Sakarya University Journal of Science

ISSN 1301-4048 | e-ISSN 2147-835X | Period Bimonthly | Founded: 1997 | Publisher Sakarya University | http://www.saujs.sakarya.edu.tr/

Title: On A Graph Of Submodules

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Recieved: 2018-11-15 00:00:00

Accepted: 2018-12-20 00:00:00

Article Type: Research Article
Volume: 23
Issue: 3
Month: June
Year: 2019
Pages: 396-402

How to cite
Ali Öztürk, Tahire Özen, Erol Yılmaz; (2019), On A Graph Of Submodules. Sakarya University Journal of Science, 23(3), 396-402, DOI: 10.16984/saufenbilder.483138
Access link
http://www.saujs.sakarya.edu.tr/issue/41686/483138

Sakarya University Journal of Science 23(3), 396-402, 2019

On A Graph of Submodules

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Abstract

Let S be an associative ring with identity and N be a right S-module. We define the non-maximal graph μ(N) of N with all non-trivial submodules of N as vertices and two distinct vertices A, B are adjacent if and only if A + B is not a maximal submodule of N. In this paper, we investigate the connectivity, completeness, girth, domination number, cut edges, perfectness and r-partite of μ(N). Moreover, we give connections between the graph-theoretic properties of μ(N) and algebraic properties of N.

Keywords: Non-maximal submodule, connected and complete graph, clique and chromatic number.

1. INTRODUCTION

Throughout this paper, S will be an associative ring with identity. Let N be a right S-module. If X is a proper submodule of N and there exists no Y such that $X < Y < N$, then X is called a maximal submodule of N. If X is not a maximal submodule of N, then $X = N$ or there exists $Y < N$ such that $X < Y < N$. (See [2] for unknown concepts in module theory.)

An undirected graph G is defined as the pair (V(G), E(G)), where V(G) is the set of vertices of G and E(G) is the set of edges of G which have no orientation. For two distinct vertices A and B, $A - B$ means that A and B are adjacent. By the null-graphs we mean that with no edges. If $|V(G)| \geq 2$, a path from A to B is a series of adjacent vertices $A - V_1 - V_2 - \ldots - V_n - B$. The distance between two vertices A and B in a graph is the number of edges in a shortest path connecting them and denoted by d(A,B). If there is no path between A and B, $d(A,B) = \infty$. $\text{diam}(G) = \sup\{d(A,B) : A,B \in V(G)\}$ is a diameter of a graph G. A graph is connected if for any vertices A and B there is a path between A and B. If there is no path, then G is disconnected. The girth of G is the length of the shortest cycle of G and denoted by g(G). A complete graph G is a graph with an edge between every two vertices. A clique of a graph is its maximal complete subgraph and the number of vertices in the largest clique is called the clique number, denoted by w(G). $D \subset G$ is a dominating set if for all $A \in V(G)$, there exists at least one $B \in D$ such that A and B are adjacent. The domination number, γ(G), of G is the minimum cardinality of a dominating set of G. The chromatic number, χ(G), is the minimum number of colours which can be assigned to the vertices of G such that every two adjacent vertices have different colours. If $\chi(G) = w(G)$, then G is called a perfect graph. (See [3] for unknown concepts in graph theory.)

We define the non-maximal graph μ(N) of N with all non-trivial submodules of N as vertices and two distinct vertices A, B are adjacent if and only if A+B is not a maximal submodule of N. Firstly, we investigate the connectivity, completeness and girth of μ(N).

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Secondly, we study the domination number, cut edges, pendent vertex of $\mu(N)$. Moreover, we give a counter-example such that $\mu(N)$ may not be perfect and $\mu(N)$ cannot be a complete r-partite graph if $N \neq N_1 \oplus N_2$ where N_1 and N_2 are simple. This graph satisfies us to use module algebraic properties in graph theory. Zero divisor graphs, intersection graphs and their generalizations were investigated in [1, 4, 5, 7].

2. NON-MAXİMAL GRAPH $\mu(N)$

Definition 2.1. Let N be a module. $\mu(N)$ is called a non-maximal graph where the set of vertices of $\mu(N)$ is all non-trivial submodules of N denoted by $V(\mu(N))$ and A and B are adjacent if $A + B$ is not a maximal submodule of N. $E(\mu(N))$ is the set of edges of $\mu(N)$.

Example 2.2

i) Let N have no maximal submodules (for example $\mathbb{Z}_{p^\infty}$). Then $\mu(N)$ is complete.

ii) $\mu(\mathbb{Z}_4)$ and $\mu(\mathbb{Z}_8)$ are null-graphs. For $n \geq 4$, $\mu(\mathbb{Z}_{p^n})$ is not complete and not connected, $w(\mu(\mathbb{Z}_{p^n})) = n-2$.

iii) $\mu(\mathbb{Z}_p \oplus \mathbb{Z}_q)$ ($p \neq q$ primes) is a complete and connected graph.

iv) $\mu(\mathbb{Z}_{p^n} \oplus \mathbb{Z}_q)$ is connected, not complete and clique number is $2n-3$ for $n \geq 3$. $\mu(\mathbb{Z}_{p^2} \oplus \mathbb{Z}_q)$ is not connected, not complete and clique number is 2.

v) $\mu(\mathbb{Z}_4 \oplus \mathbb{Z}_2)$ is not complete, not connected and $w(\mu(\mathbb{Z}_4 \oplus \mathbb{Z}_2))$ = the number of maximal submodules of $\mathbb{Z}_4 \oplus \mathbb{Z}_2 = 3$.

Proposition 2.3. Let N be a right S-module.

i) If $w(\mu(N)) < \infty$, then $l_S(N) < \infty$.

ii) İf N is a non-maximal vertex and $\deg(N) < \infty$, then $l_S(N) < \infty$.

Proof. It follows from the proof of Lemma 3.1. and Lemma 3.4. in [6]. □

Remark 2.4. Let soc(N) be a proper essential submodule of N or soc(N) be a maximal submodule of N. Then $\mu(N)$ may not be connected. For example, in $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_q$ which is cyclic, there exists no path between $\mathbb{Z}_p$ and $\mathbb{Z}_q$. Moreover, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ is not connected and it is two generated.

Now, we investigate the connectivity of some special modules in the following three theorems by considering Remark 2.4.

Theorem 2.5.

i) If Rad (N) = 0, then $\mu(N)$ is connected and diam $(\mu(N)) \leq 3$.

ii) If N is semisimple, then $\mu(N)$ is connected.

iii) Let N be not cyclic, but Artinian module. If soc(N) and cyclic submodules are not maximal in N, then $\mu(N)$ is connected.

Proof.

İ) Let $\mu(N)$ be not connected. Then there exists no path between vertices X and Y.

a) Let $X+Y <^{max} N$. Since Rad (N) = 0, there exists a maximal submodule Y_1 such that $X+Y_1 = N$. If Y is not a submodule of Y_1, then we have a path $X - Y_1 - Y$. Otherwise, there exists a maximal submodule Y_2 such that $Y + Y_2 = N$. Then we have a path $X - Y_1 - Y_2 - Y$.

b) Let $X < Y <^{max} N$. If X is in every maximal submodule of N, then $Rad(N) \neq 0$ which is a contradiction. So there exists a maximal submodule $X \not< Y_1 \neq Y$. By $X - Y_1 - Y$, there is a contradiction.

ii) It follows from part (i).

iii) Let $\mu(N)$ be not connected. Then there exists no path between vertices X and Y.

a) Let $X + Y <^{max} N$. Then $X \cap Y = 0$ and $X \oplus Y <^{max} N$.

a1) Let X not simple. Then there exists $U < X$ and $U \oplus Y < X \oplus Y <^{max} N$. By $Y - U - X$, there is a contradiction.

a2) Let Y not simple. Similarly, there is a contradiction.

a3) Let X and Y be simple. Then $X \oplus Y <^{max} N$ where X and Y are simple. Moreover, soc $(N) <^{ess} N$ and $soc(N) <^{max} N$ by the part (ii), which is a contradiction.

b) Let $X < Y <^{max} N$ and $a \in N - Y$. Then we have simple submodules E, F such that $E \leq X$ and $F \leq aS$ such that $E + F \leq soc(N)$. So, we have a path, which is a contradiction.□

Theorem 2.6. Let N be a module whose number of generators ≥ 3. Then $\mu(N)$ is connected and diam $(\mu(N)) \leq 3$.

Proof. Assume that $\mu(N)$ is not connected. Then there exists no path between vertices X and Y.

a) Let $X + Y <^{max} N$. Then $X \cap Y = 0$ and $X \oplus Y <^{max} N$ where X and Y are simple. Therefore, $soc(N)= X \oplus Y <^{max} N$ and $soc(N)= X \oplus Y <^{ess} N$. Let $a \in N - soc(N)$. Then $soc(N) + aS = N$ and $aS \neq N$. Since $X \not< aS$ and $Y \not< aS$, we have a path $X - Y \oplus aS - X \oplus aS - Y$, which is a contradiction.

b) Let $X < Y <^{max} N$.

b1) Let $X \ll N$. If $x \in N - Y$, then $xS \neq N$ and $Y+ xS = N$. Moreover, xS cannot be maximal. If X+xS is maximal, then there exists $y \in N - (X+xS)$ and then $X + xS + yS = N$. Since $X \ll N$, $xS + yS = N$, which can not be. So, we have a path, which is a contradiction.

b2) Let X be not small in N. Then there exists $T < N$ such that $X + T = N$. Then $Y +T = N$. This gives a contradiction. □

Theorem 2.7. Let N be not cyclic, and it has only one maximal. Then $\mu(N)$ is connected.

Proof. Assume $\mu(N)$ is not connected. Then there exists no path between vertices X and Y.

a) Let $X + Y <^{max} N$. Then $X \cap Y = 0$ and hence $X \oplus Y <^{max} N$ where X and Y are simple. Therefore, soc $(N)= X \oplus Y <^{max} N$ and $soc(N)= X \oplus Y <^{ess} N$. Let $a \in N -$ soc (N). Then we have a path $(X - aS - Y)$, which is a contradiction.

b) Let $X < Y <^{max} N$ and $a \in N - Y$. Then we have a path $(X - aS - Y)$, which is a contradiction. □

Now, we investigate the completeness of $\mu(N)$. If N has no maximal, then $\mu(N)$ is complete for example $\mathbb{Z}_{p^\infty}$.

Theorem 2.8. Let $\mu(N)$ be a complete graph and N has a maximal submodule. Then either $N = N_1 \oplus N_2$ where N_1 and N_2 are simple or N has only one submodule.

Proof. Let $A <^{max} N$ and $A_1 \subsetneq A$. There exists no $A_1 - A$. So, A is simple. Then A is both maximal and simple submodule of N. Let $A \neq Y \subsetneq N$. Then $A \oplus Y = N$ implies that Y is simple and maximal otherwise N has only one submodule A. □

Now, we prove that $g(\mu(N)) \leq 4$.

Theorem 2.9. Let $\mu(N)$ have a cycle.

1) If N has at least three maximal submodules, then $g(\mu(N)) = 3$.

2) Let N_1, N_2 be not maximal and $N = N_1 + N_2$, then $g(\mu(N)) = 3$.

3) If N has two submodules such that $B + C \neq N$ is not maximal, then $g(\mu(N)) = 3$.

4) If N has only one maximal submodule, then $g(\mu(N)) = 3$.

5) If N has only two maximal submodules, then $g(\mu(N)) \leq 4$.

Proof.

1) Let N_1, N_2 and N_3 be maximals. Then $(N_1 - N_2 - N_3)$, which says $g(\mu(N)) = 3$.

2) Let $N = N_1 +N_2$ where N_1 and N_2 are not maximal. If $N_1 \cap N_2 \neq 0$ then $N_1 - N_1 \cap N_2 - N_2$. Assume $g(\mu(N)) > 3$. Then, $N_1 \cap N_2 = 0$. Since N_1 and N_2 are not simple, there exists $Y_1 < N_1$ and $Y_2 < N_2$. So $Y_1 \oplus Y_2 < N_1 \oplus Y_2 < N_1 \oplus N_2 = N$ and hence $Y_1 \oplus Y_2$ is not maximal. Therefore $g(\mu(N)) = 3$.

3) It is straightforward.

4) Let U be the only maximal submodule of N. Assume that $g(\mu(N)) > 3$.

a) Let every proper submodule of N is in U. Then we have a cycle $(A_1 - A_2 - A_3 - A_4 - \ldots)$ of length greater than three and $A_i \neq U$, for all i.

i) Let $A_1 < A_2$. Since $A_2 < A_3$ or $A_3 < A_2$ cannot be, we have a contradiction

ii) Let $A_2 < A_1$ where $A_1 +A_3 = U$.

ii1) If $A_2 < A_3$, then $A_2 + A_4 = U$ gives a contradiction.

Ii2) If $A_3 < A_2$, then $A_3 < A_2 < A_1$ implies a contradiction. Thus $g(\mu(N)) = 3$.

b) Let $A < N$ and $A \not< U$. Let $(A_1 - A_2 - A_3 - A_4 - \ldots)$ of length greater than three.

i) $A_1 < A_2$ and A_2 is not maximal. Then $A_2 < A_3$ or $A_3 < A_2$ cannot be. Let $A_2 +A_3 = N$. If A_3 is not maximal, then there is a contradiction. So $A_3 = U$ and we have a cycle $(A_1 - A_2 - U - A_4 - \ldots)$. Then $A_2 \not< U$ and $A_4 \not< U$. $U = A_2 + A_4 <^{max} N$ which is a contradiction.

ii) $A_2 < A_1$ and A_1 is not maximal. $A_3 < A_2$ and $A_3 + A_2 = N$ cannot be. Let $A_2 < A_3$. Then $A_3 < A_4$ or $A_4 < A_3$ cannot be. Thus $A_3 + A_4 = N$ where $A_3 \neq U$. If $A_4 = U$, $A_1 + A_3 = U$ and $A_1 + A_3 = A_4$. Then $A_3 < A_4$ which is a contradiction.

iii) $A_1 + A_2 = N$.

iii1) Let $A_2 = U$. Then we have a three cycle $A_1 - U - A_3$

iii2) Let $A_1 = U$, we get $(U - A_2 - A_3 - A_4 - \ldots)$. Since $A_2 + A_4 = U$, we have a contradiction.

5) Assume that $g(\mu(N)) > 4$. Let A and B be maximal submodules of N and $0 \neq X < N$. Then either $X \leq A$ or $X \leq B$. Assume that $A \cap B = 0$. Then $A \oplus B = N$ and A and B are simple. Since we have a cycle, there exists at least one submodule $C \neq A$ and B. Thus $C \oplus A = N$ and C is maximal, which is a contradiction. Therefore $A \cap B \neq 0$.

a) Let every nonzero proper submodule of N (except for A and B) is in $A \cap B$. Since we have a cycle $(A_1 - A_2 - A_3 - A_4 - \ldots)$ such that $A_i \neq A$ and B, $g(\mu(N)) = 3$.

b) Let there exists at least one submodule C such that ($C \not< A$ and $C < B$) or ($C \not< B$ and $C < A$). Assume that $C \not< B$ and $C < A$. Then B has no submodules except for $A \cap B$ and its submodules. Otherwise, if $D < B$ and $D \not< A$, then $D + C \neq A, B$ which says 3 - cycle.

b1) Let $A \cap B$ be simple. Since $N/B = (A + B) / B \cong A / (A \cap B)$ simple, $A \cap B <^{max} A$. Then $A \cap B \not< C$, so $(A \cap B) \oplus C = A$ where C is simple. Thus, $B \oplus C = N$ and B has no proper submodule except for $A \cap B$. If $D < A$ where $A \cap B \not< D$, then D is simple and $C \oplus D = A$. So, we have no cycle.

b2) Let $A \cap B$ be not simple. Then we have a submodule $Y < A \cap B$ where Y is simple and $A \cap B$ has no submodule except for Y.

b21) Let $Y \not< C$. Then $Y \oplus C = A$ where C simple. Thus, $Y \oplus (A \cap B \cap C) = A \cap B$ which is a contradiction.

b22) Let $Y < C$. Since we have a cycle, we have $D \neq A$, B, C, $A \cap B$ and Y. Then $D \not< B$ and $D < A$.

i) Let $Y < D$. Then $(D - B - C - Y)$ is a cycle.

ii) Let $Y \not< D$. Thus $D \oplus Y = A$ where D and Y are simple. So, $Y \oplus (A \cap B \cap D) = A \cap B$, which is a contradiction. □

Corollary 2.10. Let $\mu(N)$ have a cycle. If N has only two maximals A and B such that simple $= Y < A \cap B$ and there exist proper submodules C, D of A only containing Y and C, $D \not< B$ where B has only two proper submodules $A \cap B$ and Y, then we have a 4-cycle. Otherwise, we have a 3-cycle.

3. DOMINATION NUMBER, CUT EDGES, PENDENT VERTEX AND PERFECTNESS OF μ(N)

Lemma 3.1. Let N be a module with $Rad(N) = 0$. If $N = A \oplus B$ or $N = A \oplus B \oplus C$ cannot be where A, B and C are simple, then $\mu(N)$ has no cut edge.

Proof. Let $A - B$ be cut edge.

i) Let $A + B$ ($\neq N$) be not maximal. Since $Rad(N) = 0$, there exist at least one maximal submodule T_1 and T_2 such that $A \not< T_1$ and $B \not< T_2$. So, there is a path $A - T_1 - T_2 - B$, which is a contradiction.

ii) Let $A + B = N$.

İi1) Let A and B not maximal. Then $A \cap B = 0$. So, $A \oplus B = N$ where A and B are not simple. So, $A - A_1 - B_1 - B$ (where $A_1 < A$ and $B_1 < B$) is a path, which is a contradiction.

Ii2) Let A and B maximal. If there is another maximal submodule, then $A - C - B$ is a path. Otherwise, $A \cap B = 0$ and $A \oplus B = N$ where A and B are simple. If there is $C \neq A$ and B, then $A \oplus C = B \oplus C = N$, where C is maximal, which is a contradiction.

Ii3) Let A be not maximal and B be maximal. Assume that every maximal except for B contains A. Then $A \cap B \leq Rad(N) = 0$. Therefore, $A \oplus B = N$ and A is simple. Assume that $A \oplus B_1 <^{max} A \oplus B = N$ where $B_1 < B$ and B_1 is simple. Since B_1 is not small in B, $B_1 \oplus B_2 = B$ where B_2 is simple. Then $N = A \oplus B_1 \oplus B_2$, which is a contradiction.□

Lemma 3.2. Let N be a module whose number of generators is greater than or equal to four, then $\mu(N)$ has not a cut edge.

Proof. Let $A - B$ be cut edge.

i) Let $A + B$ ($\neq N$) be not maximal.

i1) Let $A < B$ where A is simple. If B is cyclic and $y \in N - B$, then $A - yS - B$ is a path. If B is not cyclic and $x \in B - A$, then $A - xS - B$ is a path, which gives a contradiction.

i2) Let $A \not< B$ and $B \not< A$. If $A \cap B \neq 0$, then we have a path $A - A \cap B - B$ except for $A - B$. So, $A \cap B = 0$ such that A and B are simple. Let $x \in N - A \oplus B$ (where $xS \neq N$). Thus, $A + xS$ is not maximal and $B + xS$ is not maximal and $A - xS - B$ is a path. Therefore, we have a contradiction.

ii) Let $A + B = N$.

ii1) Let A and B be not maximal. Then $A \cap B = 0$, $A \oplus B = N$ where A and B are not simple. So, $A - A_1 - B_1 - B$ (where $A_1 < A$ and $B_1 < B$) is a path, which is a contradiction.

ii2) Let A and B be maximal. Let $x \in N - A$ and $y \in N - B$. Thus xS, $yS \neq A$ and B. Then $A - xS - yS - B$ is a path, which is a contradiction.

iii3) Let A be not maximal and B be maximal. If A is not cyclic and $x \in A$ and $y \in N - B$, then we have a path $A - xS - yS - B$. If A is cyclic and $x \in B - A$, then $A - A + xS - B$ is a path, which gives a contradiction. □

A vertex of a graph is said to be pendent if its neighbourhood contains exactly one vertex. Le $N = \mathbb{Z}_2 \oplus \mathbb{Z}_3$. Then $\mathbb{Z}_2$ is a pendent vertex. Let $N = \mathbb{Z}_4 \oplus \mathbb{Z}_3$. Then $\mathbb{Z}_3$ is a pendent vertex.

Lemma 3.3. If N has no maximal submodules, then N has no pendent vertex. Moreover, N has only one maximal submodule U, then U cannot be pendent.

Proof. Since N has no maximal, N has at least three non-trivial submodules, say C, D, E. Then $C + D$ and $C + E$ can not be maximal. So, C cannot be pendent. Similarly, all non-trivial submodules can not be pendent. Let every submodule be in U, then U cannot be pendent. Let there exist a submodule V such that $V \not< U$. Since V is not maximal, there exist a nontrivial submodule T such that $V < T$. So, we have a path $U - V$ and $U - T$, hence U cannot be pendent. □

Proposition 3.4 Let N have only one maximal submodule U and $V \not< U$. Then $\mu(N)$ has no pendent vertex.

Proof. Let A be a pendent vertex.

i) If $A = U$, then by Lemma 3.3, we have a contradiction.

ii) Let $A \neq U$.

a) If $A \not< U$, then we have a nontrivial submodule T such that $A < T$, so we have paths $A - U$ and $A - T$, which gives a contradiction.

b) Let $A < U$. Then we have $V < T$ and we have paths $A - V$ and $A - T$, which gives a contradiction. □

Example 3.5. Let $N = \mathbb{Z}_{16}$ which has only one maximal submodule $\mathbb{Z}_8$ and $\mathbb{Z}_8$ contains all nontrivial submodules. $\mathbb{Z}_4$ is a pendent vertex in $\mu(N)$. Moreover, $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$ is semisimple and every simple is pendent. If $N = N_1 \oplus N_2 \oplus \dots \oplus N_n$ $(n \geq 4)$ is semisimple, then N has no pendent vertex.

Lemma3.6. Let $Rad(N) = 0$. If N has generators whose number greater than two, then N has no pendent except for simple submodules. If N has generators whose number greater than three, then $\mu(N)$ has no pendent.

Proof. Let $\mu(N)$ have pendent, say P. By Lemma 3.3, N has maximals.

a) Let P be not simple and not maximal. Since $Rad(N) = 0$, there exist a maximal N_1 such that $P \not< N_1$. Moreover, there exist $P_1 < P$. So, we get a contradiction.

b) Let P be maximal. Since $Rad(N) = 0$, there exist a maximal $N_1 \neq P$. If there exist no maximal N_1 and P, then this contradicts that $Rad(N) = 0$ and N has generators whose number greater than two. So, we have another maximal N_2. So, we have a path $N_1 - P - N_2$, which gives a contradiction.

c) Let P be simple where N has generators whose number greater than three. Since $Rad(N) = 0$, there exist a maximal N_1 such that $P \not< N_1$. Let $x \in N - P$. Then, we have a path $xS - P - N_1$. □

Now, we investigate the dominating set and $\gamma(\mu(N))$ in $\mu(N)$.

Lemma 3.7. Let X be a dominating set for N and $U < N$. Let $B = \{T \cap U : T \in X\}$ is a dominating set for U.

Proof. Let $0 \neq Y < U$. Then there exist $T_1 \in X$ such that $Y + T_1$ is not maximal. Let $Y + (T_1 \cap U) < Y + T_1$ implies that $Y + (T_1 \cap U)$ is not maximal. So, B is a dominating set for U. □

Lemma 3.8 Let $Rad(N) = 0$. Then $\gamma(\mu(N)) <$ the number of maximals if N has maximals. If N has no maximals, then $\gamma(\mu(N)) = 1$.

Proof. If N has no maximals, then for all $0 \neq U < N$, $\{U\}$ is a dominating set, and so $\gamma(\mu(N)) = 1$. Let N have maximals. Since $Rad(N) = 0$, for all $0 \neq X < N$, there exist at least one maximal submodule such that it does not contain X. So, the set of all maximals is a dominating set, hence $\gamma(\mu(N)) <$ the number of maximals. □

Lemma 3.9. Let $\{M_i : i \in I\}$ be the set of all maximals. If there exist $U \not< M_i$ for all $i \in I$ then $\gamma(\mu(N)) = 1$.

Proof. Since $\{U\}$ is dominating set, $\gamma(\mu(N)) = 1$. □

Proposition 3.10. $\{U\}$ is a dominating set if and only if one of the following is satisfied:

i) N has no maximal submodules.

ii) If N has at least one maximal submodule, then every maximal does not contain U.

Example 3.11. Let $N = \mathbb{Z}_2 \oplus \mathbb{Z}_3$ then $\gamma(\mu(N)) = 2$. Let $N = \mathbb{Z}_4 \oplus \mathbb{Z}_3$ then $\gamma(\mu(N)) = 3$ where N has two maximal and $Rad(N) = \mathbb{Z}_2$ and minimal dominating set is $\{\mathbb{Z}_4, \mathbb{Z}_3, \mathbb{Z}_2\}$.

Lemma 3.12. Let $Rad(N) = 0$ and $B = \{M_i: i \in I\}$ be set of all maximals. Then B is a dominating set. Moreover, if $\{M_j: j \in J\} = C$ and $\bigcap_{j \in J} M_j = 0$ and there exist no U, $U \not< M_j$, for all $j \in J$, then C is a dominating set.

Lemma 3.13. Let $Rad(N) \neq 0$ and N has only two maximals N_1 and N_2. $\{U, N_i\}$ is a dominating set where i = 1 or 2 if and only if $U < N_j$, $U \not< N_i$ and $U + Rad(N) < N_j$ where j = 1 or 2.

Proof. Let $\gamma(\mu(N)) = 2$. Then $U \neq N_j$, since $N_1 \cap N_2 \neq 0$. And $U \not< N_i$ since $\{U, N_i\}$ is a dominating set. If $U \not< N_j$ and $U \not< N_i$, then $\gamma(\mu(N)) = 1$. So, $U < N_j$. If $U + Rad(N) = N_j$, then $\{U, N_i\}$ cannot be a dominating set. For the converse, let $0 \neq X < N$. Assume that i = 1.

a) If $X \not< N_1 \cap N_2$, then $X \not< N_1$ or $X \not< N_2$.

a1) If $X \not< N_1$, then $N_1 + X = N$.

a2) If $X \not< N_2$, then $X < N_1$. $U + X \neq N_1$ and $U + X \neq N_2$.

b) If $X \leq N_1 \cap N_2$, then $U + X \leq U + N_1 \cap N_2 < N_2$. So $\{U, N_i\}$ is a dominating set. □

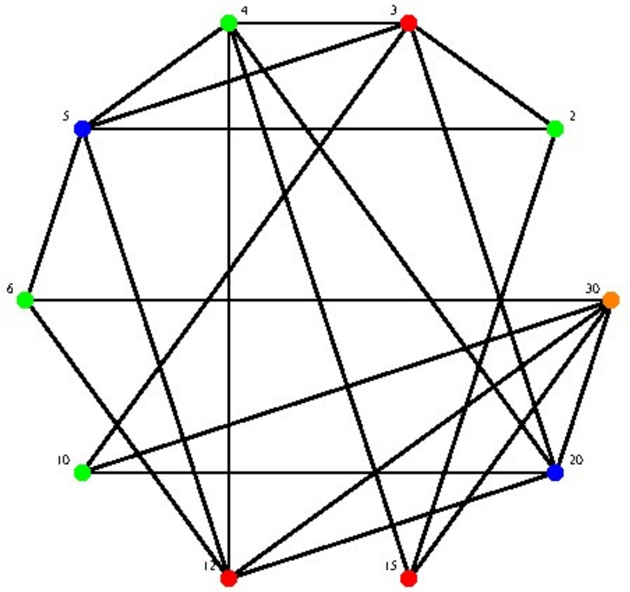
Proposition 3.14 Let N have only two maximals and $Rad(N) \neq 0$. $\{U_1, U_2\}$ is a minimal dominating set where U_1 and U_2 are not N_1 and N_2 if and only if $U_1 < N_1$ and $U_1 \not< N_2$, $U_2 < N_2$ and $U_2 \not< N_1$ and $N = U_1 + U_2$ and $U_1 + Rad(N) < N_1$ or $U_2 + Rad(N) < N_2$.

Proof. Let $\{U_1, U_2\}$ be a minimal dominating set such that U_1 and U_2 are not N_1 and N_2. Assume that $U_1 \not< N_1$ and $U_1 \not< N_2$. Then $\gamma(\mu(N)) = 1$ by Lemma 3.9. So $U_1 < N_1$ or $U_1 < N_2$. Assume that $U_1 < N_1$. Then $U_2 \not< N_1$, otherwise $\{U_1, U_2\}$ cannot be a dominating set. Thus, $U_2 < N_2$ and $U_1 \not< N_2$. Assume that $U_1 + U_2 \neq N$. Then $U_1 + U_2 \not< N_1$ and $U_1 + U_2 \not< N_2$, then we have a contradiction. Moreover, $U_1 + Rad(N) < N_1$ or $U_2 + Rad(N) < N_2$. For the converse, the proof is the same as the proof of Lemma 3.13.

Theorem 3.15. Let $N \neq N_1 \oplus N_2$ where N_1 and N_2 are simple. Then a non-maximal graph $\mu(N)$ cannot be a complete r-partite graph, where $2 \leq r \in \mathbb{N}$.

Proof. Assume that $\mu(N)$ is a complete r-partite graph. If N has no maximal elements, then there exists a non-zero submodule $M < N$. Since M is not maximal, there exists a submodule M_1 such that $M < M_1$. Similarly, if we can continue then we can find some submodules of N such that $M < M_1 < M_2 < \ldots < M_{r-1} < \ldots$ Since $\mu(N)$ is a complete r-partite graph a partition of $\mu(N)$ must be $\{\{X_i\}$: i = 1, 2, …, r where $0 \neq X_i < N\}$. But N has infinitely many submodules, it cannot be. Let N have only one maximal submodule N_1. If N_1 contains all nonzero proper submodules of N, then $\mu(N)$ cannot be complete r-partite graph. Then there exists at least one $0 \neq Y < N$ such that $Y \not< N_1$. Since $\mu(N)$ is a complete r-partite graph, we have a partition $\{N_1, X: 0 \neq X \leq N_1\}$, $\{\{M_1\}, \ldots, \{M_{r-1}\}\}$ where $0 \neq M_i < N$ and $M_i \not< N_1$ for $i \in \{1, \ldots, r\text{-}1\}$. Since M_1 is not maximal and $0 \neq M_1 \not< N_1$, there exists a non-maximal submodule T_1, $M_1 < T_1$. If we continue in this way, we can find non-maximal submodules of N such that $M_1 < T_1 < T_2 < T_3 < \ldots$ So there exists at least one $T_k \notin \{M_1, \ldots, M_{r-1}\}$. Then $T_k = N_1$ or $T_k = X$. But these cannot be. Assume that N has at least two maximal submodules N_1 and N_2. Since $\mu(N)$ is a complete r-partite graph, $\mu(N)$ has a partition which contains $T_1 = \{N_1, X: 0 \neq X \leq N_1\}$ and $T_2 = \{N_2, Y: 0 \neq Y \leq N_2\}$. If $N_1 \cap N_2 \neq 0$ then $N_1 \cap N_2 \in T_1$ and $N_1 \cap N_2 \in T_2$. But this cannot be. Then $N_1 \cap N_2 = 0$, and hence $N = N_1 \oplus N_2$ where N_1 and N_2 are simple. But this is a contradiction. □

Example 3.16. $\mu(N)$ may not be a perfect graph. $\chi(\mu(\mathbb{Z}_{60})) = 4$ and $w(\mu(\mathbb{Z}_{60})) = 3$ in the following diagram where 30, 20, 15, 12, 10, 6, 5, 4, 3, 2 means that the number of elements of submodules of $\mathbb{Z}_{60}$.



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