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# Topological Characterizations for Sheaf of the Groups Formed by Topological Generalized Group

Hatice Aslan<sup>\*1</sup>, Hakan Efe<sup>2</sup>

## Abstract

In the present paper, we provide the some algebraic topological characterizations for an algebraic sheaf by means of the topological generalized group constructed in [1] by considering both homotopy and sheaf theory.

**Keywords:** generalized group, topological generalized group, Whitney sum, sheaf, characterization.

## 1. INTRODUCTION

Generalized group is an algebraic structure which has a deep physical background in the unified gauge theory. The unified theory offers a new insight into the structure, order and measures of the quantum world of the entire universe. Because of this physical forces mathematicians and physicists have been trying to construct a suitable unified theory for unified gauge theory, twistor theory, isotopies theory and so on. For special manifolds and metrics a mechanism for constructing one metric from another one has been presented in [2-4]. Santilli isotopies or axiom preserving maps have essential roles in this mechanism. The main point of isothory is the identity of an isofield for each local chart. When Molaei decided to extend this theory even for points of  $\mathbb{R}$ , one method is: to associate an identity to each element of  $\mathbb{R}$ . This method was a reason for Molaei to define generalized groups [5] in 1999.

Generalized groups are agitative extension of groups because Molaei's generalized groups established the uniqueness of the identity element of each element in a generalized group and where the identity element is not unique for each element. This new concept studied in various areas of mathematics in connection with algebraic, topological and differentiable [5-11]. Moreover this kind of structure appears in genetic codes. Therefore generalized groups have been applied to DNA analysis by transforming the set of DNA sequences to generalized group in [12].

One more important concept in this paper are sheaves which were originally defined by Leray [13] in 1946. The modified definition of sheaves now used was given by Lazard, and appeared first in the Cartan Sem. [14] 1950-51. Sheaf theory provides a language for the discussion of geometric objects of many different kinds. Its main applications in topology and in modern algebraic geometry, where it has been used with great success as a tool in the solution of several longstanding problems.

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Yıldız constructed an algebraic sheaf by means of the topological group in [15]. This was our motivation for constructing a sheaf by the means of the topological generalized group in the paper [1]. We replaced topological group with topological generalized group and constructed an algebraic sheaf by means of the topological generalized group in [1]. In the present paper our aim is giving some algebraic topological characterizations for algebraic sheaf constructed in [1].

**2. PRELIMINARIES**

In order to construct the sheaf in section 3 we need some fundamental definitions and concepts related to the generalized groups, topological generalized groups and sheaves. We can start by giving some basic recalls of the concept of generalized group which was first introduced by Molaei [5].

**Definition 2.1. [5]** A generalized group  $G$  is a non-empty set admitting an operation called multiplication subject to the set of rules given below:

- (i)  $(ab)c = a(bc)$ , for all  $a, b, c \in G$  (associative law);
- (ii) For each  $a \in G$ , there exists a unique  $e(a) \in G$  such that  $ae(a) = e(a)a = a$  (existence and uniqueness of identity);
- (iii) For each  $a \in G$ , there exists  $a^{-1} \in G$ , such that  $aa^{-1} = a^{-1}a = e(a)$  (existence of inverse).

**Example 2.2. [8]** Let  $G = \mathbb{R} \times \{\mathbb{R} \setminus \{0\}\}$ . Then with the multiplication  $(a, b)(c, d) = (bc, bd)$  is a generalized group in which for all  $(a, b) \in G$ ,  $e(a, b) = (a/b, 1)$  and  $(a, b)^{-1} = (a/b^2, 1/b)$ .

**Example 2.3. [16]** Let  $G$  with the multiplication  $m$  be generalized group. Then  $G \times G$  with the multiplication

$$m_1((a, b), (c, d)) = (m(a, c), m(b, d))$$

is a generalized group. For any element  $(a, b) \in G \times G$  the identity element is  $e_1(a, b) = (e(a), e(b))$  and inverse element is  $(a, b)^{-1} = (a^{-1}, b^{-1})$ .

**Theorem 2.4. [10]** Each  $a$  in a generalized group  $G$  has a unique inverse in  $G$ .

**Example 2.5. [10]** Let  $S = \{1, 2\}$ . Then  $S$  with the binary operation:  $2 \cdot 2 = 2, 2 \cdot 1 = 1 \cdot 2 = 2, 1 \cdot 1 = 1$  is a semigroup. Then  $S$  is semigroup which is not a generalized group, because the identity of 2 is not unique.

It is easily seen from Definition 1 that every group is a generalized group. But it is not true in general that every generalized group is a group.

**Lemma 2.6. [11]** Let  $G$  be a generalized group and  $ab = ba$  for all  $a, b \in G$ . Then,  $G$  is an abelian group.

Let us give some results related to the structure of generalized groups via following lemma.

**Lemma 2.7. [6]** Let  $G$  be a generalized group. Then,

- (i)  $e(a) = e(a^{-1})$  and  $e(e(a)) = e(a)$  where  $a \in G$ .
- (ii)  $(a^{-1})^{-1} = a$  where  $a \in G$ .
- (iii) The set  $\{G_a = e^{-1}\{e(a)\}; a \in G\}$  is a partition of groups for  $G$ .

We here state definition of a topological generalized group which was defined by Molaei [17] and set fourth simplest properties of this structure from topological point of view was presented in [8] and [10] thereof.

**Definition 2.8. [10]** A topological generalized group  $G$  is a set which satisfies the following conditions:

- (i)  $G$  is generalized group;
- (ii)  $G$  is a Hausdorff topological space;
- (iii) The mappings

$$m_1: G \times G \rightarrow G, (a, b) \rightarrow ab$$

and

$$m_2: G \rightarrow G, a \rightarrow a^{-1}$$

are continuous mappings.

If  $a \in G$  then  $G_a = e^{-1}\{e(a)\}$  with the product of  $G$  is a topological group, and  $G$  is disjoint union of such topological groups i.e.  $G = \bigcup_{a \in G} G_a$ .

**Example 2.9. [10]** Every non-empty Hausdorff topological space  $G$  with the operation:

$$m: G \times G \rightarrow G, (a, b) \mapsto a$$

is a topological generalized group.

**Example 2.10. [12]** Let  $\mathcal{D} = \{A, G, C, T\}$  to encode information in genetic where the information in DNA is stored as a code made up of four chemical bases: adenine (A), guanine (G), cytosine (C), and thymine (T). Define the space of DNA sequences as  $\Sigma = \{a = (a_i) : a_i \in \mathcal{D}, i \in \mathbb{Z}\}$ . Then  $\Sigma$  is a topological generalized group with the operation

$$*: \Sigma \times \Sigma \rightarrow \Sigma, (a, b) \rightarrow c$$

where

$$c_{[n_i+1, n_{i+1}]} = \begin{cases} a_{[n_i+1, n_{i+1}]} & i \text{ is even} \\ b_{[n_i+1, n_{i+1}]} & i \text{ is odd} \end{cases}$$

and  $\xi = \{n_i\}_{i \in \mathbb{Z}}$  be a strictly increasing sequence of integer numbers so that  $n_{-1} < 0 \leq n_0$ .

**Definition 2.11 [17]** Let  $X, S$  both topological spaces, and  $\pi: S \rightarrow X$  be a locally topological map. Then the pair  $S = (S, \pi)$  or shortly  $S$  is called a sheaf over  $X$ .

In the definition of a sheaf,  $X$  is not assumed to satisfy any separation axioms (See in [5]).  $S$  is called the sheaf space,  $\pi$  the projection map, and  $X$  the base space. Let  $x$  be an arbitrary point in  $X$  and  $V$  be an open neighborhood of  $x$ . A section over  $V$  is a continuous map  $s: V \rightarrow S$  such that  $\pi \circ s = id_V$ .

Let us denote the collection of all sections of  $S$ , by  $\Gamma(V, S)$  and recall the Whitney sum.

**Definition 2.12. [18], [19]** Let  $(S_1, \pi_1), (S_2, \pi_2)$  be  $, \dots, (S_k, \pi_k)$  sheaves on  $X$ . Construct product  $M_W = \Gamma(W, S_1) \times \Gamma(W, S_2) \times \dots \times \Gamma(W, S_k)$  for  $V, W \subset X$  open sets. Let  $\Gamma_V^W: M_W \rightarrow M_V$  defined by  $\Gamma_V^W(s) = (s_1|_V, s_2|_V, \dots, s_k|_V)$  for  $(s_1, s_2, \dots, s_k) \in M_W$  and  $V \subset W$ . Then  $\{M_W, \Gamma_V^W\}$  is a presheaf. The Whitney sum of  $S_1, S_2, \dots, S_k$  sheaves is a sheaf defined by this presheaf and denoted by  $S^* = S_1 \oplus S_2 \oplus \dots \oplus S_k$ .

Now we can say that the Whitney sum of sheaves  $(S_1, \pi_1), (S_2, \pi_2), \dots, (S_k, \pi_k)$ :

$$S^* = S_1 \oplus \dots \oplus S_k$$

$$:= \{\sigma = (\sigma_1, \dots, \sigma_k) \in S_1 \times \dots \times S_k : \pi_1(\sigma) = \dots = \pi_k(\sigma)\}$$

$$= \bigvee_{x \in X} ((S_1)_x \times \dots \times (S_k)_x)$$

is a set over  $X$  topological spaces. Then the map  $\pi: S^* = S_1 \oplus S_2 \oplus \dots \oplus S_k \rightarrow X, \pi(\sigma) = (\pi_i \circ P_i)(\sigma)$  is a local homeomorphism, hence  $S^* = S_1 \oplus S_2 \oplus \dots \oplus S_k$  is a sheaf over  $X$ .

**Theorem 2.13. [20]** Let  $(S_i, \pi_i), i = 1, \dots, k$  be sheaves and  $S^* = S_1 \oplus S_2 \oplus \dots \oplus S_k$  be Whitney sum of  $S_1, S_2, \dots, S_k$ . Then there is bijection

$$\pi: S_x^* \rightarrow (S_1)_x \times (S_2)_x \times \dots \times (S_k)_x$$

defined by

$$(W, (s_1, s_2, \dots, s_k))_x \rightarrow (s_1(x), s_2(x), \dots, s_k(x)).$$

**Theorem 2.14. [20]** Let  $(S_i, \pi_i), i = 1, \dots, k$  be sheaves on  $X$ . Then the canonic projection

$$P_i: S_1 \oplus S_2 \oplus \dots \oplus S_k \rightarrow S_i, P_i(\sigma_1, \sigma_2, \dots, \sigma_k) = \sigma_i$$

is a sheaf morphism.

Let  $s_i \in \Gamma(W_i, S_i)$  for  $i = 1, \dots, k$ . Define  $s_1 \oplus \dots \oplus s_k: W \rightarrow S^* = S_1 \oplus S_2 \oplus \dots \oplus S_k$ , such that

$$(s_1 \oplus \dots \oplus s_k)(x) = (s_1(x), s_2(x), \dots, s_k(x)).$$

Clearly  $(s_1, s_2, \dots, s_k) \in M_W$  and

$$\begin{aligned} r(s_1, s_2, \dots, s_k) &= (W, (s_1, s_2, \dots, s_k))_x \\ &= (s_1(x), s_2(x), \dots, s_k(x)) \\ &= (s_1 \oplus \dots \oplus s_k)(x). \end{aligned}$$

Therefore since  $s_1 \oplus \dots \oplus s_k = r(s_1, s_2, \dots, s_k)$

$\in \Gamma(W, S_1 \oplus S_2 \oplus \dots \oplus S_k)$  we have

$$\begin{aligned} \Gamma(W, S_1 \oplus S_2 \oplus \dots \oplus S_k) \\ = \Gamma(W, S_1) \times \Gamma(W, S_2) \times \dots \times \Gamma(W, S_k). \end{aligned}$$

Furthermore since  $W \subset X$  is open set and  $\pi$  is a local homeomorphism  $s(W)$  is open set in  $S$ , and  $S$  is union of these type of open sets. Also if  $s_1, s_2 \in \Gamma(W, S)$  and  $s_1(x) = s_2(x)$  for  $x \in X$  then  $s_1 = s_2$  in  $W$ . So we can say that every element of  $S$  can be seen as a substance of sections in  $S$ .

### 3. THE SHEAF OF THE GROUPS FORMED BY TOPOLOGICAL GENERALIZED GROUP OVER TOPOLOGICAL SPACES

Let  $\mathcal{C}$  be the category of the topological spaces  $X$  satisfying the property that all pointed spaces  $(X, x)$  with  $x \in X$  have same homotopy type. This category includes all topological vector spaces.

Let us take  $X \in \mathcal{C}$  as a base set if  $P$  is any topological group with identity element  $p_0$  as base point. Then the set of homotopy class of homotop maps preserving the base point from  $(X, x)$  to  $(P, p_0)$  obtained for each  $x \in X$ ,  $(X, x)$  pointed topological spaces i.e.  $S(X) = \bigvee_{x \in X} [(X, x), (P, p_0)]$ . Thus  $S(X)$  is a set over  $X$ .

If  $(P, p_0)$  is any topological group with the identity element of the group is  $p_0$ , we can construct a sheaf over  $X$  by using following theorem which is given by Yildiz [15].

**Theorem 3.1 [15]** Let  $(P, p_0)$  be any pointed topological group with the identity element  $p_0$  and  $X \in \mathcal{C}$ . If  $\pi: S(X) \rightarrow X$  such that  $\pi(\sigma) = \pi([f]_x) = x$  for  $\sigma = [f]_x \in S(X)$ ,  $x \in X$  then there is the natural topology over  $S(X)$  such that  $\pi$  is locally topological with respect to this topology. Thus the pair  $(S, \pi)$  is a sheaf over  $X$ .

In Theorem 3.1, Yildiz by defining  $S(X) = \bigvee_{x \in X} [(X, x), (P, p_0)]$  and  $\pi: S(X) \rightarrow X$  such that  $\pi(\sigma) = x$ ,  $x \in X$  and a mapping  $s: V \rightarrow S(X)$  as follows:

If  $x_0 \in X$ , then there exists a group  $[(X, x_0), (P, p_0)]$  in  $S(X)$ . If  $y$  is any point in  $V$ , Then  $(X, x_0)$  and  $(X, y)$  are having same homotopy type where  $V = V(x_0)$  open neighborhood of  $x_0$  in  $X$ . Therefore, there is a homotopy equivalence map  $\Phi: (X, x_0) \rightarrow (X, y)$ .

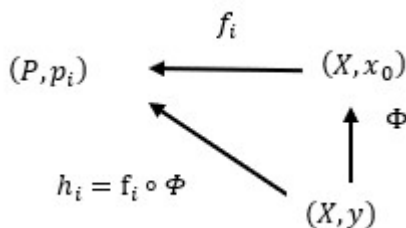


Figure 1

Hence from the diagram in Figure 1, the map  $h = f \circ \Phi: (X, y) \rightarrow (P, p_0)$  is continuous and base-point preserving.  $[h]_y \in [(X, y), (P, p_0)]$  is a homotopy class of map  $f \circ \Phi = h$ . Therefore, we define  $s(y) = [h]_y$ . In this way  $s$  is well defined and  $(\pi \circ s)(y) = \pi(s(y)) = y$  for each  $y \in V$ . Therefore  $\pi \circ s = I_V$ . Thus  $s$  is called a section of  $S(X)$  over  $V$ .

Let us denote the collection of all sections of  $S(X)$ , by  $\Gamma(V, S)$ . A topology-base is constructed on  $S(X)$  by using  $s(V) = \bigvee_{y \in V} [h]_y$ ,

$$\beta = \{s(V) : V = V(x) \subset X, x \in X, s \in \Gamma(V, S)\}.$$

Thus gives a natural topology on  $S(X)$ . Therefore  $S(X)$  is a topological space.

Therefore the sheaf  $(S, \pi)$  given by Theorem 3.1 is a sheaf of the homotopic groups formed by topological group  $P$  over  $(X, x)$  pointed topological spaces [15]. The stalk of the sheaf  $(S, \pi)$  over  $X$  is the group  $[(X, x), (P, p_0)] = \pi^{-1}$  denoted by  $S(X)_x$  for every  $x \in X$ .

$\Gamma(V, S)$  is a group with pointwise multiplication defined by

$$(s_1 s_2)(y) = s_1(y) s_2(y), s_1, s_2 \in \Gamma(V, S) \text{ and } y \in V.$$

Therefore  $(S, \pi)$  is an algebraic sheaf with the operation  $(\cdot): S(X) \otimes S(X) \rightarrow S(X)$  (that is,  $(\sigma_1, \sigma_2) \rightarrow \sigma_1 \cdot \sigma_2$  for every  $\sigma_1, \sigma_2 \in S(X)$  is continuous [15].

In the case of the finite pointed topological generalized group  $P$ , the Whitney sum of sheaves  $S_1, \dots, S_k$  i.e.  $S^* = S_1 \oplus \dots \oplus S_k$  used for constructing the sheaf over  $X$ . For this sheaf a map  $\pi: S^*(X) \rightarrow X$ ,  $\pi(\sigma) = (\pi_{i_0} \circ P_{i_0})(\sigma)$  for fixed  $i_0$  and canonic projection  $P_{i_0}$  is defined.

If  $x_0 \in X$ , then there exists groups  $[(X, x_0), (P, p_i)]$  in  $S_i(X)$  for  $i = 1, \dots, k$ . Let

$$\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k) = ([h_1]_{x_0}, [h_2]_{x_0}, \dots, [h_k]_{x_0})$$

be a homotopy class in the group  $\prod_{i=1, \dots, k} [(X, x_0), (P, p_i)]$

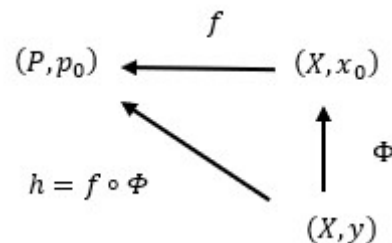


Figure 2

where the map in the Figure 2  $h_i = f_i \circ \Phi: (X, x) \rightarrow (P, p_i)$  is continuous base point preserving.  $[h_i]_y \in$

$[(X, y), (P, p_i)]$  for  $i = 1, \dots, k$  is a homotopy class of map  $f_i \circ \Phi = h_i$ .

If  $x_0 \in X$  is an arbitrarily fixed point, then let us denote  $V = V(x_0)$  open neighborhood of  $x_0$  in  $X$ . Now, we can define a mapping  $s = (s_1, \dots, s_k): V \rightarrow S^*(X)$ , as follows:

If  $y$  is any point in  $V$ , then we define  $s(y) = (s_1, s_2, \dots, s_k)(y) = (s_1(y), s_2(y), \dots, s_k(y))$  for

$s_i(y) = [h_i]_y, i = 1, \dots, k$ . In this way  $s$  is well defined and  $(\pi \circ s)(y) = y$  for each  $y \in V$ . Therefore  $\pi \circ s = I_V$ . Also if,  $x_0$  is an arbitrary fixed point in  $V, s(x_0) = ([h_1]_{x_0}, \dots, [h_k]_{x_0})$  for  $V = V(x_0)$ . Hence it can be written as

$$s(V) = \prod_{i=1, \dots, k} s_i(V) = \prod_{i=1, \dots, k} (V_{y \in V} [h_i]_y).$$

If  $s(V)$  is defined as an open set, then it can be easily shown that the family

$$\beta = \left\{ s(V) := \prod_{i=1, \dots, k} s_i(V) : V = V(x) \subset X, x \in X, s_i \in \Gamma(V, S_i) \right\}$$

is a topology-base on  $S^*(X)$ . Thus  $S^*(X)$  is a topological space [1].

**Theorem 3.2. [1]** Let  $(P, p_i)_{i=1, \dots, k}$  be any pointed finite topological generalized group with the identity elements  $p_1, p_2, \dots, p_k$  and  $X \in \mathcal{C}$ . If

$$S^* = S_1 \oplus S_2 \oplus \dots \oplus S_k \text{ and } \pi: S^*(X) \rightarrow X$$

such that

$$\pi(\sigma) = (\pi_i \circ P_i)([h_1]_x, [h_2]_x, \dots, [h_k]_x) = x, i = 1, \dots, k,$$

for  $\sigma \in S^*(X)$  and  $x \in X$ , then there is the natural topology based on  $S^*(X)$ , such that  $\pi$  is locally topological with respect to this natural topology. Thus the pair  $(S^*, \pi)$  is a sheaf over  $X$ .

**Definition 3.3. [1]** The sheaf  $(S^*, \pi)$  given by Theorem 3.2 is called sheaf of the homotopic groups

formed by generalized groups over  $(X, x), x \in X$  pointed topological spaces.

**Definition 3.4. [1]** The group

$$\prod_{i=1, \dots, k} [(X, x), (P, p_i)] = \pi^{-1}$$

is called the stalk of the sheaf  $(S^*, \pi)$  over  $X$  and denoted by  $S^*(X)_x$  for every  $x \in X$ .

Now, if  $x \in X$  is an arbitrarily fixed point and  $V$  is open neighborhood of  $x$  in  $X$ , the mapping  $s: V \rightarrow S^*(X)$  as defined in the construction of topology of  $S^*(X)$ , is called section of  $S^*(X)$ , over  $V$ . Let us denote the collection of all sections of  $S^*(X)$ , by  $\Gamma(V, S^*)$ .

**Theorem 3.5. [1]**  $\Gamma(V, S^*)$  is a group with the operation

$$(s_1 s_2)(y) = s_1(y) s_2(y), s_1, s_2 \in \Gamma(V, S^*)$$

where  $y \in V$ .

In this group the operation of production is well-defined and closed. Clearly, the operation of production is associative and the mapping  $I: V \rightarrow S^*(X)$  is identity element which is obtained by means of the identity element of  $\prod_{i \in I} [(X, x), (P, p_i)]$ . On the other hand, the any inverse element of  $s \in \Gamma(V, S^*)$ , namely,  $s^{-1} \in \Gamma(V, S^*)$  which is obtained by means of the homotopy inverses of pointed groups  $(P, p_i)$  for  $i = 1, \dots, k$ .

**Theorem 3.6. [1]**  $(S^*, \pi)$  is an algebraic sheaf with the continuous operation

$$\begin{aligned} (\cdot): S^*(X) \otimes S^*(X) &\rightarrow S^*(X), \\ (\sigma_1, \sigma_2) &\rightarrow \sigma_1 \cdot \sigma_2 \end{aligned}$$

where  $\sigma_1, \sigma_2 \in S^*(X)$ .

#### 4. THE CHARACTERIZATIONS

Let  $P$  is a finite pointed topological generalized group and  $X_1, X_2$  be two pointed topological spaces in the category  $\mathcal{C}$ . Let  $S^*(X_1), S^*(X_2)$  be the corresponding sheaves respectively. Let us denote these as the pairs  $(X_1, S^*(X_1))$  and  $(X_2, S^*(X_2))$ .

**Definition 4.1.** Let the pairs  $(X_1, S^*(X_1))$  and  $(X_2, S^*(X_2))$  be given. We say that there is a homomorphism between these pairs and write

$F = (\alpha^*, \alpha): (X_1, S^*(X_1)) \rightleftharpoons (X_2, S^*(X_2))$ , if there exists a pair  $F = (\alpha^*, \alpha)$  such that

1.  $\alpha: X_1 \rightarrow X_2$  is an open and continuous mapping;
2.  $\alpha^*: S^*(X_2) \rightarrow S^*(X_1)$  is an open and continuous mapping;
3.  $\alpha^*$  preserves the stalks with respect to  $\alpha$ , so  $\alpha^*(S^*(X_2)_{\alpha(x_1)}) \subset S^*(X_1)_{x_1} \ni \alpha(x_1) = x_2$  for  $S^*(X_2)_{\alpha(x_1)} \subset S^*(X_2)$ ;
4. For every  $x_1 \in X_1$  the restricted map

$$\alpha^*|_{S^*(X_2)_{\alpha(x_1)}}: S^*(X_2)_{\alpha(x_1)} \rightarrow S^*(X_1)_{x_1}$$

is homomorphism.

**Definition 4.2.** Let the pairs  $(X_1, S^*(X_1))$  and  $(X_2, S^*(X_2))$  be given such map  $F(\alpha^*, \alpha): (X_1, S^*(X_1)) \rightleftharpoons (X_2, S^*(X_2))$  is homomorphism. Then the map  $F = (\alpha^*, \alpha)$  is called isomorphism, so the pairs  $(X_1, S^*(X_1))$  and  $(X_2, S^*(X_2))$  are called isomorphic and can be written  $(X_1, S^*(X_1)) \cong (X_2, S^*(X_2))$ , if the maps  $\alpha^*$  and  $\alpha$  are topological maps (See in Figure 3).

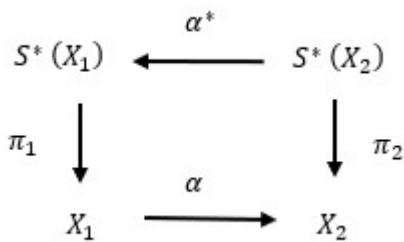


Figure 3

**Theorem 4.3.** Let the pairs  $(X_1, S^*(X_1))$  and  $(X_2, S^*(X_2))$  be given. If the map  $\alpha: X_1 \rightarrow X_2$  is given as open and continuous map, then there exists a homomorphism between pairs  $(X_1, S^*(X_1))$  and  $(X_2, S^*(X_2))$ .

**Proof.** Let  $x_1 \in X_1$  be an arbitrarily fixed point.  $\alpha(x_1) = x_2$  and for  $i = 1, \dots, k$ ,

$$\begin{aligned} \pi_1^{-1}(x_1) &= \prod_{i=1, \dots, k} [(X_1, x_1), (P, p_i)] \\ &= S^*(X_1)_{x_1} \subset S^*(X_1) \end{aligned}$$

$$\begin{aligned} \pi_2^{-1}(\alpha(x_1)) &= \prod_{i=1, \dots, k} [(X_2, \alpha(x_1)), (P, p_i)] \\ &= S^*(X_2)_{\alpha(x_1)} \subset S^*(X_2) \end{aligned}$$

are corresponding stalks.

If  $(X_1, x_1), (X_2, \alpha(x_1))$  are pointed topological spaces and  $f, g$  are base-points preserving continuous maps that have components  $f_i, g_i$  such that base-points preserving and continuous maps from  $(X_1, x_1)$  to  $(P, p_i)$ , so we have  $f_{i2}, g_{i2}$  are base-points preserving continuous maps from  $(X_2, \alpha(x_1))$  to  $(P, p_i)$  can be defined as  $f_i = f_{i2} \circ \alpha, g_i = g_{i2} \circ \alpha, i = 1, \dots, k$  respectively. Furthermore, if  $f_{i2} \sim g_{i2}$  rel.  $\alpha(x_1)$ , then it can be easily shown that  $f_i \sim g_i$  rel.  $x_1$  for  $i = 1, \dots, k$ . Thus correspondence

$$\begin{aligned} [f]_{\alpha(x_1)} &= ([f_{12}]_{\alpha(x_1)}, \dots, [f_{k2}]_{\alpha(x_1)}) \rightarrow \\ [f \circ \alpha]_{x_1} &= ([f_{12} \circ \alpha]_{x_1}, \dots, [f_{k2} \circ \alpha]_{x_1}) \end{aligned}$$

is well defined, and it maps homotopy classes of base points components are preserving continuous maps from  $(X_1, x_1)$  to  $(P, p_i)$ , to the homotopy classes of base-points preserving continuous maps from  $(X_2, \alpha(x_1))$  to  $(P, p_i), i = 1, \dots, k$ . That is, to each element  $[f]_{x_2}$  there corresponds a unique element  $[f \circ \alpha]_{x_1}$ .

Since the point  $x_1 \in X_1$  can be an arbitrarily fixed, the above correspondence gives us a map:  $\alpha^*: S^*(X_2) \rightarrow S^*(X_1)$  such that  $\alpha^*([f]_{x_2}) = [f \circ \alpha]_{x_1} \in S^*(X_1)$  for every  $[f]_{x_2} \in S^*(X_2)$ .

1.  $\alpha^*$  is continuous. Because if  $U_1 \subset S^*(X_1)$  is an open set, then it can be shown that  $\alpha^{*-1}(U_1) = U_2 \subset S^*(X_2)$  is an open set. In fact, if  $U_1 \subset S^*(X_1)$  is an open set, then  $U_1 = \cup_{j \in I} (\prod_{i=1}^k s_{ij}^1(V_j))$  and  $\pi_1(U_1) = \cup_{j \in I} V_j$  where the  $V_j \subset X_1$  are open neighborhoods and the  $s_j^1$  are sections over  $V_j$ . Thus  $\cup_{j \in I} V_j \subset X_1$  is an open set and  $\alpha(\cup_{j \in I} V_j) = \cup_{j \in I} \alpha(V_j) \subset X_2$  is an open set since  $\alpha$  is open and continuous map. Furthermore, since  $\alpha(V_j), j \in I$  are open neighborhoods in  $X_2$ , there exists sections  $s_j^2: \alpha(V_j) \rightarrow S^*(X_1)$  such that  $\cup_{j \in I} (\prod_{i=1}^k s_{ij}^2(\alpha(V_j))) \subset S^*(X_2)$  is an open set.

Let us now show that  $U_2 = \cup_{j \in I} (\prod_{i=1}^k s_{ij}^2(\alpha(V_j)))$ . If  $\sigma_1 = [f]_{x_1} = ([f_1]_{x_1}, \dots, [f_k]_{x_1}) \in U_1$  is any element, then there exists

$$\sigma_2 = [f]_{x_2} = ([f_{12}]_{x_2}, \dots, [f_{k2}]_{x_2}) \in U_2$$

such that  $\alpha^*(\sigma_2) = \sigma_1$  and  $\pi_1(\sigma_1) = \pi_1([f]_{x_1}) = x_1$ . Hence, if  $\alpha(x_1) = x_2 \in \alpha(V_j)$  for at least one  $j \in I$ , then  $x_2 \in \alpha(V_j)$  and  $\sigma_2 = [f]_{x_2} = \cup_{j \in I} (\prod_{i=1}^k s_{ij}^2(\alpha(V_j)))$ . Hence

$$U_2 \subset \cup_{j \in I} \left( \prod_{i=1}^k s_{ij}^2(\alpha(V_j)) \right).$$

On the other hand  $\sigma_2 \in \cup_{j \in I} (\prod_{i=1}^k s_{ij}^2(\alpha(V_j)))$  implies that  $\sigma_2 \in \prod_{i=1}^k s_{ij}^2(\alpha(V_j))$  for at least one  $j \in I$ . From here if  $\sigma_2 = [f]_{x_2} = ([f_{12}]_{x_2}, \dots, [f_{k2}]_{x_2})$ , then  $\pi_2(\sigma_2) = (\pi_{2i} \circ P_i)(\sigma_2) = x_2$  and  $f_{i2} \circ \alpha$  is a base point preserving continuous map from  $(X_2, x_2)$  to  $(P, p_i)$ , where  $\alpha(x_1) = x_2 \in V_j$ . Thus  $([f_{12} \circ \alpha]_{x_1}, \dots, [f_{k2} \circ \alpha]_{x_1}) = [f \circ \alpha]_{x_1} = \sigma_1$ . Hence  $\sigma_2 \in U_2$  and  $\cup_{j \in I} (\prod_{i=1}^k s_{ij}^2(\alpha(V_j))) \subset U_2$ . Therefore,  $U_2 = \cup_{j \in I} (\prod_{i=1}^k s_{ij}^2(\alpha(V_j)))$ . Hence  $\alpha^*$  is continuous map.

2.  $\alpha^*$  is open. Because if  $U_2 \subset S^*(X_2)$  is an open set, then it can be shown that  $\alpha^*(U_2) = U_1 \subset S^*(X_1)$  is an open set. In fact, if  $U_2 = \cup_{j \in I} (\prod_{i=1}^k s_{ij}^2(W_j))$  and  $\pi_2(U_2) = \cup_{j \in I} W_j$  where  $W_j \subset X_2, j \in I$  open neighborhoods and  $s_j^2: W_j \rightarrow S^*(X_2)$  are the sections on  $W_j$ . Since  $\cup_{j \in I} W_j \subset X_2$  open and  $\alpha$  is continuous,  $\alpha^{-1}(\cup_{j \in I} W_j) = \cup_{j \in I} \alpha^{-1}(W_j) \subset X_1$  is open. Furthermore since  $\alpha$  is continuous  $\alpha^{-1}(W_j), j \in I$  are open neighborhoods and there exists  $s_j^1$  such that  $\cup_{j \in I} (\prod_{i=1}^k s_{ij}^1(\alpha^{-1}(W_j))) \subset S^*(X_1)$  is an open.

Let us now show that  $U_1 = \cup_{j \in I} (\prod_{i=1}^k s_{ij}^1(\alpha^{-1}(W_j)))$ . If  $\sigma_1 = [f]_{x_1} = ([f_1]_{x_1}, \dots, [f_k]_{x_1}) \in U_1$  is any element, then there exists

$$\sigma_2 = [f]_{\alpha(x_1)} = ([f_{12}]_{\alpha(x_1)}, \dots, [f_{k2}]_{\alpha(x_1)}) \in U_2$$

such that  $\alpha^*(\sigma_2) = \sigma_1$  and

$$\pi_2(\sigma_2) = \pi_2([f]_{\alpha(x_1)}) = \alpha(x_1) = x_2.$$

Hence, if  $\alpha(x_1) = x_2 \in \alpha(W_j)$  for at least one the  $x_1 \in \alpha^{-1}(W_j)$  and

$$\sigma_1 = [f \circ \alpha]_{x_1} = ([f_{12} \circ \alpha]_{x_1}, \dots, [f_{k2} \circ \alpha]_{x_1}) \in \cup_{j \in I} \left( \prod_{i=1}^k s_{ij}^1(\alpha^{-1}(W_j)) \right).$$

$$\text{Hence } U_1 \subset \cup_{j \in I} \left( \prod_{i=1}^k s_{ij}^1(\alpha^{-1}(W_j)) \right).$$

On the other hand  $\sigma_1 \in \cup_{j \in I} (\prod_{i=1}^k s_{ij}^1(\alpha^{-1}(W_j)))$  implies that  $\sigma_1 \in \prod_{i=1}^k s_{ij}^1(\alpha^{-1}(W_j))$  for at least one  $j \in I$ . From here if then  $\pi_1(\sigma_1) = (\pi_1 \circ P_i)(\sigma_1) = x_1$  and  $f_{i2} \circ \alpha$  is a base point preserving continuous map from  $(X_1, x_1)$  to  $(P, p_i)$ , where  $\alpha(x_1) = x_2 \in W_j$ . Thus

$$([f_{12}]_{\alpha(x_1)}, \dots, [f_{k2}]_{\alpha(x_1)}) = \sigma_2.$$

$$\text{Hence } \sigma_1 \in U_1 \text{ and } \cup_{j \in I} \left( \prod_{i=1}^k s_{ij}^1(\alpha^{-1}(W_j)) \right) \subset U_1.$$

Therefore,

$$U_1 = \cup_{j \in I} \left( \prod_{i=1}^k s_{ij}^1(\alpha^{-1}(W_j)) \right).$$

Hence  $\alpha^*$  is open map.

3.  $\alpha^*$  preserves the stalks with respect to  $\alpha$ . In fact,

$$\begin{aligned} \alpha^*([f]_{x_2}) &= [f \circ \alpha]_{x_1} \\ &= ([f_{12} \circ \alpha]_{x_1}, \dots, [f_{k2} \circ \alpha]_{x_1}) \in S^*(X_1)_{x_1} \end{aligned}$$

for any  $\sigma_2 = [f]_{x_2} = ([f_{12}]_{x_2}, \dots, [f_{k2}]_{x_2}) \in S^*(X_2)_{x_2} \subset S^*(X_2)$  and  $\alpha(x_1) = x_2$ . Thus  $\alpha^*(S^*(X_2)_{x_2}) \subset S^*(X_1)_{x_1}$  such that  $\alpha(x_1) = x_2$ .

4. For every  $\alpha(x_1) = x_2$  the restricted map  $\alpha^*|_{S^*(X_2)_{x_2}}: S^*(X_2)_{x_2} \rightarrow S^*(X_1)_{x_1}$  is homomorphism. In fact, if the maps  $f_i, g_i$  are the base points preserving continuous map from  $(X_2, x_2)$  to  $(P, p_i)$  for  $x_1 \in X_1$  and  $f_i \circ \alpha, g_i \circ \alpha: (X_1, x_1) \rightarrow (P, p_i)$  are the corresponding maps, then  $[f]_{x_2} = ([f_1]_{x_2}, \dots, [f_k]_{x_2}), [g]_{x_2} = ([g_1]_{x_2}, \dots, [g_k]_{x_2}) \in S^*(X_2)_{x_2}$  and  $[f \circ \alpha]_{x_1} = ([f_1 \circ \alpha]_{x_1}, \dots, [f_k \circ \alpha]_{x_1}), [g \circ \alpha]_{x_1} = ([g_1 \circ \alpha]_{x_1}, \dots, [g_k \circ \alpha]_{x_1}) \in S^*(X_1)_{x_1}$ .



Now, if  $[f]_{x_1}, [g]_{x_1} \in S^*(X_1)_{x_1}$ , then

$$\begin{aligned} \alpha^*([f]_{x_2} [g]_{x_2}) &= \alpha^*([fg]_{x_2}) = [(fg) \circ \alpha]_{x_1} \\ &= [(f \circ \alpha)(g \circ \alpha)]_{x_1} = [f \circ \alpha]_{x_1} [g \circ \alpha]_{x_1} \\ &= \alpha^*([f]_{x_2}) \alpha^*([g]_{x_2}). \end{aligned}$$

Thus  $F = (\alpha^*, \alpha)$  is homomorphism.

**Theorem 4.4.** Let the pairs  $(X_1, S^*(X_1))$ ,  $(X_2, S^*(X_2))$ ,  $(X_3, S^*(X_3))$  and surjective and continuous maps  $\alpha_1: X_1 \rightarrow X_2$ ,  $\alpha_2: X_2 \rightarrow X_3$  be given. Then, there exists a homomorphism  $F = (\alpha^*, \alpha): (X_1, S^*(X_1)) \rightleftharpoons (X_3, S^*(X_3))$  such that  $\alpha = \alpha_2 \circ \alpha_1$ ,  $\alpha^* = \alpha_1^* \circ \alpha_2^*$ .

**Proof.** Since  $\alpha = \alpha_2 \circ \alpha_1: X_1 \rightarrow X_3$  is a surjective and open map, there exists a homomorphism  $F = (\alpha^*, \alpha): (X_1, S^*(X_1)) \rightarrow (X_3, S^*(X_3))$  from Theorem 4.3 To prove this theorem it is sufficient to show that  $\alpha^* = \alpha_2^* \circ \alpha_1^*$ . In fact, for any  $[f]_{x_3} = ([f_1]_{x_3}, \dots, [f_k]_{x_3}) \in S^*(X_3)$ , we must show that  $\alpha^*([f]) = (\alpha_1^* \circ \alpha_2^*)([f])$ . However

$$\begin{aligned} \alpha^*([f]_{x_3}) &= \alpha^*([f]_{\alpha_2(x_2)}) \\ &= [f \circ \alpha_2]_{x_2} = [f \circ \alpha_2]_{\alpha_1(x_1)} \\ &= \alpha_1^*([f \circ \alpha_2]_{x_2}) = \alpha_1^*(\alpha_2^*([f]_{x_3})) \\ &= (\alpha_1^* \circ \alpha_2^*)([f]_{x_3}) = (\alpha_1^* \circ \alpha_2^*)([f]_{x_3}) \end{aligned}$$

Therefore  $\alpha^* = \alpha_1^* \circ \alpha_2^*$ .

Now we can state the following theorem:

**Theorem 4.5.** There is a contravariant functor from the category  $\mathcal{C}$  of pointed topological spaces to the category of pairs.

**Proof.** Let  $\mathcal{C}$  be the category of of pointed topological spaces and surjective continuous mappings and  $\mathcal{D}$  be the category of the pairs and their homomorphisms. Then we can define a mapping  $\delta: \mathcal{C} \rightarrow \mathcal{D}$  as follows:

For any pointed topological space  $X$  and every open-continuous map  $\alpha: X_1 \rightarrow X_2$ , let  $\delta(X) = \bigvee_{x \in X} \prod_{i=1, \dots, k} [(X, x), (P, p_i)] = S^*(X)$  and  $\delta(\alpha) = \alpha^*: S^*(X_2) \rightarrow S^*(X_1)$ . Then, from Theorem 4.4,

1. If  $X_1 = X_2$  ve  $\alpha = 1_{X_1}$ , then  $1_{X_1}^* = 1_{S^*(X_1)}$  dir.

2.  $\alpha_1: X_1 \rightarrow X_2$  and  $\alpha_2: X_2 \rightarrow X_3$  are two open and continuous maps, then

$$\delta(\alpha_2 \circ \alpha_1) = \alpha_1^* \circ \alpha_2^* = \delta(\alpha_1) \delta(\alpha_2).$$

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