$ and $\widehat{\beta}$ for both data sets. The coverage lengths for $\widehat{a}$, $\widehat{b}$, $\widehat{c}$, $\widehat{\alpha}$ and $\widehat{\beta}$ can be expected to be less than 0.3, 0.01, 0.2, 0.2 and 0.2, respectively, for both data sets. Hence, the point as well as interval estimates given in Section 10 can be considered accurate enough.

10. Applications

In this section, we use two data sets to compare the fits of the KBBS distribution with those of two sub-models (i.e., the beta-BS (BBS) and BS distributions) and also to the following non-nested models: the McDonald-Birnbaum-Saunders (McBS) (Cordeiro et al., [19]), the McDonald-gamma (McGa) (Marciano et al., [50]), the length-biased-Birnbaum-Saunders (LBS) (Leiva et al., [38]), the extended Birnbaum-Saunders (ExBS) (Leiva et al., [41]), the Marshall-Olkin extended Birnbaum-Saunders (MOEBS) (Lemonte, [42]) and the generalized Birnbaum-Saunders (GBS) (Owen, [60]) distributions. All the computations were performed using the R statistical software. Obviously, due to the genesis of the BS and

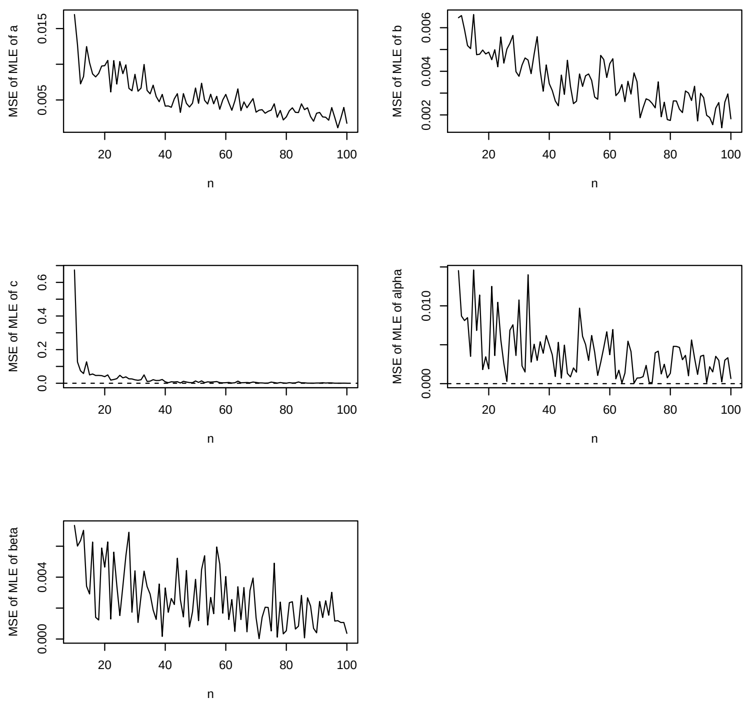

Figure 6. Mean squared errors of the MLEs of (a, b, c, α, β) versus $n = 10, 11, \ldots, 100$.

gamma distributions, the fatigue processes are ideally modeled by these distributions. Thus, the use of the KBBS distribution and its sub-models and also other lifetime distributions for fitting the data sets is justified.

10.1. Breaking stress of carbon fibres data. Here, we shall compare the fitted KBBS, BBS, BS, McBS, McGa, LBS, ExBS, MOEBS and GBS distributions to the data from Nichols and Padgett [58] on the breaking stress of carbon fibres (in Gba). Nichols and Padgett [58] described the data from a process which produces carbon fibers to be used in constructing fibrous composite materials. The carbon fiber fifty millimeters in length were sampled ($n = 66$) from the process, tested and their tensile strength were observed.

Firstly, in order to estimate the parameters, we consider the maximum likelihood estimation method discussed in Section 9. We take the estimates of α and β from the fitted BS distribution as starting values for the numerical iterative procedure. Table 2 lists the MLEs of the parameters and the values of the following statistics: Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC) and Bayesian Information Criterion (BIC). The results indicate that the KBBS distribution has the smallest values of these statistics, and so, it could be chosen as the more suitable distribution.

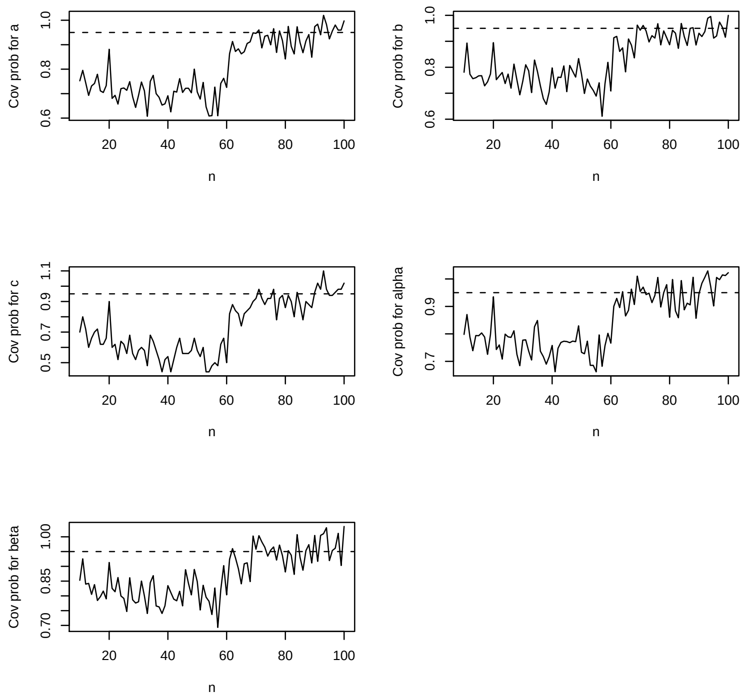

Figure 7. Coverage probabilities of the MLEs of (a, b, c, α, β) versus $n = 10, 11, \ldots, 100$.

Table 2. MLEs (standard errors in parentheses) and information criteria for breaking stress of carbon fibres data.

Model	α	β	a	b	c	AIC	BIC	CAIC
KBBS	0.6770	2.9944	0.3430	11.4176	-22.2353	179.6	190.6	181.6
	(0.5479)	(0.2732)	(0.2079)	(3.7988)	(5.4008)			
BBS	1.0452	57.5997	0.1990	1876.8935	0	191.6	200.4	193.0
	(0.0039)	(0.3413)	(0.0219)	(605.0512)				
BS	0.43712	2.51540	1	1	0	204.3	208.7	205.0
	(0.0380)	(0.1321)						

A comparison of the proposed distribution with some of its sub-models using LR statistics is given in Table 3. We reject the null hypotheses of the two LR tests in favor of the KBBS distribution. The rejection is extremely highly significant. This gives a clear evidence of the potential of the three skewness parameters when modeling real data.

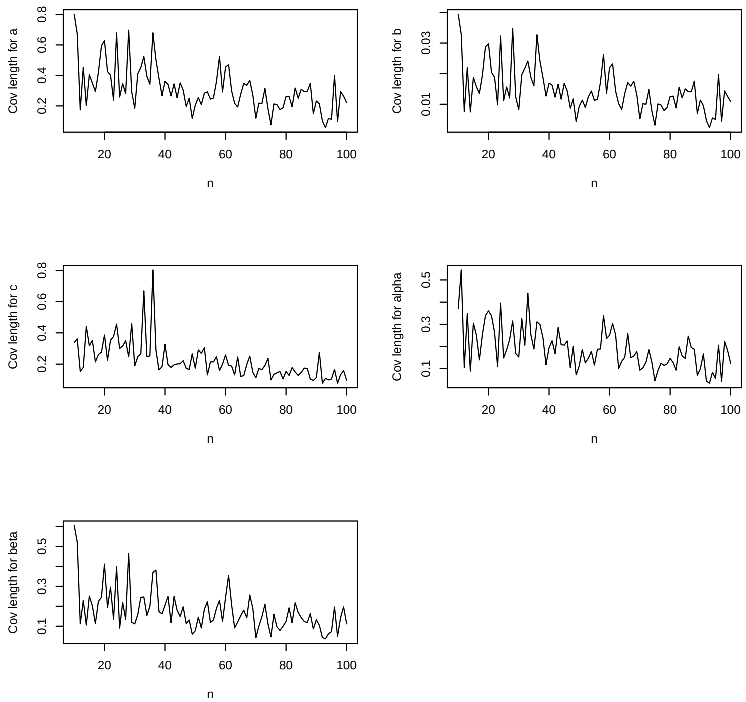

Figure 8. Coverage lengths of the MLEs of (a, b, c, α, β) versus $n = 10, 11, \ldots, 100$.

Table 3. LR statistics for breaking stress of carbon fibres data.

Model	Hypotheses	Statistic w	p-value
KBBS vs BBS	$H_0 : c = 0$ vs $H_1 : H_0$ is false	30.69	< 0.0001
KBBS vs BS	$H_0 : a = b = 1$ and $c = 0$ vs $H_1 : H_0$ is false	13.08	0.00029

In order to assess if the distribution is appropriate, Figures 9a and 9b display plots of the estimated pdfs and survival functions of the KBBS distribution and its sub-models. We can conclude that the KBBS distribution is a very suitable distribution to fit the data.

Secondly, we shall apply formal goodness-of-fit tests in order to verify which distribution fits the data better. We consider the Cramér-Von Mises (W^*) and Anderson-Darling (A^*) statistics. In general, the smaller the values of the statistics, W^* and A^*, the better the fit to the data. Let $F(x; \boldsymbol{\theta})$ be the cdf, where the form of F is known but $\boldsymbol{\theta}$ (a k-dimensional parameter vector, say) is unknown. To obtain the statistics, W^* and A^*, we proceed as follows:

(a) (b)

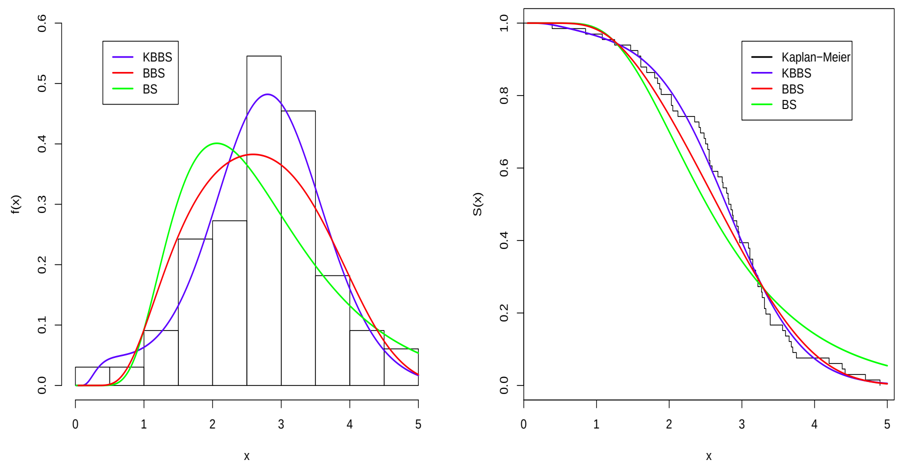

Figure 9. (a) Estimated pdfs of the KBBS distribution and its sub-models for breaking stress of carbon fibres data. (b) Empirical and estimated survival functions of the KBBS distribution and its sub-models for breaking stress of carbon fibres data.

(i) compute $v_i = F\left(x_i; \widehat{\boldsymbol{\theta}}\right)$, where the x_i's are in ascending order, $y_i = \Phi^{-1}(v_i)$ is the standard normal quantile function and $u_i = \Phi\left\{(y_i - \overline{y})/s_y\right\}$, where $\overline{y} = n^{-1}\sum_{i=1}^{n} y_i$ and $s_y^2 = (n-1)^{-1}\sum_{i=1}^{n}(y_i - \overline{y})^2$;

(ii) compute

$$W^2 = \sum_{i=1}^{n}\left\{u_i - (2i-1)/(2n)\right\}^2 + 1/(12n)$$

and

$$A^2 = -n - n^{-1}\sum_{i=1}^{n}\left\{(2i-1)\log(u_i) + (2n+1-2i)\log(1-u_i)\right\};$$

(iii) modify W^2 into $W^* = W^2(1+0.5/n)$ and A^* into $A^* = A^2\left(1 + 0.75/n + 2.25/n^2\right)$.

For further details, the reader is referred to Chen and Balakrishnan [16]. The values of the statistics, W^* and A^*, for the distributions are given in Table 4. Thus, according to these formal goodness-of-fit tests, the KBBS distribution fits the data better than its sub-models.

The MLEs (standard errors in parentheses) of the parameters of the LBS, MOEBS, GBS, ExBS, McBS and McGa distributions are listed in Table 5. On the basis of the statistics given in this table, the ExBS distribution yields a better fit than others. Overall, by comparing the measures in Tables 4 and 5, we conclude that the KBBS distribution outperforms all the distributions considered in Table 5. So, the proposed distribution can yield better fits than the LBS, MOEBS, GBS,

Table 4. Formal goodness-of-fit tests for breaking stress of carbon fibres data.

Model	Statistic	
	W^*	A^*
KBBS	0.0498	0.3123
BBS	0.2115	1.2216
BS	0.4603	2.5896

ExBS, McBS and McGa distributions and therefore may be an interesting alternative to these distributions for modeling fatigue lifetime data sets. These results illustrate the potentiality of the new distribution and the necessity for additional shape parameters.

Table 5. MLEs (standard errors in parentheses) and the measures, W^* and A^*, for breaking stress of carbon fibres data.

Model	Estimates					Statistic	
						W^*	A^*
ExBS	$\widehat{\alpha} = 3.3418$	$\widehat{\beta} = 0.5840$	$\widehat{\sigma} = 0.7586$	$\widehat{v} = -2.3019$	$\widehat{\lambda} = 0.0179$	0.0599	0.3825
	(1.8483)	(0.3675)	(0.1922)	(0.1145)	(0.0024)		
McBS	$\widehat{\alpha} = 3.8736$	$\widehat{\beta} = 0.1487$	$\widehat{a} = 18.8160$	$\widehat{\eta} = 35.5380$	$\widehat{c} = 29.00002$	0.0935	0.5223
	(0.1444)	(0.0923)	(0.4067)	(4.5916)	(1.2795)		
McGa	$\widehat{\alpha} = 28.5769$	$\widehat{\beta} = 2.3734$	$\widehat{a} = 0.1240$	$\widehat{b} = 48.0712$	$\widehat{c} = 0.2335$	0.0812	0.5173
	(4.0265)	(0.9942)	(0.5479)	(2.7540)	(0.1044)		
GBS	$\widehat{\alpha} = 0.5409$	$\widehat{\beta} = 2.6613$	$\widehat{\kappa} = 0.0009$			0.0742	0.4176
	(0.0249)	(0.1687)	(0.0001)				
MOEBS	$\widehat{\alpha} = 0.4358$	$\widehat{\beta} = 2.4723$	$\widehat{\eta} = 1.1187$			0.4453	2.5048
	(0.0379)	(0.1536)	(0.2439)				
LBS	$\widehat{\alpha} = 0.4410$	$\widehat{\beta} = 2.0919$				0.4192	2.3516
	(0.0396)	(0.1314)					

The QQ plots of the normalized quantile residuals was introduced by Dunn and Smyth [22] and more recently used by Cordeiro et al. [17]. Figures 10 and 11 show the improved fit achieved using the KBBS distribution over other distributions. We also emphasize the gain yielded by the KBBS distribution in relation to the BS, BBS, McBS, McGa, LBS, ExBS, MOEBS and GBS distributions.

10.2. Aluminum alloy fatigue data. The data refer to the fatigue life of 6061 - T6 aluminum coupons cut parallel to the direction of rolling and oscillated at eighteen cycles per second. It was reported and analyzed by Birnbaum and Saunders [12]. The KBBS distribution seems to be an appropriate distribution for fitting these data. Table 6 lists the MLEs (standard errors in parentheses) of the parameters. The results indicate that the KBBS distribution has the smallest values of the statistics (AIC and CAIC) in relation to its sub-models.

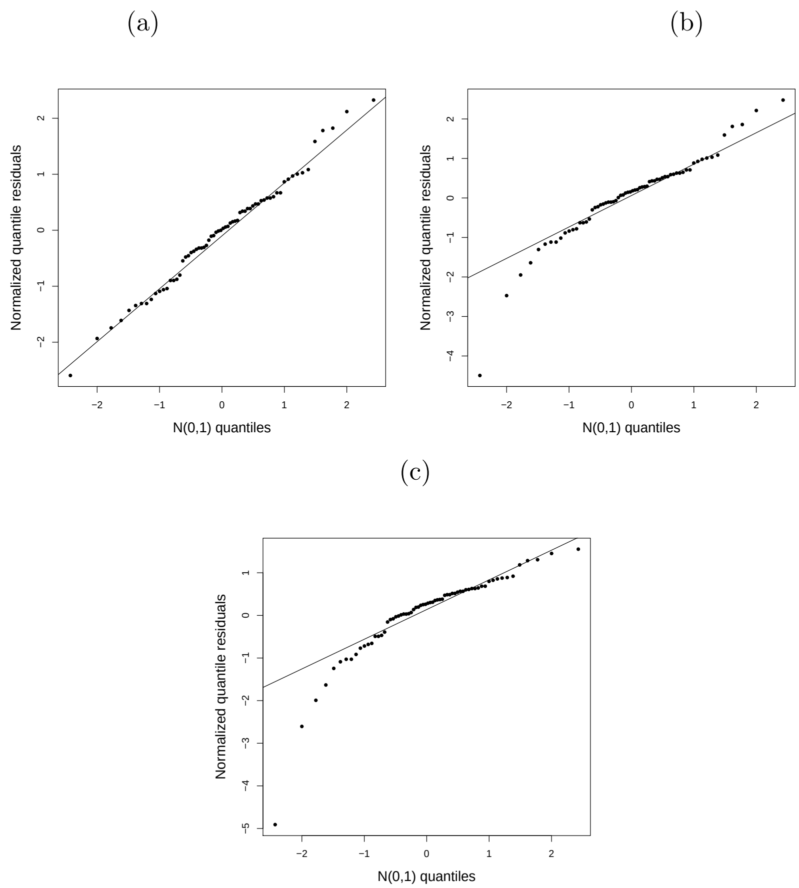

Figure 10. QQ plot of the normalized quantile residuals for the distributions: (a) KBBS, (b) BBS and (c) BS for breaking stress of carbon fibres data.

Table 6. MLEs (standard errors in parentheses) and information criteria for aluminum alloy fatigue data.

Model	α	β	a	b	c	AIC	BIC	CAIC
KBBS	0.9654	2065.821	0.9161	38.5452	-58.0575	1501.1	1514.2	1502.3
	(0.0809)	(24.1268)	(0.1069)	(1.4887)	(1.8296)			
BBS	0.2817	1600.382	0.6278	1.2967	0	1508.0	1518.5	1508.9
	(0.0114)	(29.1456)	(0.0457)	(0.0926)				
BS	0.3103	1336.377	1	1	0	1506.7	1512.0	1507.1
	(0.0218)	(40.7665)						

A test for the need for the third skewness parameter in the KBBS distribution can be based on the LR statistic described in Section 9. Applying the LR statistics to these data, the results are listed in Table 7. The p-values show that the proposed distribution yields the best fit to the data.

Table 7. LR statistics for aluminum alloy fatigue data.

Model	Hypotheses	Statistic w	p-value
KBBS vs BBS	$H_0 : c = 0$ vs $H_1 : H_0$ is false	8.92	0.0028
KBBS vs BS	$H_0 : a = b = 1$ and $c = 0$ vs $H_1 : H_0$ is false	11.65	0.0086

In Figure 12a, we provide the histogram of the data and the fitted KBBS, BBS and BS pdfs while in Figure 12b we display plots of the empirical and estimated survival functions of the KBBS distribution and some of its sub-models. We note that the KBBS distribution provides a satisfactory fit.

We can also perform formal goodness-of-fit tests in order to verify which distribution fits the data better. We apply the Cramér-Von Mises (W^*) and Anderson-Darling (A^*) statistics. The values of the statistics, W^* and A^*, for the KBBS distribution and its sub-models are given in Table 8. Thus, according to these formal tests, the KBBS distribution fits the data better than its sub-models.

Table 8. Formal goodness-of-fit test for aluminum alloy fatigue data.

Model	Statistic	
	W^*	A^*
KBBS	0.0249	0.1719
BBS	0.0758	0.5169
BS	0.1022	0.6806

The MLEs (standard errors in parentheses) of the parameters of the LBS, MOEBS, GBS, ExBS, McBS and McGa distributions are listed in Table 9. On the basis of the statistics given in this table, the ExBS distribution yields a better fit than others. Overall, by comparing the measures in Tables 8 and 9, we conclude that the KBBS distribution outperforms all the distributions considered in Table 9. So, the proposed distribution can yield a better fit than the LBS, MOEBS, GBS, ExBS, McBS and McGa distributions.

The QQ plots of the normalized quantile residuals in Figures 13 and 14 reveal the improvement in the fit achieved by the KBBS distribution over the others.

11. Concluding remarks

The Birnbaum-Saunders (BS) distribution is widely used to model times to failure for materials subject to fatigue. We proposed the Kummer beta generalized Birnbaum-Saunders (KBBS) distribution to extend the BS distribution introduced by Birnbaum and Saunders [11]. We provided a mathematical treatment of the new distribution including expansions for the cdfs and pdfs. We derived

Table 9. MLEs (standard errors in parentheses) and the measures, W^* and A^*, for aluminum alloy fatigue data.

Model	Estimates					Statistic	
						W^*	A^*
ExBS	$\widehat{\alpha} = 2.7389$	$\widehat{\beta} = 2.7518$	$\widehat{\sigma} = 2.0811$	$\widehat{v} = -7.2496$	$\widehat{\lambda} = -0.0586$	0.0292	0.2057
	(0.4777)	(0.9933)	(1.3579)	(0.4584)	(1.7623)		
McBS	$\widehat{\alpha} = 0.3624$	$\widehat{\beta} = 1600.3540$	$\widehat{a} = 0.8001$	$\widehat{\eta} = 18.4695$	$\widehat{c} = 16.6618$	0.0420	0.3119
	(0.0898)	(45.7822)	(0.4383)	(0.8048)	(0.3104)		
McGa	$\widehat{\alpha} = 21.2870$	$\widehat{\beta} = 0.0134$	$\widehat{a} = 0.4765$	$\widehat{b} = 0.9109$	$\widehat{c} = 0.0007$	0.0283	0.2038
	(3.2584)	(0.0021)	(0.0249)	(0.0397)	(0.0002)		
GBS	$\widehat{\alpha} = 10.2155$	$\widehat{\beta} = 1514.9490$	$\widehat{\kappa} = 0.00098$			0.0308	0.2074
	(2.4781)	(32.1458)	(0.00001)				
MOEBS	$\widehat{\alpha} = 0.3101$	$\widehat{\beta} = 1332.9960$	$\widehat{\eta} = 1.0379$			0.0999	0.6669
	(0.0218)	(48.3511)	(0.1850)				
LBS	$\widehat{\alpha} = 0.3109$	$\widehat{\beta} = 1216.4778$				0.4192	2.3516
	(0.0220)	(40.6364)					

expansions for the ordinary and incomplete moments, generating function, mean deviations and the moments of the order statistics. The estimation of parameters is approached by the method of maximum likelihood and the observed information matrix was derived. We considered likelihood ratio (LR) statistics and formal goodness-of-fit tests to compare the KBBS distribution with some of its sub-models and non-nested models. Applications of the KBBS distribution to two real data sets indicated that the new distribution provides consistently better fits than its sub-models and other lifetime models. We hope that this generalization may attract wider applications in the literature of the fatigue life distributions.

Appendix: Elements of the observed information matrix

The elements of the observed information matrix, $\mathbf{J}(\boldsymbol{\theta})$, for the parameters α, β, a, b and c are:

$$
\begin{aligned}
U_{\alpha\alpha} =& \frac{n}{\alpha^2} + \frac{6n}{\alpha^3} - \frac{3}{\alpha^4}\sum_{i=1}^{n}\left(\frac{x_i}{\beta} + \frac{\beta}{x_i}\right) + \frac{2(a-1)}{\alpha^2}\sum_{i=1}^{n}\frac{\nu_i\phi(\nu_i)}{\Phi(\nu_i)} \\
&-\frac{2(b-1)}{\alpha^2}\sum_{i=1}^{n}\frac{\nu_i\phi(\nu_i)}{1-\Phi(\nu_i)} + \frac{a-1}{\alpha^3}\sum_{i=1}^{n}\left\{\frac{\nu_i^4\phi(\nu_i)}{\Phi(\nu_i)} - \frac{\alpha\nu_i^2\phi^2(\nu_i)}{\Phi(\nu_i)}\right\} \\
&-\frac{b-1}{\alpha^3}\sum_{i=1}^{n}\left\{\frac{\nu_i^4\phi(\nu_i)}{1-\Phi(\nu_i)} - \frac{\alpha\nu_i^2\phi^2(\nu_i)}{1-\Phi(\nu_i)}\right\} - \frac{4c}{\alpha}\sum_{i=1}^{n}\nu_i\phi(\nu_i)\left[1+\phi(\nu_i)\right], \\
U_{\alpha\beta} =& -\frac{1}{\alpha^3\beta}\sum_{i=1}^{n}\left(\frac{x_i}{\beta} - \frac{\beta}{x_i}\right) \\
&+\frac{a-1}{2\alpha^2\beta}\sum_{i=1}^{n}\left\{\frac{\alpha\nu_i\phi(\nu_i)}{\Phi(\nu_i)} + \frac{\nu_i^4\phi(\nu_i)}{\Phi(\nu_i)} - \frac{\alpha\nu_i^2\phi^2(\nu_i)}{\Phi^2(\nu_i)}\right\} \\
&-\frac{b-1}{2\alpha^2\beta}\sum_{i=1}^{n}\left\{\frac{\alpha\nu_i\phi(\nu_i)}{1-\Phi(\nu_i)} + \frac{\nu_i^4\phi(\nu_i)}{1-\Phi(\nu_i)} - \frac{\alpha\nu_i^2\phi^2(\nu_i)}{\left[1-\Phi(\nu_i)\right]^2}\right\} \\
&-\frac{c}{\alpha\beta}\sum_{i=1}^{n}\nu_i\phi(\nu_i)\,\tau\left(\sqrt{x_i/\beta}\right), \\
U_{\beta\beta} =& \frac{n}{2\beta^2} - \sum_{i=1}^{n}(x_i+\beta)^{-2} - \frac{1}{\alpha^2\beta^3}\sum_{i=1}^{n}x_i \\
&+\frac{a-1}{2\alpha\beta^2}\sum_{i=1}^{n}\frac{\phi(\nu_i)\,\tau\left(\sqrt{x_i/\beta}\right)}{\Phi(\nu_i)} - \frac{b-1}{2\alpha\beta^2}\sum_{i=1}^{n}\frac{\phi(\nu_i)\,\tau\left(\sqrt{x_i/\beta}\right)}{1-\Phi(\nu_i)} \\
&-\frac{a-1}{4\alpha\beta^2}\sum_{i=1}^{n}\left\{\frac{-\alpha\nu_i\phi(\nu_i)}{\Phi(\nu_i)} + \frac{\nu_i\phi(\nu_i)\,\tau^2\left(\sqrt{x_i/\beta}\right)}{\alpha\Phi(\nu_i)} + \frac{\nu_i\phi^2(\nu_i)\,\tau^2\left(\sqrt{x_i/\beta}\right)}{\alpha\Phi^2(\nu_i)}\right\} \\
&-\frac{c}{\alpha\beta^2}\sum_{i=1}^{n}\left\{\alpha\nu_i\phi(\nu_i) + \alpha^{-1}\tau^2\left(\sqrt{x_i/\beta}\right)\phi^2(\nu_i)\right\}, \\
U_{aa} =& \frac{n}{K}\left\{\frac{\partial^2 K}{\partial a^2} - \frac{1}{K}\left(\frac{\partial K}{\partial a}\right)^2\right\}, \\
U_{bb} =& \frac{n}{K}\left\{\frac{\partial^2 K}{\partial b^2} - \frac{1}{K}\left(\frac{\partial K}{\partial b}\right)^2\right\}, \\
U_{cc} =& \frac{n}{K}\left\{\frac{\partial^2 K}{\partial c^2} - \frac{1}{K}\left(\frac{\partial K}{\partial c}\right)^2\right\}, \\
U_{ab} =& \frac{n}{K}\left\{\frac{\partial^2 K}{\partial a\partial b} - \frac{1}{K}\frac{\partial K}{\partial a}\frac{\partial K}{\partial b}\right\}, \\
U_{ac} =& \frac{n}{K}\left\{\frac{\partial^2 K}{\partial a\partial c} - \frac{1}{K}\frac{\partial K}{\partial a}\frac{\partial K}{\partial c}\right\}, \\
U_{bc} =& \frac{n}{K}\left\{\frac{\partial^2 K}{\partial b\partial c} - \frac{1}{K}\frac{\partial K}{\partial b}\frac{\partial K}{\partial c}\right\},
\end{aligned}
$$

where $\frac{\partial^2 K}{\partial a^2}$, $\frac{\partial^2 K}{\partial b^2}$, $\frac{\partial^2 K}{\partial c^2}$, $\frac{\partial^2 K}{\partial a \partial b}$, $\frac{\partial^2 K}{\partial a \partial c}$ and $\frac{\partial^2 K}{\partial b \partial c}$ are defined in Pescim et al. [63].

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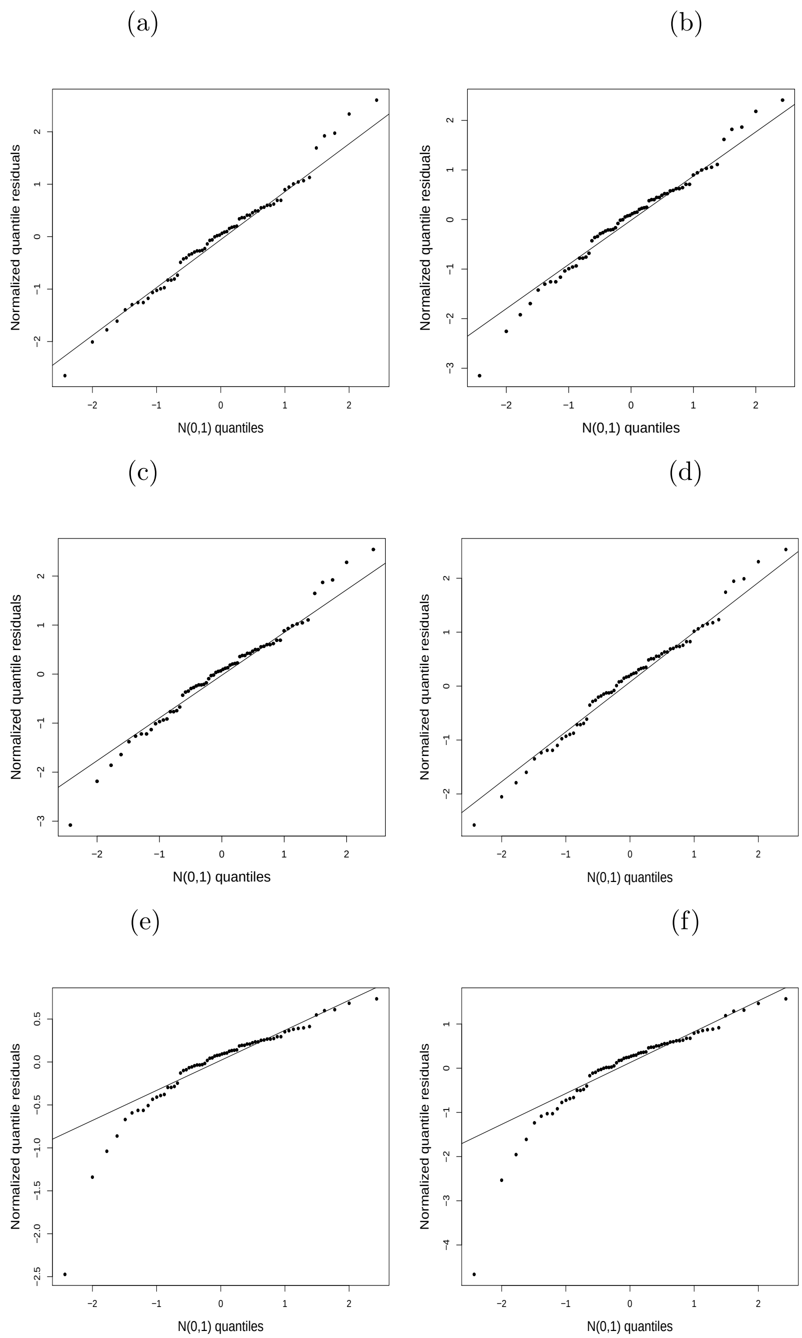
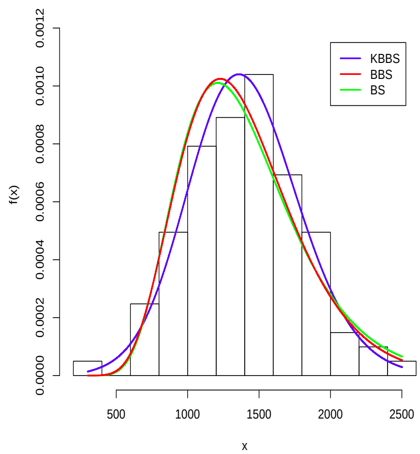


Figure 11. QQ plot of the normalized quantile residuals for the distributions: (a) ExBS, (b) McBS, (c) McGa, (d) GBS, (e) MOEBS and (f) LBS for breaking stress of carbon fibres data.

(a) (b)



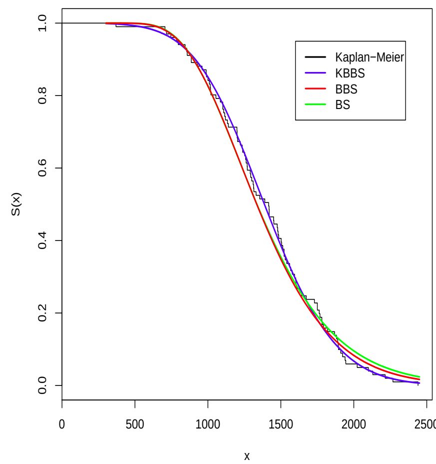


Figure 12. (a) Estimated pdfs of the KBBS distribution and its sub-models. (b) Empirical and estimated survival functions of the KBBS distribution and its sub-models.

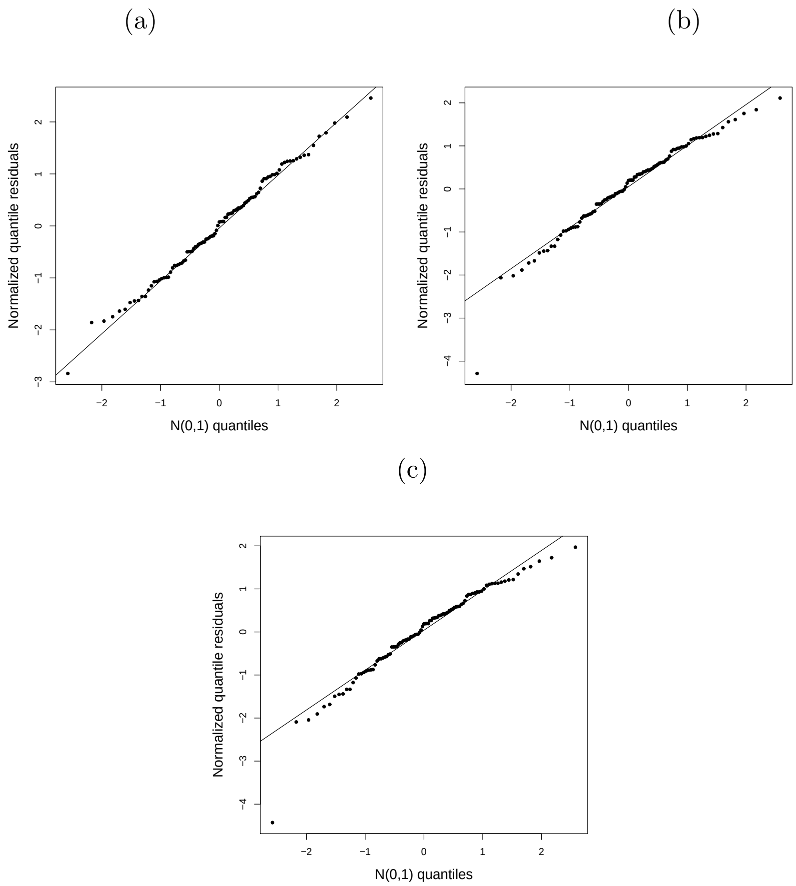


Figure 13. QQ plot of the normalized quantile residuals for the distributions: (a) KBBS, (b) BBS and (c) BS for aluminum alloy fatigue data.

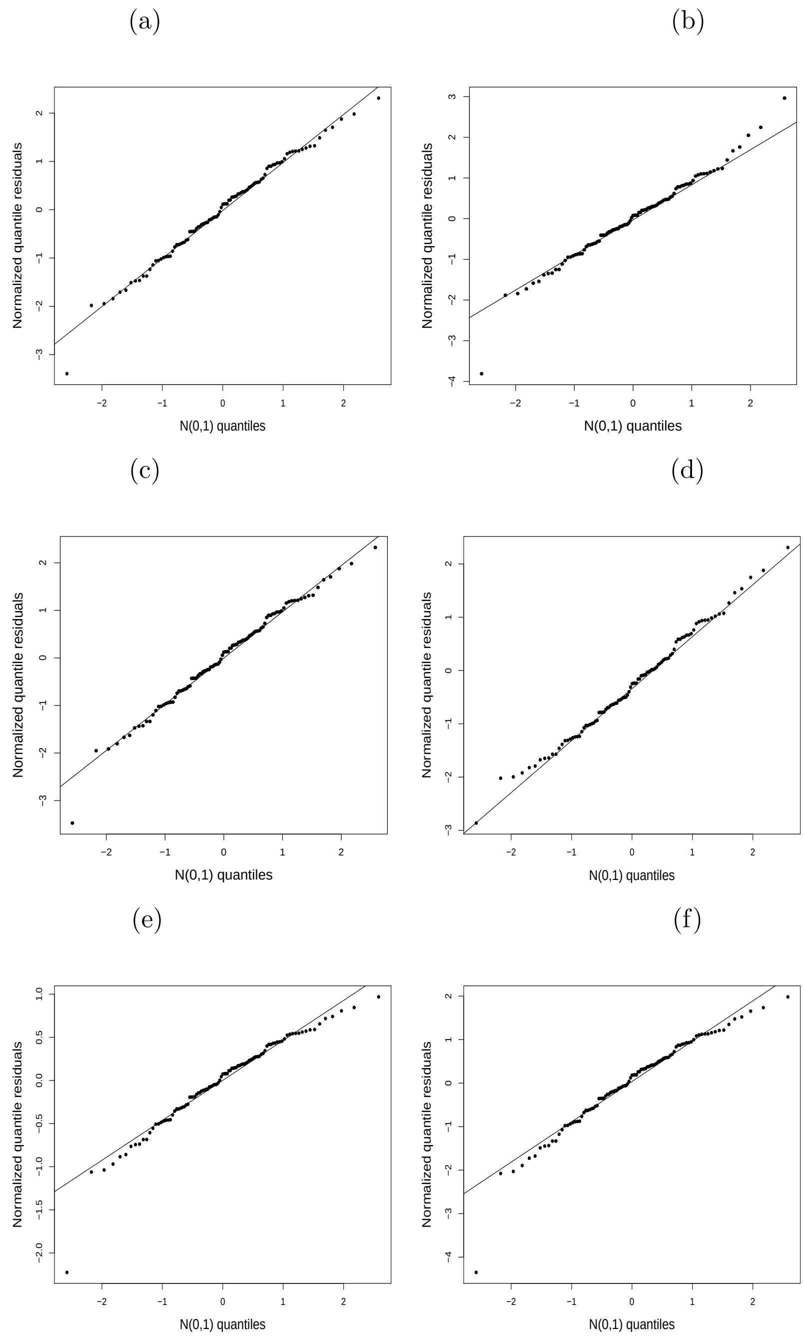


Figure 14. QQ plot of the normalized quantile residuals for the distributions: (a) ExBS, (b) McBS, (c) McGa, (d) GBS, (e) MOEBS and (f) LBS for aluminum alloy fatigue data.