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Compact operators on the sets of fractional difference sequences

Faruk Özger*[1]

Abstract

The sets of fractional difference sequences have been studied in the literature recently. In this work, some identities or estimates for the operator norms and the measure of noncompactness of certain operators on difference sets of sequences of fractional orders are established. Some classes of compact operators on those spaces are characterized.

Keywords: fractional operators, measure of noncompactness, compact operators

1. INTRODUCTION

The sets of difference sequences are probably the most common type of sets among the sets of sequences studied. The sets of difference sequences were first introduced in Kızmaz's study [21]. Many authors have made efforts to investigate the topological structures of these spaces during the past decade (see [5, 7, 8, 17, 18, 23, 24, 28]). Compact operators on the sets of difference sequences have been characterized in [9, 10, 23]. We refer to [4, 11, 12, 14-18, 27-33] for further studies in theory of FK-spaces and its applications. In order to give full knowledge on the measure of noncompactness in the sequence spaces and the sets of fractional difference sequence spaces we refer to [34-50].

More recently, certain difference sequence spaces of fractional orders have been introduced by Baliarsingh [30]. Certain Euler difference sequence spaces of fractional order and related dual properties have been studied by Kadak and Baliarsingh [32]. Topological properties of certain sequence spaces that are combined by the mean operator and the fractional difference operator are investigated by Furkan [19]. Geometric characterizations of a fractional Banach set is given by Özger in [16].

The rest of the paper is organized as follows: In the rest of this section, we consider fractional operators, their properties and the sets $c_0(\Delta^{(\alpha)})$, $c(\Delta^{(\alpha)})$ and $\ell_\infty(\Delta^{(\alpha)})$ of fractional difference sequences. In section 2, we will focus on some preliminary results, such as the determination of β duals of the sets of fractional difference sequences. In section 3, we will characterize the corresponding matrix transformations and find their operator norms. In section 4, we will establish some identities or estimates for the Hausdorff measure of noncompactness (HMN) of certain operators on fractional difference sequence spaces. Finally in section 5, we will conclude the paper with some notes and also with a table that includes main results about the compact operators on fractional Banach sets.

The gamma function of a real number x (except zero and the negative integers) is defined by an improper integral:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

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It is known that for any natural number n, $\Gamma(n+1)=n!$ and $\Gamma(n+1)=n\Gamma(n)$ holds for any real number $n\notin\{0,-1,-2,\dots\}$. The fractional difference operator for a positive fraction α have been defined in [30] as

$$\Delta^{(\alpha)}(\rho_k)=\sum_{i=0}^{\infty}(-1)^i\frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)}\rho_{k-i}. \qquad (1.1)$$

It is assumed that the series defined in (1.1) is convergent for $\rho\in\omega$. The infinite sum defined in (1.1) becomes a finite sum if α is a nonnegative integer. We use the usual convention that any term with a negative subscript is equal to naught, throughout the paper.

The inverse of fractional difference matrix

$$\Delta_{nk}^{(\alpha)}=\begin{cases}(-1)^{n-k}\dfrac{\Gamma(\alpha+1)}{(n-k)!\,\Gamma(\alpha-n+k+1)}, & 0\le k\le n\\ 0, & k>n\end{cases}$$

is given in [30] as

$$\Delta_{nk}^{(-\alpha)}=\begin{cases}(-1)^{n-k}\dfrac{\Gamma(-\alpha+1)}{(n-k)!\,\Gamma(-\alpha-n+k+1)}, & 0\le k\le n\\ 0, & k>n.\end{cases}$$

For some values of α, we have

$$\Delta^{1/2}\rho_k=\rho_k-\frac{1}{2}\rho_{k-1}-\frac{1}{8}\rho_{k-2}-\frac{1}{16}\rho_{k-3}-\frac{5}{128}\rho_{k-4}-\dots$$

$$\Delta^{-1/2}\rho_k=\rho_k+\frac{1}{2}\rho_{k-1}+\frac{3}{8}\rho_{k-2}+\frac{5}{16}\rho_{k-3}+\frac{35}{128}\rho_{k-4}+\dots$$

$$\Delta^{2/3}\rho_k=\rho_k-\frac{2}{3}\rho_{k-1}-\frac{1}{9}\rho_{k-2}-\frac{4}{81}\rho_{k-3}-\frac{7}{243}\rho_{k-4}-\dots$$

Remark 1.1 *[30, Theorem 2]The following results hold:*

- $\Delta^{(\alpha)}\circ\Delta^{(-\alpha)}=I$, where I is identity on $\rho\in\omega$.
- $\Delta^{(\alpha)}\Delta^{(\beta)}=\Delta^{(\alpha+\beta)}$.

Note that the studied fractional difference operator includes some special cases. We refer to [30,32] for further results about these operators.

2. PRELIMINARY RESULTS

We state the known results that are used here and in the sequel for the reader's convenience.

Let ω denote the set of all complex sequences $\rho=(\rho_k)_{k=0}^{\infty}$. We write ℓ_∞, c, c_0 and ϕ for the bounded, convergent, null and finite sequence spaces, respectively; also cs and ℓ_1 denote convergent and absolutely convergent series spaces.

A subspace λ of ω is said to be a BK space if it is a Banach space with continuous coordinates $P_n:\lambda\to\mathbb{C}$ $(n=0,1,\dots)$, where $P_n(\rho)=\rho_n$ for all $\rho\in\lambda$. A BK space $\lambda\supset\phi$ is said to have AK if every sequence $\rho=(\rho_k)_{k=0}^{\infty}\in\lambda$ has a unique representation $\rho=\lim_{m\to\infty}\rho^{[m]}$, where $\rho^{[m]}=\sum_{n=0}^{m}\rho_n e^{(n)}$ is the m section of the sequence ρ. Let λ be a normed space. By N_r we denote any subset of $\mathbb{N}_0$ with elements greater or equal to r.

Let ρ and σ be sequences and λ and μ be subsets of ω, then we write $\rho\cdot\sigma=(\rho_k\sigma_k)_{k=0}^{\infty}$, $\rho^{-1}*\mu=\{a\in\omega: a\cdot\rho\in\mu\}$ and $M(\lambda,\mu)=\bigcap_{\rho\in\lambda}\rho^{-1}*\mu=\{a\in\omega: a\cdot\rho\in\mu\ \ for\ all\ \ \rho\in\lambda\}$ for the multiplier spaces of λ and μ; in particular, we use the notation $\rho^{\beta}=\rho^{-1}*cs$ and $\lambda^{\beta}=M(\lambda,cs)$ for the β dual of λ.

Let $A=(a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of complex or real numbers and ρ be a sequence, then we write $A_n=(a_{nk})_{k=0}^{\infty}$ for the sequence in the n^{th} row of A, $A_n\rho=\sum_{k=0}^{\infty}a_{nk}\rho_k$ $(n=0,1,\dots)$ and $A\rho=(A_n\rho)_{n=0}^{\infty}$, provided $A_n\in\rho^{\beta}$ for all n. If λ and μ are subsets of ω, then $\lambda_A=\{\rho\in\omega: A\rho\in\lambda\}$ is called matrix domain of A in λ. Class of all infinite matrices that map λ into μ is denoted by (λ,μ) and $A\in(\lambda,\mu)$ if and only if $\lambda\subset\mu_A$. An infinite matrix $T=(t_{nk})_{n,k=0}^{\infty}$ is said to be a triangle if $t_{nk}=0$ $(k>n)$ and $t_{nn}\neq 0$ for all n. We denote its inverse by S. We have $\|a\|_{\lambda}^{*}=\sup_{\rho\in S_\lambda}\left|\sum_{k=1}^{\infty}a_k\rho_k\right|$ for $a\in\omega$, provided the expression on the right hand side is defined and finite which is the case whenever λ is a BK space and $a\in\lambda^{\beta}$ ([4], Theorem 7.2.9, p. 107).

Consider now the fractional difference sequence spaces $c_0(\Delta^{(\alpha)})=\{\rho\in\omega: \lim_k\Delta^{(\alpha)}(\rho_k)=0\}$, $c(\Delta^{(\alpha)})=\{\rho\in\omega: \lim_k\Delta^{(\alpha)}(\rho_k)\ \ \text{exists}\}$ and $\ell_\infty(\Delta^{(\alpha)})=\{\rho\in\omega: \sup_k\left|\Delta^{(\alpha)}(\rho_k)\right|<\infty\}$.

Note that the sequence $\sigma=(\sigma_k)$ can be considered as the $\Delta^{(\alpha)}$-transform of a sequence $\rho=(\rho_k)$, that is,

$$\sigma_k = \rho_k - \alpha\rho_{k-1} + \frac{\alpha(\alpha-1)}{2!}\rho_{k-2} - \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\rho_{k-3} + \cdots$$
$$= \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\,\Gamma(\alpha-i+1)}\rho_{k-i}.$$

The defined spaces can be considered as the matrix domains of the triangle $\Delta^{(\alpha)}$ in the classical sequence spaces c_0, c, ℓ_∞.

The set λ_T is a BK space with $\|.\|_T = \|T(.)\|$ whenever $(\lambda, \|.\|)$ is a BK space. By this fact, defined fractional difference sequence spaces are complete, linear BK–spaces with the norm $\|\rho\| = \sup_k |\Delta^{(\alpha)}(\rho_k)|$.

A sequence $(b_n)_{n=0}^{\infty}$ in a linear metric space λ is called a Schauder basis if for each $\rho \in \lambda$ there exists a unique sequence $(\lambda_n)_{n=0}^{\infty}$ of scalars such that $\rho = \sum_{n=0}^{\infty} \lambda_n b_n$. Also λ_T has a basis if and only if λ has a basis.

Theorem 2.1 *If we write* $\eta^{(m)}$ *and* $\eta^{(-1)}$ *as*

$$\eta_k^{(m)} = \begin{cases} 0, & 0 \le k \le m \\ (-1)^{k-m}\dfrac{\Gamma(-\alpha+1)}{(k-m)!\,\Gamma(-\alpha+m-k+1)}, & k \ge m \end{cases}$$

for $n = 0,1,\ldots$ and for k=1,2,…

$$\eta_k^{(-1)} = \sum_{n=0}^{k} (-1)^{k-n}\frac{\Gamma(-\alpha+1)}{(k-n)!\,\Gamma(-\alpha+n-k+1)}.$$

- Then $\left(\eta^{(m)}\right)_{m=0}^{\infty}$ is a *Schauder* basis for $c_0(\Delta^{(\alpha)})$ and every sequence $\rho = (\rho_m)_{m=0}^{\infty} \in c_0(\Delta^{(\alpha)})$ has a unique representation $\rho = \sum_m (\Delta_m^{(\alpha)}\rho)\eta^{(m)}$ for all m.
- Then $\left(\eta^{(m)}\right)_{m=-1}^{\infty}$ is a *Schauder* basis for $c(\Delta^{(\alpha)})$, and every sequence $\rho = (\rho_m)_{m=0}^{\infty} \in c(\Delta^{(\alpha)})$ has a unique representation $\rho = \xi\eta^{(-1)} + \sum_m (\sigma_m - \xi)\eta^{(m)}$, where $\xi = \lim_{m\to\infty} \sigma_m$.
- The set $\ell_\infty(\Delta^{(\alpha)})$ has no *Schauder* basis.

Proof. The proof is an immediate consequence of [1, Lemma 2.3 and Corollary 2.5].

We now focus on the β duals and operator norms of fractional sets of sequences.

Lemma 2.2 *[10, Lemma 3.4] Let* T *be a triangle,* $S = T^{-1}$ *and* $R = S^t$.

- If λ is a BK set with AK property or $\lambda = \ell_\infty$, we have $a \in (\lambda_T)^\beta$ if and only if $a \in \left(\lambda^\beta\right)_R$ and $W \in (\lambda, c_0)$ where the triangle W is defined for $n = 0,1,2,\ldots$ by $w_{nk} = \sum_{j=n}^{\infty} a_j s_{jk}$ $(0 \le k \le n)$ and $w_{nk} = 0$ $(k > n)$. Furthermore, if $a \in (\lambda_T)^\beta$ then

$$\sum_k a_k z_k = \sum_k (R_k a)(T_k z) \ \ \forall z \in \lambda_T. \quad (2.1)$$

- We have $a \in (c_T)^\beta$ if and only if $a \in (\ell_1)_R$ and $W \in (c, c)$. Furthermore, if $a \in (c_T)^\beta$ then we have

$$\sum_k a_k z_k = \sum_k (R_k a)(T_k z) - \lim_k T_k z \lim_n \sum_{k=0}^{n} w_{nk} \ \ \forall z \in c_T. \quad (2.2)$$

Remark 2.3 *[10, Remark 3.5] We have the following results:*

- The condition $W \in (\lambda, c_0)$ in Lemma 2.2(i) can be changed by $W \in (\lambda, \ell_\infty)$ if λ is a BK set with AK property.
- The condition $W \in (c, c)$ in Lemma 2.2(ii) can be changed by the conditions $W \in (c_0, \ell_\infty)$ and $\lim_n W_n e = \gamma$ exists.

Theorem 2.4 *We have*

- $a \in \left(c_0(\Delta^{(\alpha)})\right)^\beta$ if and only if

$$\sum_k \left|\sum_{j=k}^{\infty} (-1)^{j-k}\frac{\Gamma(-\alpha+1)}{(j-k)!\,\Gamma(-\alpha-j+k+1)} a_j\right| < \infty \quad (2.3)$$

and

$$\sup_\iota \left(\sum_{k=0}^{n}\left|\sum_{j=\iota}^{\infty} (-1)^{j-k}\frac{\Gamma(-\alpha+1)}{(j-k)!\,\Gamma(-\alpha-j+k+1)} a_j\right|\right) < \infty; \quad (2.4)$$

furthermore, if $a \in \left(c_0(\Delta^{(\alpha)})\right)^\beta$ then $\forall \rho \in c_0(\Delta^{(\alpha)})$ we have

$$\sum_k a_k\rho_k = \sum_k \left(\sum_{j=k}^{\infty} (-1)^{j-k}\frac{\Gamma(-\alpha+1)}{(j-k)!\,\Gamma(-\alpha-j+k+1)} a_j\right)\sigma_k. \quad (2.5)$$

- $a \in \left(c(\Delta^{(\tilde{\alpha})})\right)^\beta$ if and only if (2.3), (2.4) and

$$\lim_{\iota} \sum_{k=0}^{\iota} \left(\sum_{j=\iota}^{\infty} (-1)^{j-k} \frac{\Gamma(-\alpha+1)}{(j-k)!\Gamma(-\alpha-j+k+1)} a_j \right) = \rho; \quad (2.6)$$

furthermore, if $a \in \left(c(\Delta^{(\alpha)})\right)^{\beta}$ then $\forall \rho \in c(\Delta^{(\alpha)})$ we have

$$\sum_k a_k \rho_k = \sum_k \left(\sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\alpha+1)}{(j-k)!\Gamma(-\alpha-j+k+1)} a_j \right) \sigma_k - \rho \lim_k \sigma_k. \quad (2.7)$$

• $a \in \left(\ell_\infty(\Delta^{(\widetilde{\alpha})})\right)^{\beta}$ if and only if (2.3) and

$$\lim_{\iota} \sum_{k=0}^{\iota} \left| \sum_{j=\iota}^{\infty} (-1)^{j-k} \frac{\Gamma(-\alpha+1)}{(j-k)!\Gamma(-\alpha-j+k+1)} a_j \right| = 0; \quad (2.8)$$

furthermore, if $a \in \left(\ell_\infty(\Delta^{(\alpha)})\right)^{\beta}$ then (2.5) holds $\forall \rho \in \ell_\infty(\Delta^{(\alpha)})$.

Proof. We apply Lemma 2.2 and Remark 2.3.

The triangles R and W are defined for $n = 0,1,\ldots$ by

$$R_k a = \sum_{j=k}^{\infty} s_{jk} a_j = \sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\alpha+1)}{(j-k)!\Gamma(-\alpha-j+k+1)} a_j \;\; (k = 0,1,\ldots) \quad (2.9)$$

and

$$w_{\iota k} = \begin{cases} \sum_{j=\iota}^{\infty} (-1)^{j-k} \frac{\Gamma(-\alpha+1)}{(j-k)!\Gamma(-\alpha-j+k+1)} a_j & (0 \le k \le \iota) \\ 0 & (k > \iota). \end{cases} \quad (2.10)$$

The condition $Ra \in \ell_1$ of Lemma 2.2 holds in each part because $c_0^{\beta} = c^{\beta} = \ell_\infty^{\beta}$.

By Lemma 2.2 (i) and Remark 2.3 (i), we must add the condition $W \in (c_0, \ell_\infty)$ which is equivalent to

$$\sup_{\iota} \sum_{k=0}^{\iota} |w_{\iota k}| < \infty$$

and this is the condition in (2.4).

By Lemma 2.2 (ii), Remark 2.3 (ii) and the well-known characterization of the class (c, c) we obtain the condition in (2.6).

By Lemma 2.2, we must add the condition $W \in (\ell_\infty, c_0)$ which is equivalent to

$$\lim_{\iota} \sum_{k=0}^{\iota} |w_{\iota k}| = 0$$

and this is the condition in (2.8). Note that, (2.5) in Parts (i) and (iii) and (2.7) come from (2.1) and (2.2), respectively.

3. OPERATOR NORMS AND MATRIX TRANSFORMATIONS ON FRACTIONAL SEQUENCE SPACES

Let us now establish identities and inequalities of operator norms for fractional sequence spaces. We need following results to characterize some classes of matrix mappings on the sets of fractional sequences and for determination of the operator norms of defined sets.

Lemma 3.1 *If λ and μ are BK sets.*

• Every matrix $A \in (\lambda, \mu)$ defines an operator $L_A \in \mathcal{B}(\lambda, \mu)$, where $L_A(\rho) = A\rho$ for all $\rho \in \lambda$ [6, Theorem 1.23].

• Every operator $L \in \mathcal{B}(\lambda, \mu)$ is given by a matrix $A \in (\lambda, \mu)$ such that $L(\rho) = A\rho$ for all $\rho \in \lambda$ if the set λ has AK property [1, Theorem 1.9].

Lemma 3.2 *[13, Theorem 3.4, Remark 3.5] Let μ be any subset of ω.*

• If λ is a BK set with AK property or $\lambda = \ell_\infty$, and $R = S^t$ we have $A \in (\lambda_T, \mu)$ if and only if $\hat{A} \in (\lambda, \mu)$ and $\Psi^{(A_n)} \in (\lambda, c_0)$ for all $n = 0,1,\ldots$. Here $\hat{A}$ is the matrix with rows $\hat{A}_n = RA_n$ for $n = 0,1,\ldots$, and the triangles $\Psi^{(A_n)}$ $(n = 0,1,\ldots)$ are defined as 3.2 with a_j changed by a_{nj}. Furthermore, if $A \in (\lambda, \mu)$ then we have $Az = \hat{A}(Tz)$ for all $z \in Z = \lambda_T$.

• We have $A \in (c_T, \mu)$ if and only if $\hat{A} \in (c_0, \mu)$ and $\Psi^{(A_n)} \in (c, c)$ for all $n = 0,1,\ldots$ and $\hat{A}e - (\gamma_n)_{n=0}^{\infty} \in \mu$, where $\gamma_n = \lim_m \sum_{k=0}^{m} \psi_{mk}^{(A_n)}$ for $n = 0,1,\ldots$.

Furthermore, if $A \in (c_T, \mu)$ then we have $Az = \hat{A}(Tz) - \eta(\gamma_n)_{n=0}^{\infty}$ for all $z \in c_T$, where $\eta = \lim_k T_k z$.

Theorem 3.3 *Let $\lambda = c_0(\Delta^{(\alpha)})$ or $\lambda = \ell_\infty(\Delta^{(\alpha)})$.*

• Let $\mu = c_0, c, \ell_\infty$. If $A \in (\lambda_T, \mu)$ then, putting

$$\|A\|_{(\lambda_T,\infty)} = \sup_n \sum_k \left| \sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\alpha+1)}{(j-k)!\,\Gamma(-\alpha-j+k+1)} a_{nj} \right|,$$

we have $\|L_A\| = \|A\|_{(\lambda_T,\infty)}$.

• If $A \in (\lambda_T, \ell_1)$ and $\Omega_1 = \|A\|_{(\lambda_T,1)}$. Then we have $\Omega_1 \leq \|L_A\| \leq 4\Omega_1$, where

$$\Omega_1 = \sup_{\substack{N\subset\mathbb{N}\\ N-\text{finite}}} \sum_k \times \left|\sum_{n\in N}\sum_{j=k}^{\infty} (-1)^{j-k}\frac{\Gamma(-\alpha+1)}{(j-k)!\,\Gamma(-\alpha-j+k+1)}a_{nj}\right|.$$

Proof. The proof is based on the results in [22, Theorem 2.8].

Theorem 3.4 *The operator norm of the set $c(\Delta^{(\tilde{\alpha})})$ is given.*

• Let $A \in \left(c(\Delta^{(\alpha)}), \mu\right)$, where μ is any of the spaces c_0, c or ℓ_∞. Then we have

$$\|L_A\| = \sup_n\left(\sum_k\left|\sum_{j=k}^{\infty}(-1)^{j-k}\frac{\Gamma(-\alpha+1)}{(j-k)!\,\Gamma(-\alpha-j+k+1)}a_{nj}\right| + |\gamma_n|\right),$$

where $\gamma_n = \lim_m \sum_{k=0}^{m} \psi_{mk}^{(A_n)}forn = 0,1,\ldots$

• Let $A \in \left(c(\Delta^{(\alpha)}), \ell_1\right)$ and $\Omega_2 = \|A\|_{(c(\Delta^{(\alpha)}),1)}$. Then we have $\Omega_2 \leq \|L_A\| \leq 4\Omega_2$, where

$$\Omega_2 = \sup_{\substack{N\subset\mathbb{N}\\ N\text{finite}}}\left(\sum_k\left|\sum_{n\in N}\sum_{j=k}^{\infty}(-1)^{j-k}\frac{\Gamma(-\alpha+1)}{(j-k)!\,\Gamma(-\alpha-j+k+1)}a_{nj}\right| + \left|\sum_{n\in N}\gamma_n\right|\right).$$

Proof. The proof is based on the results in [22, Theorem 2.9].

We now give the necessary and sufficient conditions for $A \in (\ell_\infty(\Delta^{(\alpha)}), \mu)$, $A \in (c_0(\Delta^{(\alpha)}), \mu)$ and $A \in (c(\Delta^{(\alpha)}), \mu)$, where $\mu \in \{\ell_\infty, c_0, c, \ell_1\}$.

Theorem 3.5 *The necessary and sufficient conditions for $A \in (\lambda(\Delta^{(\alpha)}), \mu)$, where $\mu \in \{\ell_\infty, c_0, c, \ell_1\}$ and $\lambda \in \{\ell_\infty, c_0, c\}$ can be read from the following table:*

Table 1. Necessary and sufficient conditions for $A \in (\lambda(\Delta^{(\alpha)}), \mu)$.

From/To	$\ell_\infty(\Delta^{(\alpha)})$	$c_0(\Delta^{(\alpha)})$	$c(\Delta^{(\alpha)})$
ℓ_∞	**1**	**1**	**2**
c_0	**3**	**3**	**4**
c	**5**	**5**	**6**
ℓ_1	**7**	**7**	**8**

(1) (1A) and (1B) where

(1A) - $\| A \|_{(c(\Delta^{(\alpha)}),\infty)} = \sup_n \sum_{k=0}^{\infty} |\hat{a}_{nk}| < \infty$,

(1B)- $\| \Psi^{(A_n)} \|_{(\ell_\infty,c_0)} = \lim_{m\to\infty} \sum_{k=0}^{m} \left|\psi_{mk}^{(A_n)}\right| = 0$ for all n.

(2) (1A) and (2A) where

(2A)- $\| \Psi^{(A_n)} \|_{(\ell_\infty,\ell_\infty)} = \sup_m \sum_{k=0}^{m} \left|\psi_{mk}^{(A_n)}\right| < \infty$ for all n.

(3) (1A), (2A), (3A) and (3B) where

(3A)- $\lim_{m\to\infty} \sum_{k=0}^{m} \psi_{mk}^{(n)} = \gamma_n$ exists for each n,

(3B)- $\sup_n \left|\sum_{k=0}^{\infty} \hat{a}_{nk} - \gamma_n\right| = 0$.

(4) (1B) and (4A) where

(4A)- $\lim_{m\to\infty} \sum_{k=0}^{m} |\hat{a}_{nk}| = 0$.

(5) (1A), (2A) and (5A) where

(5A)- $\lim_{n\to\infty} \hat{a}_{nk} = 0$ for each k.

(6) (1A), (2A), (3A), (5A) and (6A) where

(6A)- $\lim_{n\to\infty} \left(\sum_{k=0}^{\infty} \hat{a}_{nk} - \gamma_n\right) = 0$.

(7) (1B), (7A), (7B) and (7C) where

(7A)- $\lim_{n\to\infty} \hat{a}_{nk} = \hat{\alpha}_k$ exists for each k,

(7B)- $\sum_{k=0}^{\infty} |\hat{a}_{nk}|, \sum_{k=0}^{\infty} |\hat{\alpha}_k| < \infty$ for all n,

(7C)- $\lim_{n\to\infty} \left(\sum_{k=0}^{\infty} \hat{a}_{nk} - \hat{\alpha}_k\right) = 0$.

(8) (1A), (2B) and (7A).

(9) (1A), (2B), (3A), (7A) and (9A) where

(9A) - $\lim_{n\to\infty} \left(\sum_{k=0}^{\infty} \hat{a}_{nk} - \gamma_n\right) = \delta$ exists.

(10) (1B) and (10A) where

(10A) - $\sup_{K\subset\mathbb{N}} \sum_{n=0}^{\infty} |\sum_{k\in K} \hat{a}_{nk}| < \infty$.

(11) (2A) and (10A).

(12) (2A), (3A), (10A) and (12A) where

(12A) - $\sum_{n=0}^{\infty} |\sum_{k=0}^{\infty} \hat{a}_{nk} - \gamma_n| < \infty$,

where for a given matrix $A = (a_{nk})_{n,k=0}^{\infty}$, we define the corresponding matrices $\hat{A} = (\hat{a}_{nk})_{n,k=0}^{\infty}$ and $\Psi^{(A_n)} = (\psi_{mk}^{(A_n)})_{m,k=0}^{\infty}$ by

$$\hat{a}_{nk} = \sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\alpha+1)}{(j-k)!\Gamma(-\alpha-j+k+1)} a_{nj} \qquad (3.1)$$

for all $n, k \in \mathbb{N}_0$ and

$$\psi_{mk}^{(A_n)} = \begin{cases} \sum_{j=m}^{\infty} (-1)^{j-k} \frac{\Gamma(-\alpha+1)}{(j-k)!\Gamma(-\alpha+j-k+1)} a_{nj} & (0 \leq k \leq m) \\ 0 & (k > m) \end{cases} \qquad (3.2)$$

for $n, m \in \mathbb{N}_0$.

Proof. Note that the entries of the triangles $\hat{A}$ and $W^{(A_n)}$ are given above and assume that $\mu \in \{c_0, c, \ell_\infty, \ell_1\}$.

Taking into account Lemma 3.2(i) we have $\hat{A} \in (\ell_\infty, \mu)$ and $\Psi^{(A_n)} \in (\ell_\infty, c_0)$ if and only if $A \in (\ell_\infty(\Delta^{(\alpha)}), \mu)$ for each n. Note that $\hat{A} \in (\ell_\infty, \mu)$ satisfies (1A) in **(1)**, by [4, Theorem 1.3.3], (4A) in **(4)** by [27, 21. (21.1)], (7A), (7B) and (7C) in **(7)** by [4, Theorem 1.7.18 (ii)], and (10A) in [10] by [4, 8.4.9A]. Additionally, $\Psi^{(A_n)} \in (\ell_\infty, c_0)$ for each n satisfies (2A) in **(1), (4), (7), (10)** by [[4], Theorem 1.3.3]. Therefore, we have shown **(1), (4), (7), (10)**.

Remark 2.3(i) and Lemma 3.2(i) satisfy $A \in (\ell_\infty(\Delta^{(\alpha)}), \mu)$ if and only if $\hat{A} \in (c_0, \mu)$ and $\Psi^{(A_n)} \in (c_0, c_0)$ for each $n = 0,1, \ldots$. First $\hat{A} \in (c_0, c_0)$ satisfies (1A) in **(2)** by [4, Theorem 1.3.3], (1A) and (5A) in **(5)** by [4, 8.4.5A], (1A) and (7A) in **(8)** by [4, 8.4.5A] and (10A) in **(11)**. Additionally $\Psi^{(A_n)} \in (c_0, c_0)$ satisfies (2A) in **(2), (5), (8), (11)** for each n by [4, Theorem 1.3.3]. Hence, we have proved **(2), (5), (8), (11).**

Remark 2.3(i) and Lemma 3.2(i) satisfy $A \in (c_0(\Delta^{(\alpha)}), \mu)$ if and only if $\hat{A} \in (c_0, \mu)$ and $\Psi^{(A_n)} \in (c_0, \ell_\infty)$ for each n, and

$$\lim_{m\to\infty} \Psi_m^{(A_n)} e = \lim_m \sum_{k=0}^{m} \psi_{mk}^{(A_n)} = \gamma_n$$

satisfies for $n = 0,1, \ldots$ and also

$$\left\{\hat{A} - \left(\lim_m \sum_{k=0}^{m} \psi_{mk}^{(A_n)}\right)\right\} \in \mu$$

satisfies for $n = 0,1, \ldots$. It implies, we must add final two conditions to those for $A \in (c_0(\Delta^{(\alpha)}), \mu)$, that is, (3A) and (3B) in **(3)** to those in **(2),** (3A) and (6A) in **(6)** to those in **(5)**, (3A) and (9A) in **(9)** to those in **(8)** and (3A) and (12A) in **(12)** to those in **(11)**. It completes the proofs of **(3) , (6) , (9) , (12)**.

4. APPLICATIONS OF MEASURE OF NONCOMPACTNESS ON FRACTIONAL BANACH SETS

We first present some concepts about HMN.

Let λ and μ be infinite dimensional Banach spaces then a linear operator $L: \lambda \to \mu$ is called compact if domain of L is all of λ, and $(L(\rho_n))$ has a convergent subsequence for every bounded sequence $(\rho_n) \in \lambda$. Class of those operators is denoted by $\mathcal{C}(\lambda, \mu)$.

Let (λ, d) be a metric space, $B(x_0, \delta) = \{x \in \lambda: d(x, x_0) < \delta\}$ be an open ball and $\mathcal{M}_\lambda \in \lambda$ be the collection of bounded sets. HMN of $Q \in \mathcal{M}_\lambda$ is

$$\chi(Q) = \inf\{\varepsilon > 0: Q \subset \bigcup_{k=1}^{n} B(x_k, \delta_k): x_k \in \lambda, \delta_k < \varepsilon, \quad 1 \leq k \leq n, \quad n \in \mathbb{N}\}.$$

Let λ and μ be Banach spaces and χ_1 and χ_2 be measures of noncompactness on λ and μ. Then the operator $L: \lambda \to \mu$ is called (χ_1, χ_2)–bounded if $L(Q) \in \mathcal{M}_\mu$ for every $Q \in \mathcal{M}_\lambda$ and there exists a positive constant C such that

$$\chi_2(L(Q)) \leq C\chi_1(Q) \quad for\ every \quad Q \in \mathcal{M}_\lambda. \qquad (4.1)$$

If an operator L is (χ_1, χ_2)–bounded then the number

$$\| L \|_{(\chi_1,\chi_2)} = \inf\{C \geq 0: (4.1) \quad holds\ for\ all \quad Q \in \mathcal{M}_\lambda\}$$

is called the (χ_1, χ_2)–measure of noncompactness of L. In particular, if $\chi_1 = \chi_2 = \chi$, then we write $\| L \|_\chi$ instead of $\| L \|_{(\chi,\chi)}$.

Let λ and μ be Banach sets and L ∈ $\mathcal{B}$(λ, μ). Then

$$\|L\|_\chi = \chi(L(\bar{B}_\lambda)) = \chi(L(S_\lambda)), \quad (4.2)$$

$$\|L\|_\chi = 0 \text{ if and only if } L \in \mathcal{C}(\lambda,\mu) \quad (4.3)$$

by [6, Corollary 2.26].

If Q is a bounded subset of the normed space λ, where λ is ℓ_p for $1 \le p < \infty$ or c_0 and if $P_n: \lambda \to \lambda$ is the operator defined by $P_n(\rho) = \rho^{[n]}$ for $\rho = (\rho_k)_{k=0}^\infty \in \lambda$, then we have $\chi(Q) = \lim_n (\sup_{\rho\in Q} \|R_n(\rho)\|)$ [6, Theorem 2.8].

We now establish some identities or estimates for the HMN of certain operators on fractional difference sequence spaces to characterize compact operators in the last section.

Theorem 4.3 *The identities or estimates for L_A when $A \in (\lambda(\Delta^{(\alpha)}), \mu)$, where $\mu \in \{\ell_\infty, c_0, c, \ell_1\}$ and $\lambda \in \{\ell_\infty, c_0, c\}$ can be read from the following table:*

From/To	$\ell_\infty(\Delta^{(\alpha)})$	$c_0(\Delta^{(\alpha)})$	$c(\Delta^{(\alpha)})$
ℓ_∞	**1**	**1**	**2**
c_0	**3**	**3**	**4**
c	**5**	**5**	**6**
ℓ_1	**7**	**7**	**8**

Table 2. Identities or estimates for L_A when $A \in (\lambda(\Delta^{(\alpha)}), \mu)$.

Here

(1) $0 \le \| L_A \|_\chi \le \lim_{r\to\infty} \left(\sup_{n\ge r} \sum_{k=0}^\infty |\hat{a}_{nk}|\right)$;

(2) $0 \le \| L_A \|_\chi \le \lim_{r\to\infty} \left(\sup_{n\ge r} \sum_{k=0}^\infty |\hat{a}_{nk}| + |\gamma_n|\right)$;

(3) $\| L_A \|_\chi = \lim_{r\to\infty} \|\hat{A}^{[p]}\|_{(\ell_\infty,\ell_\infty)}$;

(4) $\| L_A \|_\chi = \lim_{r\to\infty} \left(\sup_{n\ge r} \sum_{k=0}^\infty |\hat{a}_{nk}| + |\gamma_n|\right)$;

(5) $\frac{1}{2} \cdot \lim_{r\to\infty} \|\hat{B}^{[p]}\|_{((\ell_\infty,\ell_\infty)} \le \| L_A \|_\chi \le \lim_{r\to\infty} \|\hat{B}^{[p]}\|_{((\ell_\infty,\ell_\infty)}$;

(6) $\frac{1}{2} \cdot \lim_{r\to\infty} \left(\sup_{n\ge r} \sum_{k=0}^\infty |\hat{b}_{nk}| + |\delta_n|\right) \le \| L_A \|_\chi \le \lim_{r\to\infty} \left(\sup_{n\ge r} \sum_{k=0}^\infty |\hat{b}_{nk}| + |\delta_n|\right)$;

(7) $\lim_{r\to\infty} \sup_{N\subset \mathbb{N}_0 \text{ finite}} \left\|\sum_{n\in\mathbb{N}} \hat{A}_n^{[p]}\right\|_1 \le \| L_A \|_\chi \le 4 \lim_{r\to\infty} \sup_{N\subset \mathbb{N}_0 \text{ finite}} \left\|\sum_{n\in\mathbb{N}} \hat{A}_n^{[p]}\right\|_1$;

(8) $\lim_{r\to\infty} \sup_{N\subset \mathbb{N}_0 \text{ finite}} \left(\left\|\sum_{n\in\mathbb{N}} \hat{A}_n^{[p]}\right\|_1 + |\sum_{n\in N} \gamma_n|\right) \le \| L_A \|_\chi \le 4 \lim_{r\to\infty} \sup_{N\subset \mathbb{N}_0 \text{ finite}} \left(\left\|\sum_{n\in\mathbb{N}} \hat{A}_n^{[p]}\right\|_1 + |\sum_{n\in N} \gamma_n|\right)$,

where the notations used in the theorem are defined as follows:

Here, $A^{[p]}$ represents a matrix with rows $A_n^{[p]} = 0$ for $0 \le n \le p$ and $A_n^{[p]} = A_n$ for $n \ge p+1$ where $A = (a_{nk})_{n,k=0}^\infty$ is an infinite matrix and $p \in \mathbb{N}_0$. Then

We write $\hat{A}$ for the matrix with

$$\hat{a}_{nk} = \sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\alpha+1)}{(j-k)!\,\Gamma(-\alpha-j+k+1)} a_{nj}$$

for all $n,k \in \mathbb{N}_0$; and $\hat{\alpha} = (\hat{\alpha}_k)_{k=0}^\infty$ and $\gamma = (\gamma_n)_{n=0}^\infty$ for the sequences with $\hat{\alpha}_k = \lim_{n\to\infty} \hat{a}_{nk}$ for $k = 0,1,\ldots$ and $\gamma_n = \lim_m \sum_{k=0}^m \psi_{mk}^{(A_n)} = \lim_m \sum_{k=0}^m \sum_{j=m}^\infty (-1)^{j-m} \frac{\Gamma(-\alpha+1)}{(j-m)!\Gamma(-\alpha-j+m+1)} a_{nj}$; also $\beta = \lim_{n\to\infty}(\sum_{k=0}^\infty \hat{a}_{nk} - \gamma_n)$. We also write $\hat{B} = (\hat{b}_{nk})_{n,k=0}^\infty$ for the matrix with $\hat{b}_{nk} = \hat{a}_{nk} - \hat{\alpha}_k$ for each $n,k \in \mathbb{N}_0$ and $\delta = (\delta_n)_{n=0}^\infty$ for the sequence with $\delta_n = \sum_{k=0}^\infty \hat{\alpha}_k - \gamma_n + \beta (n = 0,1,\ldots)$.

Proof. The conditions in **(1)** and **(2)** are immediate consequence of [22, Corollary 3.6(a)]. We define $P_r: \ell_\infty \to \ell_\infty$ by $P_r(\rho) = \rho^{[r]}$ for all $\rho \in \ell_\infty$ and $r = 0,1,\ldots$, $R_r = I - P_r$, and write $L = L_A$ and $\bar{B} = \bar{B}_{\ell_\infty}$ for the sake of brevity. So we have

$$0 \le \|L\|_\chi = \chi(L(\bar{B}))$$
$$\le \chi(P_r(L(\bar{B}))) + \chi(R_r(L(\bar{B})))$$
$$= \chi\left(R_r(L(\bar{B}))\right) \le \sup_{\rho\in\bar{B}} \|R_r(L(\rho))\|_\infty = \left\|\hat{A}^{[p]}\right\|_{(\lambda,\infty)}$$

by (4.1), [6, Theorem 2.12] and Lemma 3.3(i). Hence, **(3)** holds.

The conditions in **(4)** and **(6)** are immediate consequence of [22, Theorem 3.7 (b), (a)]. Part **(5)** follows by a similar argument as part **(3)**; we use Lemma 3.4(i) instead of Lemma 3.3(i).

5. CONCLUSION

Fractional difference sets of sequences have been shown up in literature like fractional derivatives and fractional integrals. The gamma function which can be written by the improper integral is used to construct the fractional difference operators. One of the main goal of this study is to consider fractional operators and fractional sets of sequences $c_0(\Delta^{(\alpha)})$, $c(\Delta^{(\alpha)})$ and $\ell_\infty(\Delta^{(\alpha)})$ in addition to determine the operator norms, find the β duals and characterize corresponding matrix transformations. The following table gives necessary and sufficient conditions for an operator from our fractional sets to classical sets ofsequences to be compact.

From/To	$\ell_\infty(\Delta^{(\alpha)})$	$c_0(\Delta^{(\alpha)})$	$c(\Delta^{(\alpha)})$
c_0	1	1	2
c	3	3	4
ℓ_1	5	5	6

Table 3. Compactness conditions

(1) $\lim_{r\to\infty}\sup_{n\geq r}\sum_{k=0}^{\infty}\left|\sum_{j=k}^{\infty}(-1)^{j-k}\frac{\Gamma(-\alpha+1)}{(j-k)!\Gamma(-\alpha-j+k+1)}a_{nj}\right| = 0;$

(2) $\lim_{r\to\infty}\sup_{n\geq r}\left(\sum_{k=0}^{\infty}\left|\sum_{j=k}^{\infty}(-1)^{j-k}\frac{\Gamma(-\alpha+1)}{(j-k)!\Gamma(-\alpha-j+k+1)}a_{nj}\right| + |\gamma_n|\right) = 0;$

(3) $\lim_{r\to\infty}\sup_{n\geq r}\sum_{k=0}^{\infty}\left|\sum_{j=k}^{\infty}(-1)^{j-k}\frac{\Gamma(-\alpha+1)}{(j-k)!\Gamma(-\alpha-j+k+1)}a_{nj} - -\hat{\alpha}_k\right| = 0;$

(4) $\lim_{r\to\infty}\sup_{n\geq r}\left(\sum_{k=0}^{\infty}\left\|\sum_{j=k}^{\infty}(-1)^{j-k}\frac{\Gamma(-\alpha+1)}{(j-k)!\Gamma(-\alpha-j+k+1)}a_{nj} - \hat{\alpha}_k\right\| + \|\sum_{k=0}^{\infty}\hat{\alpha}_k - \gamma_n - \beta\|\right) = 0;$

(5) $\lim_{r\to\infty}\sup_{\substack{N\subset\mathbb{N}_0\\ \text{finite}}}\left\|\sum_{n\in N_r}\sum_{j=k}^{\infty}(-1)^{j-k}\frac{\Gamma(-\alpha+1)}{(j-k)!\Gamma(-\alpha-j+k+1)}a_{nj}\right\|_1 = 0$;

(6) $\lim_{r\to\infty}\sup_{\substack{N\subset\mathbb{N}_0\\ \text{finite}}}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in N_r}\sum_{j=k}^{\infty}(-1)^{j-k}\frac{\Gamma(-\alpha+1)}{(j-k)!\Gamma(-\alpha-j+k+1)}a_{nj}\right| + \left|\sum_{n\in N_r}\gamma_n\right|\right) = 0.$

The final corollary of the study gives sufficient conditions when the final set is the of all bounded sequences.

Corollary 5.1 *Let λ be one of the spaces $c_0(\Delta^{(\alpha)})$ or $\ell_\infty(\Delta^{(\alpha)})$.*

• Let $A \in (\lambda, \ell_\infty)$, then L_A becomes a compact operator if the condition **(1)** given in **Table 3** satisfies.

• Let $A \in (c(\Delta^{(\alpha)}), \ell_\infty)$, then L_A is compact if the condition **(2)** given in **Table 3** satisfies.

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