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Some fixed point theorems in 2-Banach spaces and 2-normed tensor product spaces

Dipankar Das[1], *Nilakshi Goswami*[1], *Vandana*[2]

[1]Department of Mathematics, Gauhati University, Guwahati, Assam, India
[2]School of Studies in Mathematics, Pt. Ravishankar Shukla University, Raipur (C.G.), India

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Abstract: In this paper, we derive some fixed point theorems in 2-Banach spaces. Let X be a 2-Banach space and T be a self-mapping on X. Let $\psi : [0,\infty) \to [0,\infty)$; $\beta,\phi : [0,\infty)\times[0,\infty) \to [0,\infty)$ and $\gamma : [0,\infty)\times[0,\infty)\times[0,\infty) \to [0,\infty)$ be continuous mappings having some specific characteristics. Using these mappings, we define some conditions for T under which T has a unique fixed point in X. The conditions for two self-mappings T_1 and T_2 on X for having the common unique fixed point are also derived here with proper examples. Moreover, defining a 2-norm in the projective tensor product space, we derive a fixed point theorem here with a suitable example.

Keywords: 2-Banach space, fixed points, projective tensor product.

1 Introduction

In this paper, we derive some fixed point theorems for mappings on 2-Banach spaces satisfying some specific characteristics. The notion of 2-normed linear spaces and their topological structures was initiated by Gähler [10] in his paper "Linear 2-normed spaces". He studied the special class of 2-metric spaces which is linear and defined a 2-norm on those spaces. Motivated by this work, several authors namely Iseki [11], Rhoads [27], White [29], etc., studied various aspects of the fixed point theory and proved some fixed point theorems in 2-metric and 2-Banach spaces. Cho et al. [3] investigated about common fixed points of weakly compatible mappings in 2-metric spaces. In 1993, Khan and Khan [12] derived some results on fixed points of involution maps in 2-Banach spaces. In 2013 [28], Saha et al. discussed some fixed point theorems for a class of for weakly C-contractive mappings in a setting of 2-Banach Space.

2 Preliminaries

Definition 1. *Let X be a real linear space of dimension greater than 1 and let $\|.,.\|$ be a real valued function on $X\times X$ satisfying the following conditions:*

(i) $\|x,y\| = 0$ *if and only if x and y are linearly dependent,*
(ii) $\|x,y\| = \|y,x\|$ *for all* $x,y \in X$,
(iii) $\|\alpha x,y\| = |\alpha|\|y,x\|$, α *being real,* $x,y \in X$,
(iv) $\|x,y+z\| \leqslant \|x,y\| + \|x,z\|$, *for all,* $x,y,z \in X$

Then $\|.,.\|$ is called a 2-norm on X and $(X,\|.,.\|)$ is called a linear 2-normed space.

Definition 2. *A sequence $\{x_n\}$ in a 2-normed space $(X,\|.,.\|)$ is said to be a Cauchy sequence if* $\lim_{n,m\to\infty}\|x_n - x_m, a\| = 0$ *for all a in X.*

* Corresponding author e-mail: dipankardasguw@yahoo.com

Definition 3. *A sequence* $\{x_n\}$ *in a* 2*-normed space* X *is called a convergent sequence if there is an* x *in* X *such that* $\lim_{n\to\infty}\|x_n - x, a\| = 0$ *for all* a *in* X.

Definition 4. *A* 2*-normed space in which every Cauchy sequence is convergent is called a* 2*-Banach space.*

Definition 5. *Let* X *and* Y *be two linear* 2*-normed spaces. An operator* $T : X \to Y$ *is said to be continuous at* $x \in X$ *if for every sequence* $\{x_n\}$ *in* X, $\{x_n\} \to x$ *as* $n \to \infty$ *implies* $\{T(x_n)\} \to T(x)$ *in* Y *as* $n \to \infty$.

Definition 6. *Let* f *and* g *be two self-maps on a set* X. *If* $fx = gx$, *for some* x *in* X *then* x *is called coincidence point of* f *and* g.

Definition 7. *Let* f *and* g *be two self-maps defined on a set* X. *Then* f *and* g *are said to be weakly compatible if they commute at coincidence points, i.e., if* $fu = gu$ *for some* $u \in X$, *then* $fgu = gfu$.

3 Fixed point in 2-Banach spaces

Theorem 1. *Let* X *be a* 2*-Banach space and* T *be a self map on* X. *Let* $\psi : [0,\infty) \to [0,\infty)$ *and* $\beta : [0,\infty)\times[0,\infty) \to [0,\infty)$ *are continuous mappings satisfying the conditions:* $\psi(0) = 0$, ψ *is monotonically increasing;*

$$b\psi(s) \leqslant \beta(r,s) \Rightarrow bs \leqslant r, b \in \{1,2\}; \beta(s,t) = 0 \Leftrightarrow s = t = 0.$$

Let

$$\psi(\|Tx - Ty, a\|) \leqslant \beta(\|x - Tx, a\|, \|y - Ty, a\|) - \max[\psi(\|x - Tx, a\|), \psi(\|y - Ty, a\|)]$$

where $x, y, a \in X$. *Then* T *has a unique fixed point on* X.

Proof. For any fixed $x_0 \in X$, we construct a sequence $\{x_n\}$ by $x_{n+1} = Tx_n,\ n = 0,1,2,\ldots$

$$\begin{aligned}\psi(\|x_n - x_{n+1}, a\|) &\leqslant \beta(\|x_{n-1} - x_n, a\|, \|x_n - x_{n+1}, a\|)\\ &\quad - \max[\psi(\|x_{n-1} - x_n, a\|), \psi(\|x_n - x_{n+1}, a\|)]\\ &\leqslant \beta(\|x_{n-1} - x_n, a\|, \|x_n - x_{n+1}, a\|)\end{aligned}$$

Therefore we write,

$$\|x_n - x_{n+1}, a\| \leqslant \|x_{n-1} - x_n, a\|.$$

So, $\{\|x_n - x_{n+1}, a\|\}$ is a monotonic decreasing sequence of real numbers and hence it converges to some r, say, i.e., $\|x_n - x_{n+1}, a\| \to r$ as $n \to \infty$.

Now, $\|x_n - x_{n+1}, a\| = \|Tx_{n-1} - Tx_n, a\|$. So,

$$\begin{aligned}\psi(r) &= \psi(\lim_{n\to\infty}\|x_n - x_{n+1}, a\|) = \lim_{n\to\infty}\psi(\|Tx_{n-1} - Tx_n, a\|)\\ &\leqslant \lim_{n\to\infty}[\beta(\|x_{n-1} - Tx_{n-1}, a\|, \|x_n - Tx_n, a\|) - \max(\psi(\|x_{n-1} - Tx_{n-1}, a\|), \psi(\|x_n - Tx_n, a\|))]\\ &= \lim_{n\to\infty}[\beta(\|x_{n-1} - x_n, a\|, \|x_n - x_{n+1}, a\|) - \max(\psi(\|x_{n-1} - x_n, a\|), \psi(\|x_n - x_{n+1}, a\|))]\\ &= \beta(r,r) - \max[\psi(r), \psi(r)].\end{aligned}$$

Thus, $2\psi(r) \leqslant \beta(r,r) \Rightarrow 2r \leqslant r$, possible for $r = 0$. Hence, $\|x_n - x_{n+1}, a\| \to 0$ as $n \to \infty$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in X. If possible, let $\{x_n\}$ be not a Cauchy sequence, and so, there exists

$\varepsilon > 0$ such that there exists sub sequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ such that $\|x_{m_k} - x_{n_k}, a\| \geqslant \varepsilon$ and $\|x_{m_k} - x_{n_k-1}, a\| < \varepsilon$ Then,

$$\begin{aligned}\psi(\varepsilon) &\leqslant \psi\|x_{m_k} - x_{n_k}, a\|) = \psi(\|Tx_{m_k-1} - Tx_{n_k-1}, a\|)\\ &\leqslant \beta(\|x_{m_k-1} - Tx_{m_k-1}, a\|, \|x_{n_k-1} - Tx_{n_k-1}, a\|)\\ &\quad - \max[\psi(\|x_{m_k-1} - Tx_{m_k-1}, a\|), \psi(\|x_{n_k-1} - Tx_{n_k-1}, a\|)]\end{aligned}$$

Taking n_k and $m_k \to \infty$ and using the continuity of β and ψ

$$\psi(\varepsilon) \leqslant \beta(0,0) - \max[\psi(0), \psi(0)] = 0 = \psi(0) \Rightarrow \varepsilon \leqslant 0$$

which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in X, and so, it converges to some z, in X. Now,

$$\begin{aligned}&\psi(\|x_n - Tz, a\|) \leqslant \beta(\|x_{n-1} - Tx_{n-1}, a\|, \|z - Tz, a\|) - \max[\psi(\|x_{n-1} - Tx_{n-1}, a\|), \psi(\|z - Tz, a\|)]\\ &\Rightarrow 2\psi(\|z - Tz, a\|) \leqslant \beta(0, \|z - Tz, a\|); [\text{taking } n \to \infty]\\ &\Rightarrow 2\|z - Tz, a\| \leqslant 0, (\forall a \in X)\\ &\Rightarrow \|z - Tz, a\| = 0\end{aligned}$$

Since a is arbitrary, taking $a = 0$, we get, $z = Tz$.

To show the uniqueness: Let $Tz_1 = z_1$ and $Tz_2 = z_2$. Then

$$\begin{aligned}\psi(\|Tz_1 - Tz_2, a\|) &\leqslant \beta(\|z_1 - Tz_1, a\|, \|z_2 - Tz_2, a\|) - \max[\psi(\|z_1 - Tz_1, a\|), \psi(\|z_2 - Tz_2, a\|)]\\ &= \beta(0,0) - \max[\psi(0), \psi(0)].\end{aligned}$$

Therefore we write,

$$\|Tz_1 - Tz_2, a\| = 0 \ \forall a \in X \Rightarrow Tz_1 = Tz_2 \Rightarrow z_1 = z_2.$$

The proof is completed.

Example 1. Let $X = \mathbb{R}^3$ and we consider the following 2-norm on X (refer to [1])

$$\|x, y\| = \left| det \begin{pmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \right|$$

where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$. Then $(X, \|.,.\|)$ is a 2-Banach space.

We fix $(e, f, g) \in \mathbb{R}^3$ and let T be a self mapping on $\mathbb{R}^3$ defined by $T(x,y,z) = (e, f, g)\ \forall (x,y,z) \in \mathbb{R}^3$.

Let $\psi(s) = 2s$, $\beta(r,s) = \dfrac{r}{2} + s$; where $(r,s) \in [0,\infty)$. Now, $Tx = (e, f, g) = Ty$ therefore $\|Tx - Ty, a\| = 0$.

Hence all the conditions of Theorem 1 are satisfied. So, T has a unique fixed point $(e, f, g) \in \mathbb{R}^3$.

For common fixed point of two self maps T_1 and T_2 on X, we prove.

Theorem 2. *Let X be a 2-Banach space and T_1 and T_2 be two self maps on X. Let ψ and β be as defined in Theorem 1 with $\beta(r,s) = \beta(s,r)$. Then T_1 and T_2 have common unique fixed point, if for $x, y, a \in X$*

$$\psi(\|T_1x - T_2y, a\|) \leqslant \beta(\|x - T_1x, a\|, \|y - T_2y, a\|) - \max[\psi(\|x - T_1x, a\|), \psi(\|y - T_2y, a\|)]$$

Proof. For a fixed point $x_0 \in X$, we construct a sequence $\{x_n\}$ by

$$x_{2n+1} = T_1(x_{2n}) \text{ and } x_{2n+2} = T_2(x_{2n+1}),\ n = 0, 1, 2, \ldots$$

Now, it can be shown that $\{x_n\}$ is a Cauchy sequence in X, converging to some z in X, which is the common fixed point for T_1 and T_2.

Corollary 1. *Let X be a 2-Banach space and T be a self map on X. Let ψ, β be as defined in Theorem 3.1 satisfying*

$$\psi(\|Tx - Ty, a\|) \leqslant \frac{1}{c}[\beta(\|x - Ty, a\|, \|y - Ty, a\|) - \max(\psi(\|y - Tx, a\|), \psi(\|x - Tx, a\|))]$$

where $c > 2$ and $x, y, a \in X$. Then T has unique fixed point on X.

Corollary 2. *If ψ satisfies then also similar result holds for*

$$\psi(\|Tx - Ty, a\|) \leqslant \beta(\|x - y, a\|, \|y - Ty, a\|) - \max[\psi(\|x - y, a\|), \psi(\|x - Tx, a\|)], \forall x, y, a \in X.$$

We now establish another fixed point theorem for T using two other mappings γ and ϕ.

Theorem 3. *Let X be a 2-Banach space and T be a self mapping on X. Let $\gamma : [0,\infty) \times [0,\infty) \times [0,\infty) \to [0,\infty)$ and $\phi : [0,\infty) \times [0,\infty) \to [0,\infty)$ be continuous mapping satisfying $\gamma(r, 0, r+t) \leqslant kr$ and $\phi(r,t) \geqslant k^/ r$, where $k, k^/ \in [0,\infty)$ such that $k - k^/ < 1$. Let*

$$\|Tx - Ty, a\| \leqslant \gamma[\|x - y, a\|, \|y - Tx, a\|, \|x - Ty, a\|] - \phi[\|x - Tx, a\|, \|y - Ty, a\|], \forall x, y, a \in X$$

Then T has a fixed point.

Proof. For any fixed $x_0 \in X$, we construct a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. We get,

$$\begin{aligned}
\|x_n - x_{n+1}, a\| &\leqslant \gamma[\|x_{n-1} - x_n, a\|, \|x_n - x_n, a\|, \|x_{n-1} - x_{n+1}, a\|] - \phi[\|x_{n-1} - x_n, a\|, \|x_n - x_{n+1}, a\|] \\
&\leqslant \gamma[\|x_{n-1} - x_n, a\|, 0, \|x_{n-1} - x_n, a\| + \|x_n - x_{n+1}, a\|] - \phi[\|x_{n-1} - x_n, a\|, \|x_n - x_{n+1}, a\|] \\
&\leqslant (k - k^/)\|x_{n-1} - x_n, a\| \\
&\leqslant (k - k^/)^2\|x_{n-2} - x_{n-1}, a\| \leqslant \ldots \leqslant (k - k^/)^n\|x_0 - x_1, a\| \to 0 \text{ as } n \to \infty
\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in X and so, it converges to some z,(say) in X.

$$\begin{aligned}
\|z - Tz, a\| &\leqslant \|z - x_{n+1}, a\| + \|x_{n+1} - Tz, a\| \\
&\leqslant \|z - x_{n+1}, a\| + \gamma[\|x_n - z, a\|, \|z - Tx_n, a\|, \|x_n - Tz, a\|] - \phi[\|x_n - Tx_n, a\|, \|z - Tz, a\|] \\
&\leqslant 0 + \gamma[0, 0, 0 + \|z - Tz, a\|] - \phi[0, \|z - Tz, a\|] \text{ [taking } n \to \infty].
\end{aligned}$$

Therefore we write $\|z - Tz, a\| = 0$, for all $a \in X$. Since a is arbitrary, taking $a = 0$, we get, $z = Tz$.

Example 2. Let $\gamma(r,s,t) = k_1(r+s+t)$ and $\phi(r,s) = k_2(r+s)$, where k_1 and k_2 are two constants (> 0). Now we can find out $k, k^/ \in [0,\infty)$ with $k - k^/ < 1$ such that $\gamma(r, 0, r+t) = k_1(2r+t) \leqslant kr$ and $\phi(r,t) = k_2(r+t) \geqslant k^/ r$. Let T and X be as defined in Example 1. Now, for $x,y,a \in X$

$$\|Tx-Ty,a\| \leqslant \gamma[\|x-y,a\|,\|y-Tx,a\|,\|x-Ty,a\|]-\phi[\|x-Tx,a\|,\|y-Ty,a\|]$$

Hence by Theorem 3, T has a fixed point on X.

Depending upon k_1 and k_2, the mapping T is of different types. From the given condition,

$$\begin{aligned}\|Tx-Ty,a\| &\leqslant \gamma[\|x-y,a\|,\|y-x,a\|+\|x-Tx,a\|,\|x-y,a\|+\|y-Ty,a\|]-\phi[\|x-Tx,a\|,\|y-Ty,a\|]\\ &\leqslant k_1[3\|x-y,a\|+\|x-Tx,a\|+\|y-Ty,a\|]-k_2[\|x-Tx,a\|+\|y-Ty,a\|].\end{aligned}$$

So, if $k_1=k_2$, then $\|Tx-Ty,a\| \leqslant 3k_1\|x-y,a\|$ which is a contraction mapping for $k_1<\frac{1}{3}$ (and has a unique fixed point) and nonexpansive for $k_1=\frac{1}{3}$.

Next, we discuss common fixed point for four mappings in 2-Banach spaces.

4 2-Norm for projective tensor product

4.1 Algebric tensor product

[2]. Let X,Y be normed spaces over F with dual spaces X^* and Y^* respectively. Given $x\in X, y\in Y$, let $x\otimes y$ be the element of $BL(X^*,Y^*;F)$ (which is the set of all bounded bilinear forms from $X^*\times Y^*$ to F), defined by

$$x\otimes y(f,g)=f(x)g(y),\ (f\in X^*, g\in Y^*)$$

The algebraic tensor product of X and Y, $X\otimes Y$ is defined to be the linear span of $\{x\otimes y : x\in X, y\in Y\}$ in $BL(X^*,Y^*;F)$.

4.2 Projective tensor product

[2]. Given normed spaces X and Y, the projective tensor norm γ on $X\otimes Y$ is defined by

$$\|u\|_\gamma=\inf\{\sum_i \|x_i\|\|y_i\| : u=\sum_i x_i\otimes y_i\}$$

where the infimum is taken over all (finite) representations of u. For the normed spaces X and Y, in the projective tensor product $X\otimes_\gamma Y$, we take

$$\|u,v\|=\|u\|\|v\|,\ u,v\in X\otimes_\gamma Y$$

Following White [29], we can say that $X\otimes_\gamma Y$ is a 2-Banach space upto linear dependence (i.e., $X\otimes_\gamma Y$ satisfies all the conditions for being a 2-Banach space except u and v may be linearly dependent and yet $\|u,v\|\neq 0$).

Let D_X, D_Y and $D_{X\otimes_\gamma Y}$ denote a closed and bounded subset of X, Y and $X\otimes_\gamma Y$ respectively. Let T_1 and T_2 be two pairs of mappings where $T_1 : D_{X\otimes_\gamma Y}\to D_X$ and $T_2 : D_{X\otimes_\gamma Y}\to D_Y$ be such that for any $u,v\in D_{X\otimes_\gamma Y}$ and $a\otimes b\in D_{X\otimes_\gamma Y}$ with $\|a\|\geqslant 1$ and $\|b\|\geqslant 1$.

(E) $\|T_1(u)-T_1(v)\| \leqslant \frac{1}{KM_2}(k\|u-v,a\otimes b\|-\psi(k\|u-v,a\otimes b\|))$

(F) $\|T_2(u)-T_2(v)\| \leqslant \frac{1}{KM_1}(k^/\|u-v,a\otimes b\|-\psi(k'\|u-v,a\otimes b\|))$

where

(i) $\psi : [0,\infty) \to [0,\infty)$ is continuous and non-decreasing, $\psi(0) = 0$
(ii) $\|T_1 u\| \leqslant M_1$ and $\|T_2 u\| \leqslant M_2$, $\forall u \in D_{X\otimes_\gamma Y}$.

Here, $D_{X\otimes_\gamma Y}$ is bounded by K and k, k' are positive. From the mappings T_1 and T_2 we define a mapping $T : D_{X\otimes_\gamma Y} \to D_{X\otimes_\gamma Y}$ such that $Tu = T_1 u \otimes T_2 u$.

Theorem 4. *The mapping T derived by the pair of mappings (T_1, T_2) satisfying (E) and (F) has a unique fixed point in $D_{X\otimes_\gamma Y}$ if $k + k^{/} \leqslant 1$.*

Proof. For $u, v \in D_{X\otimes_\gamma Y}$, $a \in X$ and $b \in Y$ and $a\otimes b \in D_{X\otimes_\gamma Y}$ with $\|a\| \geqslant 1$ and $\|b\| \geqslant 1$

$$\begin{aligned}
\|Tu - Tv, a\otimes b\| &= \|T_1 u \otimes T_2 u - T_1 v \otimes T_2 v, a\otimes b\| \\
&\leqslant \|(T_1 u - T_1 v)\otimes T_2 u, a\otimes b\| + \|T_1 v \otimes (T_2 u - T_2 v), a\otimes b\| \\
&= \|T_1 u - T_1 v\|\|T_2 u\|\|a\otimes b\| + \|T_1 v\|\|T_2 u - T_2 v\|\|a\otimes b\| \\
&\leqslant \frac{1}{KM_2}[k\|u - v, a\otimes b\| - \psi(k\|u - v, a\otimes b\|)].KM_2 \\
&+ \frac{1}{KM_1}[k^{/}\|u - v, a\otimes b\| - \psi(k^{/}\|u - v, a\otimes b\|)]KM_1 \\
&= (k + k^{/})\|u - v, a\otimes b\| - \psi(k\|u - v, a\otimes b\|) - \psi(k^{/}\|u - v, a\otimes b\|) \\
&\leqslant \|u - v, a\otimes b\| - \{\psi(k\|u - v, a\otimes b\|) + \psi(k^{/}\|u - v, a\otimes b\|)\}
\end{aligned}$$

Let $x_0 \in D_{X\otimes_\gamma Y}$ be fixed. We take $x_{n+1} = Tx_n$. Now,

$$\begin{aligned}
\|x_{n+1} - x_n, a\otimes b\| &= \|Tx_n - Tx_{n-1}, a\otimes b\| \\
&\leqslant \|x_n - x_{n-1}, a\otimes b\| - \psi(k\|x_n - x_{n-1}, a\otimes b\|) - \psi(k^{/}\|x_n - x_{n-1}, a\otimes b\|) \\
&\leqslant \|x_n - x_{n-1}, a\otimes b\|
\end{aligned}$$

Hence $\{\|x_{n+1} - x_n, a\otimes b\|\}$ is a monotonically decreasing sequence of non-negative real numbers and so, is convergent to some real, say r. Taking $n \to \infty$, we get

$$r \leqslant r - \{\psi(kr) + \psi(k^{/}r)\}, (\text{by continuity of } \psi). \text{ Then}, \;\; \psi(kr) + \psi(k^{/}r) \leqslant 0,$$

this is possible only when $r = 0$. So,

$$\begin{aligned}
&\lim_{n\to\infty} \|x_{n+1} - x_n, a\otimes b\| = 0 \\
&\Rightarrow \lim_{n\to\infty} \|x_{n+1} - x_n\|\|a\otimes b\| = 0 \;\; \Rightarrow \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0 \\
&\Rightarrow \lim_{n\to\infty} \|x_{n+1} - x_n, u\| = 0 \; \forall\, u \in D_{X\otimes_\gamma Y}
\end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence in the 2-Banach space $D_{X\otimes_\gamma Y}$. Let it converge to some $z \in D_{X\otimes_\gamma Y}$. Now,

$$\begin{aligned}
\|z - Tz, u\| &\leqslant \|z - x_{n+1}, u\| + \|x_{n+1} - Tz, u\| = \|z - x_{n+1}, u\| + \|Tx_n - Tz, u\| \\
&\leqslant \|z - x_{n+1}, u\| + \|x_n - z, u\| - [\psi(k\|x_n - z, u\|) + \psi(k^{/}\|x_n - z, u\|)] \\
&\to \; 0 \; as \; n \to \infty
\end{aligned}$$

Hence, $\|z - Tz, u\| = 0 \Rightarrow z = Tz$. To show the uniqueness. Let z_1 and z_2 be two distinct fixed points for T in $D_{X\otimes_\gamma Y}$. Now,

$$\begin{aligned}\|z_1 - z_2, u\| = \|Tz_1 - Tz_2, u\| &\leqslant \|z_1 - z_2, u\| - [\psi(k\|z_1 - z_2, u\|) + \psi(k^/\|z_1 - z_2, u\|)] \\ &\Rightarrow \psi(k\|z_1 - z_2, u\|) + \psi(k^/\|z_1 - z_2, u\|) \leqslant 0\end{aligned}$$

which is contradiction. So, $z_1 = z_2$. Thus, T has a unique fixed point in the closed and bounded subset $D_{X\otimes_\gamma Y}$ of $X \otimes_\gamma Y$.

Example 3. Let $D_{l^1\otimes_\gamma \mathbb{K}}$(with the same 2-norm as defined above in the tensor product space), D_{l^1} and $D_{\mathbb{K}}$ denote a closed and bounded subset of $l^1 \otimes_\gamma \mathbb{K}$, l^1 and $\mathbb{K}$, bounded by K, $\sqrt{K}$ and $\sqrt{K}$ respectively($K > 0$).

We define $T_1 : D_{l^1\otimes_\gamma \mathbb{K}} \to D_{l^1}$ by

$$T_1(\sum_i a_i \otimes x_i) = \frac{1}{2K^3}\sum_i \{a_{i_n} x_i\}, \text{ where } a_i = \{a_{i_n}\}_n$$

and $T_2 : D_{l^1\otimes_\gamma \mathbb{K}} \to D_{\mathbb{K}}$ by $T_2(\sum_i a_i \otimes x_i) = \frac{1}{4}\sum_i \|a_i\|.|x_i|$. For arbitrary $b_k = \{b_{k_n}\} \in D_{l^1}$, $b \in D_{\mathbb{K}}$ with $\|b_k\| \geqslant 1$ and $|b| \geqslant 1$,

$$\begin{aligned}\|T_1(\sum_i a_i \otimes x_i)\| = \|\frac{1}{2K^3}\sum_i \{a_{i_n} x_i\}\| &\leqslant \frac{1}{2K^3}\sum_i \|\{a_{i_n} x_i\}\|\|b_k\||b| \\ &\leqslant \frac{1}{2K^3}\|\sum_i a_i \otimes x_i\|\|b_k \otimes b\| \; [l^1 \otimes_\gamma X = l^1(X)(\text{refer to [26]})] \\ &\leqslant \frac{1}{2K^3}K^2 = \frac{1}{2K}(= M_1),\end{aligned}$$

and

$$\|T_2(\sum_i a_i \otimes x_i)\| \leqslant \frac{1}{4}\|\sum_i a_i \otimes x_i\|\|b_k \otimes b\| \leqslant \frac{K^2}{4}(= M_2)$$

For $u = \sum_i a_i \otimes x_i$ and $v = \sum_i d_i \otimes y_i$ in $D_{l^1\otimes_\gamma \mathbb{K}}$, we have,

$$\begin{aligned}\|T_1 u - T_1 v\| &= \|\frac{1}{2K^3}\sum_i \{a_{i_n} x_i\} - \frac{1}{2K^3}\sum_i \{d_{i_n} y_i\}\| \\ &= \frac{\frac{1}{4}\|\sum_i a_i \otimes x_i - \sum_i d_i \otimes y_i\|}{\frac{K^3}{2}} \leqslant \frac{\frac{1}{4}\|u - v\|\|b_k \otimes b\|}{\frac{K^3}{2}} \\ &\leqslant 2\left[\frac{\frac{1}{2}\|u - v, b_k \otimes b\| - \frac{1}{2}\left[\frac{1}{2}\|u - v, b_k \otimes b\|\right]}{K\frac{K^2}{2}}\right] \\ &= \frac{1}{KM_2}\left[\frac{1}{2}\|u - v, b_k \otimes b\| - \psi\left(\frac{1}{2}\|u - v, b_k \otimes b\|\right)\right]; \text{ where } \psi(t) = \frac{t}{2}, k = \frac{1}{2},\end{aligned}$$

and

$$\|T_2 u - T_2 v\| = \|\frac{1}{4}\sum_i \|a_i\|.|x_i| - \frac{1}{4}\sum_i \|d_i\|.|y_i|\| \leqslant \frac{1}{4}\left|\sum_i \|a_i\|.|x_i| - \sum_i \|d_i\|.|y_i|\right| \|b_k\| \, |b|.$$

Taking the projective tensor norm,

$$\begin{aligned}\|T_2u-T_2v\| &\leqslant \frac{1}{4}\,|\|u\|-\|v\||\,\|b_k\otimes b\| \leqslant \frac{1}{4}\|u-v\|\|b_k\otimes b\| = \frac{1}{4}\|u-v,b_k\otimes b\| \\ &\leqslant \frac{\frac{1}{2}\|u-v,b_k\otimes b\| - \frac{1}{2}\left[\frac{1}{2}\|u-v,b_k\otimes b\|\right]}{K\frac{1}{2K}} \\ &\leqslant \frac{\frac{1}{2}\|u-v,b_k\otimes b\| - \psi\left(\frac{1}{2}\|u-v,b_k\otimes b\|\right)}{KM_1};\text{ where } \psi(t)=\frac{t}{2}, k^{/}=\frac{1}{2}\end{aligned}$$

Therefore, (T_1,T_2) satisfies the conditions (a) and (b). Also, $k+k^{/}=\frac{1}{2}+\frac{1}{2}=1$. So, the mapping $T:D_{l^1\otimes_\gamma\mathbb{K}}\to D_{l^1\otimes_\gamma\mathbb{K}}$ has a unique fixed point in $D_{l^1\otimes_\gamma\mathbb{K}}$.

Let T_1,S_1,P_1,T_2,S_2 and P_2 be some mappings where $T_1,S_1,P_1:D_{X\otimes_\gamma Y}\to D_X$ and $T_2,S_2,P_2:D_{X\otimes_\gamma Y}\to D_Y$ be two mappings such that for any $u,v\in D_{X\otimes_\gamma Y}$ and $a\otimes b\in X\otimes_\gamma Y$,

(G) $\|T_1(u)-S_1(v)\| \leqslant \frac{1}{MM_2}(k(\|Pu-Tv,a\otimes b\|+\|Pu-Sv,a\otimes b\|)-\psi(k\|Pu-Tv,a\otimes b\|,k\|Pu-Sv,a\otimes b\|))$

(H) $\|T_2(u)-S_2(v)\| \leqslant \frac{1}{MM_1}(k^{/}(\|Pu-Tv,a\otimes b\|+\|Pu-Sv,a\otimes b\|)-\psi(k^{/}\|Pu-Tv,a\otimes b\|,k^{/}\|Pu-Sv,a\otimes b\|))$

where

(i) $\psi:[0,\infty)\to[0,\infty)$ is continuous and non-decreasing, $\psi(0)=0$
(ii) $\max[\|T_1u\|,\|S_1v\|]\leqslant M_1$ and $\max[\|T_2u\|,\|S_2v\|]\leqslant M_2$, $\forall u,v\in D_{X\otimes_\gamma Y}$, $a\in X$ and $b\in Y$. Here, $D_{X\otimes_\gamma Y}$ is bounded by M and $k,k^{/}$ are positive.

From the mappings T_1,S_1,P_1,T_2,S_2 *and* P_2 we define some mappings $T:D_{X\otimes_\gamma Y}\to D_{X\otimes_\gamma Y}$ such that $Tu=T_1u\otimes T_2u$; $S:D_{X\otimes_\gamma Y}\to D_{X\otimes_\gamma Y}$ such that $Su=S_1u\otimes S_2u$ and $P:D_{X\otimes_\gamma Y}\to D_{X\otimes_\gamma Y}$ such that $Pu=P_1u\otimes P_2u$.

Theorem 5. *Let T,S and P be self mappings as defined above such that*

(i) *$\{T,P\}$ and $\{S,P\}$ are wealy compatible*
(ii) *$T(X\otimes_\gamma Y)\subseteq P(X\otimes_\gamma Y)$ and $S(X\otimes_\gamma Y)\subseteq P(X\otimes_\gamma Y)$*
(iii) *satisfy (G) and (H), then T,S and P have a common unique fixed point on $D_{X\otimes_\gamma Y}$ if $k+k^{/}\leqslant\frac{1}{4}$*

Proof. Let $x_0\in D_{X\otimes_\gamma Y}$ be fixed. We define

$$y_n=Tx_n=Px_{n+1},\ y_{n+1}=Sx_{n+1}=Px_{n+2}$$

Now, for any $a\otimes b\in D_{X\otimes_\gamma Y}$,

$$\begin{aligned}\|Tu-Sv,a\otimes b\| &\leqslant \|T_1u-S_1v\|\|T_2u\|\|a\otimes b\|+\|S_1v\|\|T_2u-S_2v\|\|a\otimes b\| \\ &\leqslant \frac{1}{4}(\|Pu-Tv,a\otimes b\|+\|Pu-Sv,a\otimes b\|)-\psi(k^{/}\|Pu-Tv,a\otimes b\|,k^{/}\|Pu-Sv,a\otimes b\|).\end{aligned}$$

$$
\begin{aligned}
\|y_n - y_{n+1}, a\otimes b\| &= \|Tx_n - Sx_{n+1}, a\otimes b\| \\
&\leqslant \frac{1}{4}(\|Px_n - Tx_{n+1}, a\otimes b\| + \|Px_n - Sx_{n+1}, a\otimes b\|) \\
&- \psi(k\|Px_n - Tx_{n+1}, a\otimes b\|, k\|Px_n - Sx_{n+1}, a\otimes b\|) \\
&- \psi(k^/\|Px_n - Tx_{n+1}, a\otimes b\|, k^/\|Px_n - Sx_{n+1}, a\otimes b\|) \\
&= \frac{1}{4}(\|y_{n-1} - y_{n+1}, a\otimes b\| + \|y_{n-1} - y_{n+1}, a\otimes b\|) \\
&- \psi(k\|y_{n-1} - y_{n+1}, a\otimes b\|, k\|y_{n-1} - y_{n+1}, a\otimes b\|) \\
&- \psi(k^/\|y_{n-1} - y_{n+1}, a\otimes b\|, k^/\|y_{n-1} - y_{n+1}, a\otimes b\|) \\
&\leqslant \frac{1}{2}(\|y_{n-1} - y_n, a\otimes b\| + \|y_n - y_{n+1}, a\otimes b\|) \\
&- \psi(k\|y_{n-1} - y_{n+1}, a\otimes b\|, k\|y_{n-1} - y_{n+1}, a\otimes b\|) \\
&- \psi(k^/\|y_{n-1} - y_{n+1}, a\otimes b\|, k^/\|y_{n-1} - y_{n+1}, a\otimes b\|) \\
&= \|y_{n-1} - y_n, a\otimes b\| - 2\psi(k\|y_{n-1} - y_{n+1}, a\otimes b\|, k\|y_{n-1} - y_{n+1}, a\otimes b\|) \\
&- 2\psi(k^/\|y_{n-1} - y_{n+1}, a\otimes b\|, k^/\|y_{n-1} - y_{n+1}, a\otimes b\|) \\
&\leqslant \|y_{n-1} - y_n, a\otimes b\|.
\end{aligned}
$$

Hence $\{\|y_{n+1} - y_n, a\otimes b\|\}$ is a monotonically decreasing sequence of non-negative real numbers and so, is convergent to some real, say r. If $r \neq 0$, then

$$
\begin{aligned}
\|y_n - y_{n+1}, a\otimes b\| &= \|Tx_n - Sx_{n+1}, a\otimes b\| \\
&\leqslant \frac{1}{4}(\|y_{n-1} - y_{n+1}, a\otimes b\| + \|y_{n-1} - y_{n+1}, a\otimes b\|) \\
&- \psi(k\|y_{n-1} - y_{n+1}, a\otimes b\|, k\|y_{n-1} - y_{n+1}, a\otimes b\|) \\
&- \psi(k^/\|y_{n-1} - y_{n+1}, a\otimes b\|, k^/\|y_{n-1} - y_{n+1}, a\otimes b\|) \\
&\leqslant \frac{1}{2}(\|y_{n-1} - y_{n+1}, a\otimes b\|) \leqslant \frac{1}{2}(\|y_{n-1} - y_n, a\otimes b\| + \|y_n - y_{n+1}, a\otimes b\|)
\end{aligned}
$$

Taking $n \to \infty$, we get $\|y_{n-1} - y_{n+1}, a\otimes b\| \to 2r$ and

$$r \leqslant r - 2\{\psi(2kr, 2kr) + \psi(2k^/r, 2k^/r)\}, \text{(by continuity of } \psi), \quad \text{therefore } 2\{\psi(2kr, 2kr) + \psi(2k^/r, 2k^/r)\} \leqslant 0,$$

this is possible only when $r = 0$. So,

$$\lim_{n\to\infty} \|y_{n+1} - y_n, a\otimes b\| = 0. \tag{1}$$

Now, proceeding as in Theorem 4.3 we have $\lim_{n\to\infty} \|y_{n+1} - y_n, q\| = 0 \ \forall q \in D_{X\otimes_\gamma Y}$ and $\{y_n\}$ is a Cauchy sequence in $D_{X\otimes_\gamma Y}$. Let it converge to some $z \in D_{X\otimes_\gamma Y}$ i.e.,

$$
\begin{aligned}
\lim_{n\to\infty} y_n = z \Rightarrow \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Px_{n+1} = z \text{ and} \\
\lim_{n\to\infty} Sx_{n+1} = \lim_{n\to\infty} Px_{n+2} = z.
\end{aligned}
$$

Since $S(X) \subseteq P(X)$ and $T(X) \subseteq P(X)$, so there exists a point $u \in D_{X\otimes_\gamma Y}$ such that $z = Pu$. Now,

$$\begin{aligned}
\|Tu - z, q\| &\leqslant \|Tu - Sx_{n+1}, q\| + \|Sx_{n+1} - z, q\| \\
&\leqslant \frac{1}{4}(\|Pu - Tx_{n+1}, q\| + \|Pu - Sx_{n+1}, q\|) \\
&- \psi(k\|Pu - Tx_{n+1}, q\|, k\|Pu - Sx_{n+1}, q\|) \\
&- \psi(k^{/}\|Pu - Tx_{n+1}, q\|, k^{/}\|Pu - Sx_{n+1}, q\|) + \|Sx_{n+1} - z, q\|.
\end{aligned}$$

Taking $n \to \infty$,

$$\|Tu - z, q\| \leqslant 0 \Rightarrow \|Tu - z, q\| = 0$$

Therefore $Tu = z$. So, $Pu = Tu = z$, i.e., u is a coincidence point of P and T. Since the pair of mappings are weakly compatiable, so,

$$PTu = TPu \Rightarrow Pz = Tz.$$

Again for $z = Pu$ we have,

$$\begin{aligned}
\|z - Su, q\| &= \|Tu - Su, q\| \\
&\leqslant \frac{1}{4}(\|Pu - Tu, q\| + \|Pu - Su, q\|) \\
&- \psi(k\|Pu - Tu, q\|, k\|Pu - Su, q\|) \\
&- \psi(k^{/}\|Pu - Tu, q\|, k^{/}\|Pu - Su, q\|) = 0.
\end{aligned}$$

Thus, $\|z - Su, q\| = 0$. So, $Su = z$. Thus $Pu = Su = z$, i.e., w is a coincidence point of P and S. Since the pair of mappings are weakly compatiable, so,

$$PSu = SPu \Rightarrow Pz = Sz$$

Now, we show that z is a fixed point of T

$$\begin{aligned}
\|Tz - z, q\| &= \|Tz - Su, q\| \\
&\leqslant \frac{1}{4}(\|Pz - Tu, q\| + \|Pz - Su, q\|) \\
&- \psi(k\|Pz - Tu, q\|, k\|Pz - Su, q\|) \\
&- \psi(k^{/}\|Pz - Tu, q\|, k^{/}\|Pz - Su, q\|) \\
&= \frac{1}{2}\|Tz - z, q\|
\end{aligned}$$

possible only for $\|Tz - z, q\| = 0 \Rightarrow Tz = z$ therefore $Tz = Pz = z$. Now, we show that z is a fixed point of S

$$\begin{aligned}
\|z - Sz, q\| &= \|Tz - Sz, q\| \\
&\leqslant \frac{1}{4}(\|Pz - Tz, q\| + \|Pz - Sz, q\|) \\
&\quad - \psi(k\|Pz - Tz, q\|, k\|Pz - Sz, q\|) \\
&\quad - \psi(k^{/}\|Pz - Tz, q\|, k^{/}\|Pz - Sz, q\|) = 0 \\
\Rightarrow \|Sz - z, q\| = 0
\end{aligned}$$

possible only for $Sz = z$ therefore $Sz = Pz = z$. Hence, $Tz = Pz = z = Sz$. Uniqueness can be shown in a similar manner. Thus z is a common unique fixed point for the mappings T,S and P.

5 Conclusion

Thus, in this paper, we have derived different fixed point theorems in 2-Banach spaces and also in the tensor product of normed spaces as 2-Banach spaces.

In the paper of Misiak [17], in 1989, the idea of n-normed spaces can be found. Some recent results and related works in n-normed spaces can be found in [13], [16]. Let $n \in \mathbb{N}$ and X be a real vector space of dimension $d \geqslant n$. A real-valued function $\|.,\dots,.\|$ on X^n satisfying the following four properties,

(1) $\|x_1,\dots,x_n\| = 0$ if and only if $x_1,\dots,x_n$ are linearly dependent;

(2) $\|x_1,\dots,x_n\|$ is invariant under permutation;

(3) $\|x_1,\dots,x_{n-1},\alpha x_n\| = |\alpha|\|x_1,\dots,x_{n-1},x_n\|$ for any $\alpha \in \mathbb{R}$;

(4) $\|x_1,\dots,x_{n-1},y+z\| \leqslant \|x_1,\dots,x_{n-1},y\| + \|x_1,\dots,x_{n-1},z\|$,
is called an n-norm on X and the pair $(X, \|.,\dots,.\|)$ is called an n-normed space. Considering the study of fixed points, the following problem can be raised.

Can we make analogous study concerning fixed points for a mapping T in the n-normed spaces and their tensor product?

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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