

SCALAR MULTIPLICATION AND RELATED MEASURES FOR INTERVAL-VALUED INTUITIONISTIC FUZZY MATRICES

GÖKHAN ÇUVALCIOĞLU AND GİZEM ARTUN EFE

0000-0001-5451-3336 and 0000-0001-9957-3252

ABSTRACT. In this paper, the Hamming measure, Generalized Hamming measure, Euclidean measure, and Generalized Euclidean measure are introduced for Interval-valued intuitionistic fuzzy matrices. Their fundamental properties are established. Furthermore, a scalar multiplication operation by a scalar (λ, μ) is introduced for interval-valued intuitionistic fuzzy matrices and its algebraic properties, including the distributive laws are established. Several illustrative examples are presented to demonstrate the validity of the proposed theoretical results.

1. INTRODUCTION

The concept of fuzzy sets introduced by Zadeh [7] provides a powerful mathematical framework for modeling uncertainty by allowing each element to possess a degree of membership in the unit interval $[0, 1]$. Since its introduction, fuzzy set theory has been successfully applied in a wide range of areas, including decision making, pattern recognition, control systems, image processing and optimization.

To extend the descriptive capability of fuzzy sets, Atanassov [1] introduced intuitionistic fuzzy sets (IFSs) which characterize each element by both membership and non-membership degrees. Subsequently, Atanassov and Gargov [3] proposed interval-valued intuitionistic fuzzy sets (IVIFSs), where the membership and non-membership degrees are represented by intervals instead of precise numerical values. This extension provides a more flexible framework for modeling uncertainty and incomplete information arising in practical applications.

2. PRELIMINARIES

Definition 2.1 ([1]). Let X be a universal set and let $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ be the membership and non-membership functions, respectively. Then,

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}$$

Date: Received: 2026-10-11; Accepted: 2026-10-18.

2010 Mathematics Subject Classification. 03E72, 15B15.

Key words and phrases. Intuitionistic fuzzy set, interval valued intuitionistic fuzzy set, interval valued intuitionistic fuzzy matrix, interval valued (λ, μ) scalar matrix.

is called an intuitionistic fuzzy set (IFS) on X . The family of all intuitionistic fuzzy sets on X is denoted by $\text{IFS}(X)$. Here $\mu_A : X \rightarrow [0, 1]$ is called the degree of membership of x in A , and $\nu_A : X \rightarrow [0, 1]$ is called the degree of non-membership of x in A . Moreover, $\mu_A(x)$ and $\nu_A(x)$ satisfy the following condition:

$$\forall x \in X, \quad \mu_A(x) + \nu_A(x) \leq 1.$$

Remark 2.1 ([6]). Let $I = [0, 1]$ and let $D(I)$ denote the set of all closed subintervals of I . Throughout this paper, the elements of $D(I)$ are denoted by capital letters such as M, N .

For an interval $M = [M^L, M^U]$ the values M^L and M^U are called the lower and upper endpoints of M , respectively. In particular $0 = [0, 0]$, $1 = [1, 1]$ and $\forall a \in [0, 1]$, $a = [a, a]$.

Definition 2.2 ([3]). Let X be a universal set. Suppose that $M_A, N_A \in D(I)$ satisfy $M_A^U(x) + N_A^U(x) \leq 1$. Then,

$$A = \{ \langle x, M_A(x), N_A(x) \rangle \mid x \in X \}$$

is called an interval-valued intuitionistic fuzzy set (IVIFS) on X . The family of all interval-valued intuitionistic fuzzy sets on X is denoted by $\text{IVIFS}(X)$.

Definition 2.3 ([2]). Let X be a universal set and $A, B \in \text{IVIFS}(X)$.

$$\begin{aligned} \text{(i)} \quad A^c &= \{ \langle x, N_A(x), M_A(x) \rangle \mid x \in X \}, \\ \text{(ii)} \quad A \sqcap B &= \left\{ \left\langle x, \begin{bmatrix} \min \{ M_A^L(x), M_B^L(x) \}, \\ \min \{ M_A^U(x), M_B^U(x) \}, \\ \max \{ N_A^L(x), N_B^L(x) \}, \\ \max \{ N_A^U(x), N_B^U(x) \} \end{bmatrix} \right\rangle \mid x \in X \right\}, \\ \text{(iii)} \quad A \sqcup B &= \left\{ \left\langle x, \begin{bmatrix} \max \{ M_A^L(x), M_B^L(x) \}, \\ \max \{ M_A^U(x), M_B^U(x) \}, \\ \min \{ N_A^L(x), N_B^L(x) \}, \\ \min \{ N_A^U(x), N_B^U(x) \} \end{bmatrix} \right\rangle \mid x \in X \right\}, \\ \text{(iv)} \quad A + B &= \left\{ \left\langle x, \begin{bmatrix} M_A^L(x) + M_B^L(x) - M_A^L(x) \cdot M_B^L(x), \\ M_A^U(x) + M_B^U(x) - M_A^U(x) \cdot M_B^U(x) \end{bmatrix}, \right. \right. \\ &\quad \left. \left. [N_A^L(x) \cdot N_B^L(x), N_A^U(x) \cdot N_B^U(x)] \right\rangle \mid x \in X \right\}, \\ \text{(v)} \quad A \cdot B &= \left\{ \left\langle x, \begin{bmatrix} M_A^L(x) \cdot M_B^L(x), M_A^U(x) \cdot M_B^U(x) \end{bmatrix}, \right. \right. \\ &\quad \left. \left. \begin{bmatrix} N_A^L(x) + N_B^L(x) - N_A^L(x) \cdot N_B^L(x), \\ N_A^U(x) + N_B^U(x) - N_A^U(x) \cdot N_B^U(x) \end{bmatrix} \right\rangle \mid x \in X \right\}. \end{aligned}$$

Definition 2.4 ([6]). Let $X = \{x_1, x_2, \dots, x_n\}$ be a universal set, $\tilde{\mu}_{\tilde{A}}(x_i) \subseteq [0, 1]$ and $\tilde{\nu}_{\tilde{A}}(x_i) \subseteq [0, 1]$, $x_i \in X$ such that $\inf \tilde{\mu}_{\tilde{A}}(x) = \sup \tilde{\mu}_{\tilde{A}}(x)$ and $\inf \tilde{\nu}_{\tilde{A}}(x) = \sup \tilde{\nu}_{\tilde{A}}(x)$ with $\sup \tilde{\mu}_{\tilde{A}}(x_i) + \sup \tilde{\nu}_{\tilde{A}}(x_i) \leq 1$. Then, an interval-valued intuitionistic fuzzy set $\tilde{A}$ is an IFS defined as:

$$\tilde{A} = \{ \langle x_i, \tilde{\mu}_{\tilde{A}}(x_i), \tilde{\nu}_{\tilde{A}}(x_i) \rangle \mid x_i \in X \}.$$

The following relations and operators are defined for two IVIFSs, $\tilde{A}_1$ and $\tilde{A}_2$:

$$(1) \quad \tilde{A}_1 \subseteq \tilde{A}_2 \Leftrightarrow \forall x \in X,$$

$$\left\{ \sup \tilde{\mu}_{\tilde{A}_1}(x) \leq \sup \tilde{\mu}_{\tilde{A}_2}(x) \text{ and } \inf \tilde{\mu}_{\tilde{A}_1}(x) \leq \inf \tilde{\mu}_{\tilde{A}_2}(x) \right\}$$

and

$$\left\{ \sup \tilde{\nu}_{\tilde{A}_1}(x) \geq \sup \tilde{\nu}_{\tilde{A}_2}(x) \text{ and } \inf \tilde{\nu}_{\tilde{A}_1}(x) \geq \inf \tilde{\nu}_{\tilde{A}_2}(x) \right\}.$$

$$(2) \quad \tilde{A}_1 = \tilde{A}_2 \Leftrightarrow \tilde{A}_1 \subseteq \tilde{A}_2 \text{ and } \tilde{A}_1 \supseteq \tilde{A}_2.$$

(3)

$$\begin{aligned} \tilde{A}_1 \cap \tilde{A}_2 = \left\{ \left\langle x, \left[\min \left\{ \inf \tilde{\mu}_{\tilde{A}_1}(x), \inf \tilde{\mu}_{\tilde{A}_2}(x) \right\}, \min \left\{ \sup \tilde{\mu}_{\tilde{A}_1}(x), \sup \tilde{\mu}_{\tilde{A}_2}(x) \right\} \right] \right. \right. \\ \left. \left. \left[\max \left\{ \inf \tilde{\nu}_{\tilde{A}_1}(x), \inf \tilde{\nu}_{\tilde{A}_2}(x) \right\}, \max \left\{ \sup \tilde{\nu}_{\tilde{A}_1}(x), \sup \tilde{\nu}_{\tilde{A}_2}(x) \right\} \right] \right\rangle \mid x \in X \right\}. \end{aligned}$$

(4)

$$\begin{aligned} \tilde{A}_1 \cup \tilde{A}_2 = \left\{ \left\langle x, \left[\max \left\{ \inf \tilde{\mu}_{\tilde{A}_1}(x), \inf \tilde{\mu}_{\tilde{A}_2}(x) \right\}, \max \left\{ \sup \tilde{\mu}_{\tilde{A}_1}(x), \sup \tilde{\mu}_{\tilde{A}_2}(x) \right\} \right] \right. \right. \\ \left. \left. \left[\min \left\{ \inf \tilde{\nu}_{\tilde{A}_1}(x), \inf \tilde{\nu}_{\tilde{A}_2}(x) \right\}, \min \left\{ \sup \tilde{\nu}_{\tilde{A}_1}(x), \sup \tilde{\nu}_{\tilde{A}_2}(x) \right\} \right] \right\rangle \mid x \in X \right\}. \end{aligned}$$

Definition 2.5 ([6]). Let $\tilde{P}_l = (\tilde{p}_{ij}^{(l)})_{n \times n}$ ($l = 1, 2$), that is, $\tilde{P}_1$ and $\tilde{P}_2$, be two interval-valued intuitionistic fuzzy matrices. The composition of these matrices is defined as $\tilde{P} = \tilde{P}_1 \circ \tilde{P}_2 = (\tilde{p}_{ij})_{n \times n}$, where for $i, j = 1, 2, \dots, n$ and $l = 1, 2$

$$\tilde{p}_{ij}^{(l)} = \left(\left[a_{ij}^{(l)}, b_{ij}^{(l)} \right], \left[c_{ij}^{(l)}, d_{ij}^{(l)} \right] \right),$$

$$\begin{aligned} \tilde{p}_{ij} = \bigcup_{k=1}^n \left(\tilde{p}_{ik}^{(1)} \cap \tilde{q}_{kj}^{(2)} \right) = \left(\left[\max_k \left\{ \min \left\{ a_{ik}^{(1)}, a_{kj}^{(2)} \right\} \right\}, \max_k \left\{ \min \left\{ b_{ik}^{(1)}, b_{kj}^{(2)} \right\} \right\} \right] \right. \\ \left. \left[\min_k \left\{ \max \left\{ c_{ik}^{(1)}, c_{kj}^{(2)} \right\} \right\}, \min_k \left\{ \max \left\{ d_{ik}^{(1)}, d_{kj}^{(2)} \right\} \right\} \right] \right). \end{aligned}$$

Theorem 2.1 ([6]). Let $\tilde{P}_1$ and $\tilde{P}_2$ be two interval-valued intuitionistic fuzzy matrices. Then, the matrix $\tilde{P} = \tilde{P}_1 \circ \tilde{P}_2$ is also an interval-valued intuitionistic fuzzy matrix.

Definition 2.6 ([6]). Let $\tilde{P} = (\tilde{p}_{ij})_{n \times n}$ be an $n \times n$ matrix, where $\tilde{p}_{ij} = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])$ for $i, j = 1, 2, \dots, n$. If the following conditions are satisfied, $\tilde{P}$ is called an interval-valued intuitionistic fuzzy similarity matrix and is denoted by IVIFSM.

- (1) **Reflexivity:** $\tilde{p}_{ij} = ([1, 1], [0, 0])$, $i = 1, 2, \dots, n$.
- (2) **Symmetry:** $\tilde{p}_{ij} = \tilde{p}_{ji}$, which implies that $[a_{ij}, b_{ij}] = [a_{ji}, b_{ji}]$ and $[c_{ij}, d_{ij}] = [c_{ji}, d_{ji}]$ for $i, j = 1, 2, \dots, n$.

Theorem 2.2 ([6]). Let $\tilde{P} = (\tilde{p}_{ij})_{n \times n}$ be an interval-valued intuitionistic fuzzy similarity matrix. Then, the matrix $\tilde{P}^2 = \tilde{P} \circ \tilde{P} = (\tilde{p}_{ij}^2)_{n \times n}$ is also an interval-valued intuitionistic fuzzy similarity matrix.

Definition 2.7 ([6]). Let $\tilde{A}$ be an IVIFS. The pair $(\tilde{\mu}_{\tilde{A}}, \tilde{\nu}_{\tilde{A}})$ is called an interval-valued intuitionistic fuzzy number. The family of all interval-valued intuitionistic fuzzy numbers is denoted by IVIFN. An interval-valued intuitionistic fuzzy number is of the form $\tilde{\alpha} = ([a, b], [c, d])$, where $[a, b] \subseteq [0, 1]$, $[c, d] \subseteq [0, 1]$, $b + d \leq 1$.

From the definition, $\tilde{\alpha}^* = ([1, 1], [0, 0])$ is the largest IVIFN. For $a = b$ and $c = d$, $\tilde{\alpha}$ is an intuitionistic fuzzy number.

Definition 2.8 ([6]). Let $\tilde{P} = (\tilde{p}_{ij})_{n \times n}$ be an $n \times n$ matrix. If $\forall 1 \leq i, j \leq n$, $\tilde{p}_{ij}$ is an IVIFN and

$$\tilde{p}_{ij} = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])$$

where $[a_{ij}, b_{ij}] \subseteq [0, 1]$, $[c_{ij}, d_{ij}] \subseteq [0, 1]$ and $b_{ij} + d_{ij} \leq 1$ for $i, j = 1, 2, \dots, n$, then $\tilde{P}$ is called an interval-valued intuitionistic fuzzy matrix. The family of all interval-valued intuitionistic fuzzy matrices is denoted by IVIFM.

Definition 2.9 ([6]). Let $\tilde{P} = (\tilde{p}_{ij})_{n \times n}$ be an $n \times n$ interval-valued intuitionistic fuzzy matrix. The matrix $\tilde{p}_{ij}^T = \tilde{p}_{ji}$ for $i, j = 1, 2, \dots, n$ is called the transpose of the matrix.

Definition 2.10 ([5]). Let X be a universal set and

$$A = \{\langle x_i, u_A(x_i), v_A(x_i) \rangle \mid x_i \in X\}.$$

Two types of cardinality for set A are defined as follows:

- (1) $\min \sum \text{count}(A) = \sum_{i=1}^n u_A(x_i)$,
- (2) $\max \sum \text{count}(A) = \sum_{i=1}^n (u_A(x_i) + \pi_A(x_i))$,

with

$$A_i \cap A_i^c = \{\langle x_i, \min \{u_A(x_i), v_A(x_i)\}, \max \{v_A(x_i), u_A(x_i)\} \rangle\}$$

and

$$A_i \cup A_i^c = \{\langle x_i, \max \{u_A(x_i), v_A(x_i)\}, \min \{v_A(x_i), u_A(x_i)\} \rangle\}.$$

Using these two cardinalities, the entropy measure given by Szmidt is as follows:

$$E_{SK}(A) = \frac{1}{n} \sum_{i=1}^n \frac{\max \sum \text{count}(A_i \cap A_i^c)}{\max \sum \text{count}(A_i \cup A_i^c)}.$$

Remark 2.2. Here, the minimum cardinality of the element $A_i \cap A_i^c$ is denoted as $\min \text{count}(A_i \cap A_i^c)$, and the maximum cardinality of the element $A_i \cup A_i^c$ is denoted as $\max \text{count}(A_i \cup A_i^c)$.

The entropy measure given by Wang [4] is as follows:

$$E_{WL}(A) = \frac{1}{n} \sum_{i=1}^n \frac{\min \{u_A(x_i), v_A(x_i)\} + \pi_A(x_i)}{\max \{v_A(x_i), u_A(x_i)\} + \pi_A(x_i)}.$$

The entropy measure given by Huang [5] is as follows:

$$E_{HL}(A) = \frac{1}{n} \sum_{i=1}^n \frac{1 - |u_A(x_i) - v_A(x_i)| + \pi_A(x_i)}{1 + |u_A(x_i) - v_A(x_i)| + \pi_A(x_i)}.$$

Theorem 2.3 ([5]). For every IFS $A = \{\langle x, u_A(x), v_A(x) \rangle \mid x \in X\}$, $E_{SK}(A) = E_{WL}(A) = E_{HL}(A)$ holds.

Definition 2.11 ([5]). Let X be a universal set, $x_i \in X$ and $E : \text{IVIFS}(X) \rightarrow [0, 1]$. $E(A)$ is called the entropy measure of an IVIFS A if the following conditions are satisfied:

- (E1) $E(A) = 0 \Leftrightarrow A$ is a crisp set.
- (E2) $E(A) = 1 \Leftrightarrow [u_A^-(x_i), u_A^+(x_i)] = [v_A^-(x_i), v_A^+(x_i)]$, $x_i \in X$.
- (E3) $E(A) = E(A^c)$.

- (E4) $E(A) \leq E(B) \Leftrightarrow$ Under the conditions $u_B^-(x_i) \leq v_B^-(x_i)$ and $u_B^+(x_i) \leq v_B^+(x_i)$ for $x_i \in X$, if $A \subseteq B$, then $E(A) \subseteq E(B)$
or
 $E(A) \leq E(B) \Leftrightarrow$ Under the conditions $u_B^-(x_i) \geq v_B^-(x_i)$ and $u_B^+(x_i) \geq v_B^+(x_i)$ for $x_i \in X$, if $A \subseteq B$, then $E(A) \subseteq E(B)$.

Remark 2.3 ([5]). The conditions in (E4) may not hold for every IVIFS.

Definition 2.12 ([5]). Let X be a universal set, $x_i \in X$, $A \in \text{IVIFS}(X)$ and $E : \text{IVIFS}(X) \rightarrow [0, 1]$. For

$$\begin{aligned} E(A) &= \frac{1}{n} \sum_{i=1}^n \frac{\min \{u_A^-(x_i), v_A^-(x_i)\} + \min \{u_A^+(x_i), v_A^+(x_i)\} + \pi_A^-(x_i) + \pi_A^+(x_i)}{\max \{u_A^-(x_i), v_A^-(x_i)\} + \max \{u_A^+(x_i), v_A^+(x_i)\} + \pi_A^-(x_i) + \pi_A^+(x_i)} \\ &= \frac{1}{n} \sum_{i=1}^n \frac{2 - |u_A^-(x_i) - v_A^-(x_i)| - |u_A^+(x_i) - v_A^+(x_i)| + \pi_A^-(x_i) + \pi_A^+(x_i)}{2 + |u_A^-(x_i) - v_A^-(x_i)| + |u_A^+(x_i) - v_A^+(x_i)| + \pi_A^-(x_i) + \pi_A^+(x_i)} \end{aligned}$$

the value is called the entropy measure of set A .

Definition 2.13 ([8]). Let X be a universal set, $x_i \in X$ and $E : \text{IVIFS}(X) \rightarrow [0, 1]$. $E(A)$ is called the entropy measure of an IVIFS A if the following conditions are satisfied:

- (ZP₁) If A is a crisp set, then $E(A) = 0$.
(ZP₂) If $M_A(x_i) = N_A(x_i)$ for $\forall x_i \in X$, then $E(A) = 1$.
(ZP₃) If the fuzzy value of set A is smaller than that of set B , then $E(A) \leq E(B)$:
For $M_B(x_i) \leq N_B(x_i)$; $M_A(x_i) \leq M_B(x_i)$, $N_A(x_i) \geq N_B(x_i)$,
For $M_B(x_i) \geq N_B(x_i)$; $M_A(x_i) \geq M_B(x_i)$, $N_A(x_i) \leq N_B(x_i)$,
For $M_B(x_i) \preceq N_B(x_i)$; $M_A(x_i) \preceq M_B(x_i)$, $N_A(x_i) \succeq N_B(x_i)$,
For $M_B(x_i) \succeq N_B(x_i)$; $M_A(x_i) \succeq M_B(x_i)$, $N_A(x_i) \preceq N_B(x_i)$.
(ZP₄) $E(A) = E(A^c)$.

Definition 2.14 ([8]). Let X be a universal set, $x_i \in X$ and $d : \text{IVIFS}(X) \times \text{IVIFS}(X) \rightarrow [0, 1]$. $d(A)$ is called the length measure of an IVIFS A if the following conditions are satisfied:

- (1) A is a crisp set $\Leftrightarrow d(A, A') = 1$, where A' denotes the complement of A .
(2) $d(A, B) = 0 \Leftrightarrow A = B$.
(3) $d(A, B) = d(B, A)$.
(4) If $A \subseteq B \subseteq C$ then $d(A, C) \geq d(A, B)$ and $d(A, C) \geq d(B, C)$.

Definition 2.15 ([8]). Let X be a universal set, $x_i \in X$ and $E : \text{IVIFS}(X) \rightarrow [0, 1]$. $E(A)$ is called the entropy measure of an IVIFS A if the following conditions are satisfied:

- (EP₁) $E(A) = 0$ for $\forall A \in P(X)$.
(EP₂) For $\forall x_i \in X$, if $A = ([\frac{1}{2}, \frac{1}{2}], [\frac{1}{2}, \frac{1}{2}])_x$ then $M_A(x_i) = N_A(x_i) = [\frac{1}{2}, \frac{1}{2}]$, so $E(A) = 1$.
(EP₃) For $\forall A, B \in \text{IVIFS}(X)$, where d is the length measure of the interval-valued intuitionistic fuzzy set: if

$$d(A, \langle [\frac{1}{2}, \frac{1}{2}], [\frac{1}{2}, \frac{1}{2}] \rangle_x) \geq d(B, \langle [\frac{1}{2}, \frac{1}{2}], [\frac{1}{2}, \frac{1}{2}] \rangle_x)$$

then $E(A) \leq E(B)$.

- (EP₄) $E(A) = E(A^c)$.

- Remark 2.4* ([8]). (EP_1) The interval-valued intuitionistic fuzzy set $\langle [\frac{1}{2}, \frac{1}{2}], [\frac{1}{2}, \frac{1}{2}] \rangle_x$ has the largest fuzzy value.
- (EP_2) The set closer to the interval-valued intuitionistic fuzzy set $\langle [\frac{1}{2}, \frac{1}{2}], [\frac{1}{2}, \frac{1}{2}] \rangle_x$ has a larger fuzzy value.
- (EP_3) An interval-valued intuitionistic fuzzy set has the same fuzzy value as its complement.

Definition 2.16 ([8]). Let $X = \{x_1, x_2, \dots, x_n\}$ be a universal set, $\forall A, B \in \text{IVIFS}(X)$ and d_i be the length measure of the interval-valued intuitionistic fuzzy set. The following length measures are defined for $i = 1, 2, \dots, 6$:

(1) Normalized Euclidean length

$$d_1(A, B) = \left\{ \frac{1}{4n} \sum_{i=1}^n \left[(M_{AL}(x_i) - M_{BL}(x_i))^2 + (M_{AU}(x_i) - M_{BU}(x_i))^2 + (N_{AL}(x_i) - N_{BL}(x_i))^2 + (N_{AU}(x_i) - N_{BU}(x_i))^2 \right] \right\}^{\frac{1}{2}}$$

(2) Normalized Hamming length

$$d_2(A, B) = \frac{1}{4n} \sum_{i=1}^n \left[|M_{AL}(x_i) - M_{BL}(x_i)| + |M_{AU}(x_i) - M_{BU}(x_i)| + |N_{AL}(x_i) - N_{BL}(x_i)| + |N_{AU}(x_i) - N_{BU}(x_i)| \right]$$

(3) Normalized Hamming length measure with Hausdorff metric

$$d_3(A, B) = \frac{1}{2n} \sum_{i=1}^n \left[(|M_{AL}(x_i) - M_{BL}(x_i)| \vee |M_{AU}(x_i) - M_{BU}(x_i)|) + (|N_{AL}(x_i) - N_{BL}(x_i)| \vee |N_{AU}(x_i) - N_{BU}(x_i)|) \right]$$

(4) Normalized length measure with Hausdorff metric

$$d_4(A, B) = \frac{1}{n} \sum_{i=1}^n \max \left\{ \frac{|M_{AL}(x_i) - M_{BL}(x_i)| + |M_{AU}(x_i) - M_{BU}(x_i)|}{2}, \frac{|N_{AL}(x_i) - N_{BL}(x_i)| + |N_{AU}(x_i) - N_{BU}(x_i)|}{2} \right\}$$

(5)

$$d_5(A, B) = \frac{1}{2n} \sum_{i=1}^n \left\{ \frac{1}{4} \left(|M_{AL}(x_i) - M_{BL}(x_i)| + |M_{AU}(x_i) - M_{BU}(x_i)| - |N_{AL}(x_i) - N_{BL}(x_i)| + |N_{AU}(x_i) - N_{BU}(x_i)| \right) + \frac{1}{2} \max \left(|M_{AL}(x_i) - M_{BL}(x_i)| + |M_{AU}(x_i) - M_{BU}(x_i)|, |N_{AL}(x_i) - N_{BL}(x_i)| + |N_{AU}(x_i) - N_{BU}(x_i)| \right) \right\}$$

(6) Generalized length measure for $p \geq 2$

$$d_6(A, B) = \left\{ \frac{1}{2n} \sum_{i=1}^n \left[(|M_{AL}(x_i) - M_{BL}(x_i)| \vee |M_{AU}(x_i) - M_{BU}(x_i)|)^p + (|N_{AL}(x_i) - N_{BL}(x_i)| \vee |N_{AU}(x_i) - N_{BU}(x_i)|)^p \right] \right\}^{\frac{1}{p}}$$

3. SOME MEASURE CONCEPTS OF INTERVAL-VALUED INTUITIONISTIC FUZZY MATRICES

Definition 3.1. Let $\tilde{P} = \tilde{p}_{ij}$ and $\tilde{Q} = \tilde{q}_{ij}$ be interval-valued intuitionistic fuzzy matrices, where

$$\tilde{P} = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}]), \quad \tilde{Q} = ([a'_{ij}, b'_{ij}], [c'_{ij}, d'_{ij}]).$$

The measures are defined as follows:

(1) Hamming measure

$$H(\tilde{P}, \tilde{Q}) = \sum_{j=1}^n \sum_{i=1}^n (|a_{ij} - a'_{ij}| + |b_{ij} - b'_{ij}| + |c_{ij} - c'_{ij}| + |d_{ij} - d'_{ij}|).$$

(2) Generalized Hamming measure

$$H^*(\tilde{P}, \tilde{Q}) = \frac{1}{4n^2} H(\tilde{P}, \tilde{Q}).$$

(3) Euclidean measure

$$\begin{aligned} \ddot{O}(\tilde{P}, \tilde{Q}) = \left(\sum_{j=1}^n \sum_{i=1}^n \left((a_{ij} - a'_{ij})^2 + (b_{ij} - b'_{ij})^2 \right. \right. \\ \left. \left. + (c_{ij} - c'_{ij})^2 + (d_{ij} - d'_{ij})^2 \right) \right)^{\frac{1}{2}}. \end{aligned}$$

(4) Generalized Euclidean measure

$$\ddot{O}^*(\tilde{P}, \tilde{Q}) = \frac{1}{2n} \ddot{O}(\tilde{P}, \tilde{Q}).$$

Theorem 3.1. Let $\tilde{P} = \tilde{p}_{ij}$ and $\tilde{Q} = \tilde{q}_{ij}$ be interval-valued intuitionistic fuzzy matrices, where

$$\tilde{P} = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}]), \quad \tilde{Q} = ([a'_{ij}, b'_{ij}], [c'_{ij}, d'_{ij}]).$$

Then

- (1) $H(\tilde{P}, \tilde{Q}) = H(\tilde{P}^T, \tilde{Q}^T),$
- (2) $H^*(\tilde{P}, \tilde{Q}) = H^*(\tilde{P}^T, \tilde{Q}^T),$
- (3) $\ddot{O}(\tilde{P}, \tilde{Q}) = \ddot{O}(\tilde{P}^T, \tilde{Q}^T),$
- (4) $\ddot{O}^*(\tilde{P}, \tilde{Q}) = \ddot{O}^*(\tilde{P}^T, \tilde{Q}^T).$

Proof. Let

$$\begin{aligned} \tilde{P} &= ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}]), & \tilde{Q} &= ([a'_{ij}, b'_{ij}], [c'_{ij}, d'_{ij}]), \\ \tilde{P}^T &= ([a_{ji}, b_{ji}], [c_{ji}, d_{ji}]), & \tilde{Q}^T &= ([a'_{ji}, b'_{ji}], [c'_{ji}, d'_{ji}]), \end{aligned}$$

then

(1)

$$\begin{aligned}
H(\tilde{P}, \tilde{Q}) &= \sum_{j=1}^n \sum_{i=1}^n (|a_{ij} - a_{ij}'| + |b_{ij} - b_{ij}'| + |c_{ij} - c_{ij}'| + |d_{ij} - d_{ij}'|) \\
&= \sum_{i=1}^n \sum_{j=1}^n (|a_{ij} - a_{ij}'| + |b_{ij} - b_{ij}'| + |c_{ij} - c_{ij}'| + |d_{ij} - d_{ij}'|) \\
&= \sum_{j=1}^n \sum_{i=1}^n (|a_{ji} - a_{ji}'| + |b_{ji} - b_{ji}'| + |c_{ji} - c_{ji}'| + |d_{ji} - d_{ji}'|) \\
&= H(\tilde{P}^T, \tilde{Q}^T).
\end{aligned}$$

(2)

$$\begin{aligned}
H^*(\tilde{P}, \tilde{Q}) &= \frac{1}{4n^2} \left(\sum_{j=1}^n \sum_{i=1}^n (|a_{ij} - a_{ij}'| + |b_{ij} - b_{ij}'| \right. \\
&\quad \left. + |c_{ij} - c_{ij}'| + |d_{ij} - d_{ij}'|) \right) \\
&= \frac{1}{4n^2} \left(\sum_{i=1}^n \sum_{j=1}^n (|a_{ij} - a_{ij}'| + |b_{ij} - b_{ij}'| \right. \\
&\quad \left. + |c_{ij} - c_{ij}'| + |d_{ij} - d_{ij}'|) \right) \\
&= \frac{1}{4n^2} \left(\sum_{j=1}^n \sum_{i=1}^n (|a_{ji} - a_{ji}'| + |b_{ji} - b_{ji}'| \right. \\
&\quad \left. + |c_{ji} - c_{ji}'| + |d_{ji} - d_{ji}'|) \right) \\
&= \frac{1}{4n^2} (H(\tilde{P}, \tilde{Q})) = H^*(\tilde{P}^T, \tilde{Q}^T).
\end{aligned}$$

(3)

$$\begin{aligned}
\ddot{O}(\tilde{P}, \tilde{Q}) &= \left(\sum_{j=1}^n \sum_{i=1}^n \left((a_{ij} - a_{ij}')^2 + (b_{ij} - b_{ij}')^2 \right. \right. \\
&\quad \left. \left. + (c_{ij} - c_{ij}')^2 + (d_{ij} - d_{ij}')^2 \right)^{\frac{1}{2}} \right) \\
&= \left(\sum_{i=1}^n \sum_{j=1}^n \left((a_{ij} - a_{ij}')^2 + (b_{ij} - b_{ij}')^2 \right. \right. \\
&\quad \left. \left. + (c_{ij} - c_{ij}')^2 + (d_{ij} - d_{ij}')^2 \right)^{\frac{1}{2}} \right) \\
&= \left(\sum_{j=1}^n \sum_{i=1}^n \left((a_{ji} - a_{ji}')^2 + (b_{ji} - b_{ji}')^2 \right. \right. \\
&\quad \left. \left. + (c_{ji} - c_{ji}')^2 + (d_{ji} - d_{ji}')^2 \right)^{\frac{1}{2}} \right) \\
&= \ddot{O}(\tilde{P}^T, \tilde{Q}^T).
\end{aligned}$$

(4)

$$\begin{aligned}
\ddot{O}^*(\tilde{P}, \tilde{Q}) &= \frac{1}{2n} \left(\sum_{j=1}^n \sum_{i=1}^n \left((a_{ij} - a_{ij}')^2 + (b_{ij} - b_{ij}')^2 \right. \right. \\
&\quad \left. \left. + (c_{ij} - c_{ij}')^2 + (d_{ij} - d_{ij}')^2 \right)^{\frac{1}{2}} \right) \\
&= \frac{1}{2n} \left(\sum_{i=1}^n \sum_{j=1}^n \left((a_{ij} - a_{ij}')^2 + (b_{ij} - b_{ij}')^2 \right. \right. \\
&\quad \left. \left. + (c_{ij} - c_{ij}')^2 + (d_{ij} - d_{ij}')^2 \right)^{\frac{1}{2}} \right) \\
&= \frac{1}{2n} \left(\sum_{j=1}^n \sum_{i=1}^n \left((a_{ji} - a_{ji}')^2 + (b_{ji} - b_{ji}')^2 \right. \right. \\
&\quad \left. \left. + (c_{ji} - c_{ji}')^2 + (d_{ji} - d_{ji}')^2 \right)^{\frac{1}{2}} \right) \\
&= \frac{1}{2n} \left(\ddot{O}(\tilde{P}, \tilde{Q}) \right) = H^*(\tilde{P}^T, \tilde{Q}^T).
\end{aligned}$$

□

Theorem 3.2. Let $\tilde{P}, \tilde{Q}, \tilde{S}$ be IVIFMs, then:

- (1) $H(\tilde{P} \circ \tilde{Q}, \tilde{S}) \leq H(\tilde{P}, \tilde{S}) \vee H(\tilde{Q}, \tilde{S})$,
- (2) $H^*(\tilde{P} \circ \tilde{Q}, \tilde{S}) \leq H^*(\tilde{P}, \tilde{S}) \vee H^*(\tilde{Q}, \tilde{S})$,
- (3) $H(\tilde{P}, \tilde{Q} \circ \tilde{S}) \leq H(\tilde{P}, \tilde{Q}) \vee H(\tilde{P}, \tilde{S})$,
- (4) $H^*(\tilde{P}, \tilde{Q} \circ \tilde{S}) \leq H^*(\tilde{P}, \tilde{Q}) \vee H^*(\tilde{P}, \tilde{S})$.

Proof. Let

$$\begin{aligned}
\tilde{P} &= ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}]), \\
\tilde{Q} &= ([a'_{ij}, b'_{ij}], [c'_{ij}, d'_{ij}]), \\
\tilde{S} &= ([a''_{ij}, b''_{ij}], [c''_{ij}, d''_{ij}]),
\end{aligned}$$

$$\begin{aligned}
\tilde{P}_l &= \left(\tilde{P}_{ij}^{(l)} \right)_{n \times n}, \tilde{P}_{ij}^{(l)} = \left([a_{ij}^{(l)}, b_{ij}^{(l)}], [c_{ij}^{(l)}, d_{ij}^{(l)}] \right) \text{ and } \tilde{p}_{ij} = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}]), \\
\tilde{Q}_l &= \left(\tilde{Q}_{ij}^{(l)} \right)_{n \times n}, \tilde{Q}_{ij}^{(l)} = \left([a'_{ij}^{(l)}, b'_{ij}^{(l)}], [c'_{ij}^{(l)}, d'_{ij}^{(l)}] \right) \text{ and } \tilde{q}_{ij} = ([a'_{ij}, b'_{ij}], [c'_{ij}, d'_{ij}]), \\
\tilde{S}_l &= \left(\tilde{S}_{ij}^{(l)} \right)_{n \times n}, \tilde{S}_{ij}^{(l)} = \left([a''_{ij}^{(l)}, b''_{ij}^{(l)}], [c''_{ij}^{(l)}, d''_{ij}^{(l)}] \right) \text{ and } \tilde{s}_{ij} = ([a''_{ij}, b''_{ij}], [c''_{ij}, d''_{ij}]), \\
&\text{where } [a_{ij}^{(l)}, b_{ij}^{(l)}] \subseteq [0, 1], [c_{ij}^{(l)}, d_{ij}^{(l)}] \subseteq [0, 1], b_{ij}^{(l)} + d_{ij}^{(l)} \leq 1 \text{ for } i, j = 1, 2, \dots, n \\
&\text{and } l = 1, 2,
\end{aligned}$$

$$\begin{aligned}
[a_{ij}, b_{ij}] &= \left[\max_k \left\{ \min \left\{ a_{ik}^{(1)}, a_{kj}^{(2)} \right\} \right\}, \max_k \left\{ \min \left\{ b_{ik}^{(1)}, b_{kj}^{(2)} \right\} \right\} \right] \subseteq [0, 1], \\
[c_{ij}, d_{ij}] &= \left[\min_k \left\{ \max \left\{ c_{ik}^{(1)}, c_{kj}^{(2)} \right\} \right\}, \min_k \left\{ \max \left\{ d_{ik}^{(1)}, d_{kj}^{(2)} \right\} \right\} \right] \subseteq [0, 1].
\end{aligned}$$

(1) For $\tilde{p}_{ij}^{(l)} = \left(\left[a_{ij}^{(l)}, b_{ij}^{(l)} \right], \left[c_{ij}^{(l)}, d_{ij}^{(l)} \right] \right)$, $i, j = 1, 2, \dots, n$, $l = 1, 2$,

$$\begin{aligned} \tilde{P} \circ \tilde{Q} &= \tilde{p}_{ij} = \bigcup_{k=1}^n \left(\tilde{p}_{ik}^{(1)} \cap \tilde{q}_{kj}^{(2)} \right) \\ &= \left(\left[\max_k \left\{ \min \left\{ a_{ik}^{(1)}, a_{kj}^{(2)} \right\} \right\}, \max_k \left\{ \min \left\{ b_{ik}^{(1)}, b_{kj}^{(2)} \right\} \right\} \right], \right. \\ &\quad \left. \left[\min_k \left\{ \max \left\{ c_{ik}^{(1)}, c_{kj}^{(2)} \right\} \right\}, \min_k \left\{ \max \left\{ d_{ik}^{(1)}, d_{kj}^{(2)} \right\} \right\} \right] \right) \end{aligned}$$

$$\begin{aligned} H(\tilde{P} \circ \tilde{Q}, \tilde{S}) &= \sum_{j=1}^n \sum_{i=1}^n \left(\left| a_{ij}^{(1)} - a_{ij}'' \right| + \left| b_{ij}^{(1)} - b_{ij}'' \right| \right. \\ &\quad \left. + \left| c_{ij}^{(1)} - c_{ij}'' \right| + \left| d_{ij}^{(1)} - d_{ij}'' \right| \right) \\ H(\tilde{P}, \tilde{S}) &= \sum_{j=1}^n \sum_{i=1}^n \left(\left| a_{ij} - a_{ij}'' \right| + \left| b_{ij} - b_{ij}'' \right| \right. \\ &\quad \left. + \left| c_{ij} - c_{ij}'' \right| + \left| d_{ij} - d_{ij}'' \right| \right) \\ H(\tilde{Q}, \tilde{S}) &= \sum_{j=1}^n \sum_{i=1}^n \left(\left| a_{ij}' - a_{ij}'' \right| + \left| b_{ij}' - b_{ij}'' \right| \right. \\ &\quad \left. + \left| c_{ij}' - c_{ij}'' \right| + \left| d_{ij}' - d_{ij}'' \right| \right) \end{aligned}$$

Hence, $H(\tilde{P} \circ \tilde{Q}, \tilde{S}) \leq H(\tilde{P}, \tilde{S}) \vee H(\tilde{Q}, \tilde{S})$ is obtained.

(2) For $\tilde{p}_{ij}^{(l)} = \left(\left[a_{ij}^{(l)}, b_{ij}^{(l)} \right], \left[c_{ij}^{(l)}, d_{ij}^{(l)} \right] \right)$, $i, j = 1, 2, \dots, n$, $l = 1, 2$,

$$\begin{aligned} \tilde{P} \circ \tilde{Q} &= \tilde{p}_{ij} = \bigcup_{k=1}^n \left(\tilde{p}_{ik}^{(1)} \cap \tilde{q}_{kj}^{(2)} \right) \\ &= \left(\left[\max_k \left\{ \min \left\{ a_{ik}^{(1)}, a_{kj}^{(2)} \right\} \right\}, \max_k \left\{ \min \left\{ b_{ik}^{(1)}, b_{kj}^{(2)} \right\} \right\} \right], \right. \\ &\quad \left. \left[\min_k \left\{ \max \left\{ c_{ik}^{(1)}, c_{kj}^{(2)} \right\} \right\}, \min_k \left\{ \max \left\{ d_{ik}^{(1)}, d_{kj}^{(2)} \right\} \right\} \right] \right) \end{aligned}$$

$$\begin{aligned}
H^*(\tilde{P} \circ \tilde{Q}, \tilde{S}) &= \sum_{j=1}^n \sum_{i=1}^n \left(\left| a_{ij}^{(1)} - a_{ij}'' \right| + \left| b_{ij}^{(1)} - b_{ij}'' \right| \right. \\
&\quad \left. + \left| c_{ij}^{(1)} - c_{ij}'' \right| + \left| d_{ij}^{(1)} - d_{ij}'' \right| \right) \\
H^*(\tilde{P}, \tilde{S}) &= \sum_{j=1}^n \sum_{i=1}^n \left(\left| a_{ij} - a_{ij}'' \right| + \left| b_{ij} - b_{ij}'' \right| \right. \\
&\quad \left. + \left| c_{ij} - c_{ij}'' \right| + \left| d_{ij} - d_{ij}'' \right| \right) \\
H^*(\tilde{Q}, \tilde{S}) &= \sum_{j=1}^n \sum_{i=1}^n \left(\left| a_{ij}' - a_{ij}'' \right| + \left| b_{ij}' - b_{ij}'' \right| \right. \\
&\quad \left. + \left| c_{ij}' - c_{ij}'' \right| + \left| d_{ij}' - d_{ij}'' \right| \right)
\end{aligned}$$

Hence, $H^*(\tilde{P} \circ \tilde{Q}, \tilde{S}) \leq H^*(\tilde{P}, \tilde{S}) \vee H^*(\tilde{Q}, \tilde{S})$ is obtained.

(3) For $\tilde{p}_{ij}^{(l)} = \left(\left[a_{ij}^{(l)}, b_{ij}^{(l)} \right], \left[c_{ij}^{(l)}, d_{ij}^{(l)} \right] \right)$, $i, j = 1, 2, \dots, n$, $l = 1, 2$,

$$\begin{aligned}
\tilde{Q} \circ \tilde{S} &= \tilde{p}_{ij} = \bigcup_{k=1}^n \left(\tilde{p}_{ik}^{(1)} \cap \tilde{q}_{kj}^{(2)} \right) \\
&= \left(\left[\max_k \left\{ \min \left\{ a_{ik}^{(1)}, a_{kj}^{(2)} \right\} \right\}, \max_k \left\{ \min \left\{ b_{ik}^{(1)}, b_{kj}^{(2)} \right\} \right\} \right], \right. \\
&\quad \left. \left[\min_k \left\{ \max \left\{ c_{ik}^{(1)}, c_{kj}^{(2)} \right\} \right\}, \min_k \left\{ \max \left\{ d_{ik}^{(1)}, d_{kj}^{(2)} \right\} \right\} \right] \right)
\end{aligned}$$

$$\begin{aligned}
H(\tilde{P}, \tilde{Q} \circ \tilde{S}) &= \sum_{j=1}^n \sum_{i=1}^n \left(\left| a_{ij} - a_{ij}^{(1)} \right| + \left| b_{ij} - b_{ij}^{(1)} \right| \right. \\
&\quad \left. + \left| c_{ij} - c_{ij}^{(1)} \right| + \left| d_{ij} - d_{ij}^{(1)} \right| \right) \\
H(\tilde{P}, \tilde{Q}) &= \sum_{j=1}^n \sum_{i=1}^n \left(\left| a_{ij} - a_{ij}' \right| + \left| b_{ij} - b_{ij}' \right| \right. \\
&\quad \left. + \left| c_{ij} - c_{ij}' \right| + \left| d_{ij} - d_{ij}' \right| \right) \\
H(\tilde{P}, \tilde{S}) &= \sum_{j=1}^n \sum_{i=1}^n \left(\left| a_{ij} - a_{ij}'' \right| + \left| b_{ij} - b_{ij}'' \right| \right. \\
&\quad \left. + \left| c_{ij} - c_{ij}'' \right| + \left| d_{ij} - d_{ij}'' \right| \right)
\end{aligned}$$

Hence, $H(\tilde{P}, \tilde{Q} \circ \tilde{S}) \leq H(\tilde{P}, \tilde{Q}) \vee H(\tilde{P}, \tilde{S})$ is obtained.

(4) For $\tilde{p}_{ij}^{(l)} = \left(\left[a_{ij}^{(l)}, b_{ij}^{(l)} \right], \left[c_{ij}^{(l)}, d_{ij}^{(l)} \right] \right)$, $i, j = 1, 2, \dots, n$, $l = 1, 2$,

$$\begin{aligned} \tilde{Q} \circ \tilde{S} &= \tilde{p}_{ij} = \bigcup_{k=1}^n \left(\tilde{p}_{ik}^{(1)} \cap \tilde{q}_{kj}^{(2)} \right) \\ &= \left(\left[\max_k \left\{ \min \left\{ a_{ik}^{(1)}, a_{kj}^{(2)} \right\} \right\}, \max_k \left\{ \min \left\{ b_{ik}^{(1)}, b_{kj}^{(2)} \right\} \right\} \right], \right. \\ &\quad \left. \left[\min_k \left\{ \max \left\{ c_{ik}^{(1)}, c_{kj}^{(2)} \right\} \right\}, \min_k \left\{ \max \left\{ d_{ik}^{(1)}, d_{kj}^{(2)} \right\} \right\} \right] \right) \end{aligned}$$

$$\begin{aligned} H^*(\tilde{P}, \tilde{Q} \circ \tilde{S}) &= \sum_{j=1}^n \sum_{i=1}^n \left(\left| a_{ij} - a_{ij}^{(1)} \right| + \left| b_{ij} - b_{ij}^{(1)} \right| \right. \\ &\quad \left. + \left| c_{ij} - c_{ij}^{(1)} \right| + \left| d_{ij} - d_{ij}^{(1)} \right| \right) \\ H^*(\tilde{P}, \tilde{Q}) &= \sum_{j=1}^n \sum_{i=1}^n \left(\left| a_{ij} - a_{ij}' \right| + \left| b_{ij} - b_{ij}' \right| \right. \\ &\quad \left. + \left| c_{ij} - c_{ij}' \right| + \left| d_{ij} - d_{ij}' \right| \right) \\ H^*(\tilde{P}, \tilde{S}) &= \sum_{j=1}^n \sum_{i=1}^n \left(\left| a_{ij} - a_{ij}'' \right| + \left| b_{ij} - b_{ij}'' \right| \right. \\ &\quad \left. + \left| c_{ij} - c_{ij}'' \right| + \left| d_{ij} - d_{ij}'' \right| \right) \end{aligned}$$

Hence, $H^*(\tilde{P}, \tilde{Q} \circ \tilde{S}) \leq H^*(\tilde{P}, \tilde{Q}) \vee H^*(\tilde{P}, \tilde{S})$ is obtained. $\square$

Theorem 3.3. Let $\tilde{P}, \tilde{Q}, \tilde{S}$ be IVIFMs, then:

- (1) $\ddot{O}(\tilde{P} \circ \tilde{Q}, \tilde{S}) \leq \ddot{O}(\tilde{P}, \tilde{S}) \vee \ddot{O}(\tilde{Q}, \tilde{S})$,
- (2) $\ddot{O}^*(\tilde{P} \circ \tilde{Q}, \tilde{S}) \leq \ddot{O}^*(\tilde{P}, \tilde{S}) \vee \ddot{O}^*(\tilde{Q}, \tilde{S})$,
- (3) $\ddot{O}(\tilde{P}, \tilde{Q} \circ \tilde{S}) \leq \ddot{O}(\tilde{P}, \tilde{Q}) \vee \ddot{O}(\tilde{P}, \tilde{S})$,
- (4) $\ddot{O}^*(\tilde{P}, \tilde{Q} \circ \tilde{S}) \leq \ddot{O}^*(\tilde{P}, \tilde{Q}) \vee \ddot{O}^*(\tilde{P}, \tilde{S})$.

Proof.

$$\begin{aligned} \tilde{P} &= ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}]), \\ \tilde{Q} &= ([a'_{ij}, b'_{ij}], [c'_{ij}, d'_{ij}]), \\ \tilde{S} &= ([a''_{ij}, b''_{ij}], [c''_{ij}, d''_{ij}]), \end{aligned}$$

$$\begin{aligned} \tilde{P}_l &= \left(\tilde{P}_{ij}^{(l)} \right)_{n \times n}, \tilde{P}_{ij}^{(l)} = \left(\left[a_{ij}^{(l)}, b_{ij}^{(l)} \right], \left[c_{ij}^{(l)}, d_{ij}^{(l)} \right] \right) \text{ and } \tilde{p}_{ij} = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}]), \\ \tilde{Q}_l &= \left(\tilde{Q}_{ij}^{(l)} \right)_{n \times n}, \tilde{Q}_{ij}^{(l)} = \left(\left[a'_{ij}^{(l)}, b'_{ij}^{(l)} \right], \left[c'_{ij}^{(l)}, d'_{ij}^{(l)} \right] \right) \text{ and } \tilde{q}_{ij} = ([a'_{ij}, b'_{ij}], [c'_{ij}, d'_{ij}]), \\ \tilde{S}_l &= \left(\tilde{S}_{ij}^{(l)} \right)_{n \times n}, \tilde{S}_{ij}^{(l)} = \left(\left[a''_{ij}^{(l)}, b''_{ij}^{(l)} \right], \left[c''_{ij}^{(l)}, d''_{ij}^{(l)} \right] \right) \text{ and } \tilde{s}_{ij} = ([a''_{ij}, b''_{ij}], [c''_{ij}, d''_{ij}]), \end{aligned}$$

where $[a_{ij}^{(l)}, b_{ij}^{(l)}] \subseteq [0, 1]$, $[c_{ij}^{(l)}, d_{ij}^{(l)}] \subseteq [0, 1]$, $b_{ij}^{(l)} + d_{ij}^{(l)} \leq 1$ for $i, j = 1, 2, \dots, n$ and $l = 1, 2$,

$$\begin{aligned} [a_{ij}, b_{ij}] &= \left[\max_k \left\{ \min \left\{ a_{ik}^{(1)}, a_{kj}^{(2)} \right\} \right\}, \max_k \left\{ \min \left\{ b_{ik}^{(1)}, b_{kj}^{(2)} \right\} \right\} \right] \subseteq [0, 1], \\ [c_{ij}, d_{ij}] &= \left[\min_k \left\{ \max \left\{ c_{ik}^{(1)}, c_{kj}^{(2)} \right\} \right\}, \min_k \left\{ \max \left\{ d_{ik}^{(1)}, d_{kj}^{(2)} \right\} \right\} \right] \subseteq [0, 1]. \end{aligned}$$

(1) For $\tilde{p}_{ij}^{(l)} = \left([a_{ij}^{(l)}, b_{ij}^{(l)}], [c_{ij}^{(l)}, d_{ij}^{(l)}] \right)$, $i, j = 1, 2, \dots, n$, $l = 1, 2$,

$$\begin{aligned} \tilde{P} \circ \tilde{Q} &= \tilde{p}_{ij} = \bigcup_{k=1}^n \left(\tilde{p}_{ik}^{(1)} \cap \tilde{q}_{kj}^{(2)} \right) \\ &= \left(\left[\max_k \left\{ \min \left\{ a_{ik}^{(1)}, a_{kj}^{(2)} \right\} \right\}, \max_k \left\{ \min \left\{ b_{ik}^{(1)}, b_{kj}^{(2)} \right\} \right\} \right], \right. \\ &\quad \left. \left[\min_k \left\{ \max \left\{ c_{ik}^{(1)}, c_{kj}^{(2)} \right\} \right\}, \min_k \left\{ \max \left\{ d_{ik}^{(1)}, d_{kj}^{(2)} \right\} \right\} \right] \right) \end{aligned}$$

$$\begin{aligned} \ddot{O}(\tilde{P} \circ \tilde{Q}, \tilde{S}) &= \left(\sum_{j=1}^n \sum_{i=1}^n \left((a_{ij}^{(1)} - a_{ij}'')^2 + (b_{ij}^{(1)} - b_{ij}'')^2 \right. \right. \\ &\quad \left. \left. + (c_{ij}^{(1)} - c_{ij}^{(2)})^2 + (d_{ij}^{(1)} - d_{ij}'')^2 \right)^{\frac{1}{2}} \right) \\ \ddot{O}(\tilde{P}, \tilde{S}) &= \left(\sum_{j=1}^n \sum_{i=1}^n \left((a_{ij} - a_{ij}'')^2 + (b_{ij} - b_{ij}'')^2 \right. \right. \\ &\quad \left. \left. + (c_{ij} - c_{ij}'')^2 + (d_{ij} - d_{ij}'')^2 \right)^{\frac{1}{2}} \right) \\ \ddot{O}(\tilde{Q}, \tilde{S}) &= \left(\sum_{j=1}^n \sum_{i=1}^n \left((a_{ij}' - a_{ij}'')^2 + (b_{ij}' - b_{ij}'')^2 \right. \right. \\ &\quad \left. \left. + (c_{ij}' - c_{ij}'')^2 + (d_{ij}' - d_{ij}'')^2 \right)^{\frac{1}{2}} \right) \end{aligned}$$

Hence, $\ddot{O}(\tilde{P} \circ \tilde{Q}, \tilde{S}) \leq \ddot{O}(\tilde{P}, \tilde{S}) \vee \ddot{O}(\tilde{Q}, \tilde{S})$ is obtained.

(2) For $\tilde{p}_{ij}^{(l)} = \left([a_{ij}^{(l)}, b_{ij}^{(l)}], [c_{ij}^{(l)}, d_{ij}^{(l)}] \right)$, $i, j = 1, 2, \dots, n$, $l = 1, 2$,

$$\begin{aligned} \tilde{P} \circ \tilde{Q} &= \tilde{p}_{ij} = \bigcup_{k=1}^n \left(\tilde{p}_{ik}^{(1)} \cap \tilde{q}_{kj}^{(2)} \right) \\ &= \left(\left[\max_k \left\{ \min \left\{ a_{ik}^{(1)}, a_{kj}^{(2)} \right\} \right\}, \max_k \left\{ \min \left\{ b_{ik}^{(1)}, b_{kj}^{(2)} \right\} \right\} \right], \right. \\ &\quad \left. \left[\min_k \left\{ \max \left\{ c_{ik}^{(1)}, c_{kj}^{(2)} \right\} \right\}, \min_k \left\{ \max \left\{ d_{ik}^{(1)}, d_{kj}^{(2)} \right\} \right\} \right] \right) \end{aligned}$$

$$\begin{aligned}
\ddot{O}^*(\tilde{P} \circ \tilde{Q}, \tilde{S}) &= \left(\sum_{j=1}^n \sum_{i=1}^n \left((a_{ij}^{(1)} - a_{ij}'')^2 + (b_{ij}^{(1)} - b_{ij}'')^2 \right. \right. \\
&\quad \left. \left. + (c_{ij}^{(1)} - c_{ij}^{(2)})^2 + (d_{ij}^{(1)} - d_{ij}'')^2 \right)^{\frac{1}{2}} \right) \\
\ddot{O}^*(\tilde{P}, \tilde{S}) &= \left(\sum_{j=1}^n \sum_{i=1}^n \left((a_{ij} - a_{ij}'')^2 + (b_{ij} - b_{ij}'')^2 \right. \right. \\
&\quad \left. \left. + (c_{ij} - c_{ij}'')^2 + (d_{ij} - d_{ij}'')^2 \right)^{\frac{1}{2}} \right) \\
\ddot{O}^*(\tilde{Q}, \tilde{S}) &= \left(\sum_{j=1}^n \sum_{i=1}^n \left((a_{ij}' - a_{ij}'')^2 + (b_{ij}' - b_{ij}'')^2 \right. \right. \\
&\quad \left. \left. + (c_{ij}' - c_{ij}'')^2 + (d_{ij}' - d_{ij}'')^2 \right)^{\frac{1}{2}} \right)
\end{aligned}$$

Hence, $\ddot{O}^*(\tilde{P} \circ \tilde{Q}, \tilde{S}) \leq \ddot{O}^*(\tilde{P}, \tilde{S}) \vee \ddot{O}^*(\tilde{Q}, \tilde{S})$ is obtained.

(3) For $\tilde{p}_{ij}^{(l)} = \left(\left[a_{ij}^{(l)}, b_{ij}^{(l)} \right], \left[c_{ij}^{(l)}, d_{ij}^{(l)} \right] \right)$, $i, j = 1, 2, \dots, n$, $l = 1, 2$,

$$\begin{aligned}
\tilde{Q} \circ \tilde{S} &= \tilde{p}_{ij} = \bigcup_{k=1}^n \left(\tilde{p}_{ik}^{(1)} \cap \tilde{q}_{kj}^{(2)} \right) \\
&= \left(\left[\max_k \left\{ \min \left\{ a_{ik}^{(1)}, a_{kj}^{(2)} \right\} \right\}, \max_k \left\{ \min \left\{ b_{ik}^{(1)}, b_{kj}^{(2)} \right\} \right\} \right], \right. \\
&\quad \left. \left[\min_k \left\{ \max \left\{ c_{ik}^{(1)}, c_{kj}^{(2)} \right\} \right\}, \min_k \left\{ \max \left\{ d_{ik}^{(1)}, d_{kj}^{(2)} \right\} \right\} \right] \right)
\end{aligned}$$

$$\begin{aligned}
\ddot{O}(\tilde{P}, \tilde{Q} \circ \tilde{S}) &= \left(\sum_{j=1}^n \sum_{i=1}^n \left((a_{ij} - a_{ij}^{(1)})^2 + (b_{ij} - b_{ij}^{(1)})^2 \right. \right. \\
&\quad \left. \left. + (c_{ij} - c_{ij}^{(1)})^2 + (d_{ij} - d_{ij}^{(1)})^2 \right)^{\frac{1}{2}} \right) \\
\ddot{O}(\tilde{P}, \tilde{Q}) &= \left(\sum_{j=1}^n \sum_{i=1}^n \left((a_{ij} - a_{ij}')^2 + (b_{ij} - b_{ij}')^2 \right. \right. \\
&\quad \left. \left. + (c_{ij} - c_{ij}')^2 + (d_{ij} - d_{ij}')^2 \right)^{\frac{1}{2}} \right) \\
\ddot{O}(\tilde{P}, \tilde{S}) &= \left(\sum_{j=1}^n \sum_{i=1}^n \left((a_{ij}' - a_{ij}'')^2 + (b_{ij}' - b_{ij}'')^2 \right. \right. \\
&\quad \left. \left. + (c_{ij}' - c_{ij}'')^2 + (d_{ij}' - d_{ij}'')^2 \right)^{\frac{1}{2}} \right)
\end{aligned}$$

Hence, $\ddot{O}(\tilde{P}, \tilde{Q} \circ \tilde{S}) \leq \ddot{O}(\tilde{P}, \tilde{Q}) \vee \ddot{O}(\tilde{P}, \tilde{S})$ is obtained.

(4) For $\tilde{p}_{ij}^{(l)} = \left(\left[a_{ij}^{(l)}, b_{ij}^{(l)} \right], \left[c_{ij}^{(l)}, d_{ij}^{(l)} \right] \right)$, $i, j = 1, 2, \dots, n$, $l = 1, 2$,

$$\begin{aligned} \tilde{Q} \circ \tilde{S} &= \tilde{p}_{ij} = \bigcup_{k=1}^n \left(\tilde{p}_{ik}^{(1)} \cap \tilde{q}_{kj}^{(2)} \right) \\ &= \left(\left[\max_k \left\{ \min \left\{ a_{ik}^{(1)}, a_{kj}^{(2)} \right\} \right\}, \max_k \left\{ \min \left\{ b_{ik}^{(1)}, b_{kj}^{(2)} \right\} \right\} \right], \right. \\ &\quad \left. \left[\min_k \left\{ \max \left\{ c_{ik}^{(1)}, c_{kj}^{(2)} \right\} \right\}, \min_k \left\{ \max \left\{ d_{ik}^{(1)}, d_{kj}^{(2)} \right\} \right\} \right] \right) \\ \ddot{O}^*(\tilde{P}, \tilde{Q} \circ \tilde{S}) &= \left(\sum_{j=1}^n \sum_{i=1}^n \left(\left(a_{ij} - a_{ij}^{(1)} \right)^2 + \left(b_{ij} - b_{ij}^{(1)} \right)^2 \right. \right. \\ &\quad \left. \left. + \left(c_{ij} - c_{ij}^{(1)} \right)^2 + \left(d_{ij} - d_{ij}^{(1)} \right)^2 \right)^{\frac{1}{2}} \right) \\ \ddot{O}^*(\tilde{P}, \tilde{Q}) &= \left(\sum_{j=1}^n \sum_{i=1}^n \left(\left(a_{ij} - a_{ij}' \right)^2 + \left(b_{ij} - b_{ij}' \right)^2 \right. \right. \\ &\quad \left. \left. + \left(c_{ij} - c_{ij}' \right)^2 + \left(d_{ij} - d_{ij}' \right)^2 \right)^{\frac{1}{2}} \right) \\ \ddot{O}^*(\tilde{P}, \tilde{S}) &= \left(\sum_{j=1}^n \sum_{i=1}^n \left(\left(a_{ij}' - a_{ij}'' \right)^2 + \left(b_{ij}' - b_{ij}'' \right)^2 \right. \right. \\ &\quad \left. \left. + \left(c_{ij}' - c_{ij}'' \right)^2 + \left(d_{ij}' - d_{ij}'' \right)^2 \right)^{\frac{1}{2}} \right) \end{aligned}$$

Hence, $\ddot{O}^*(\tilde{P}, \tilde{Q} \circ \tilde{S}) \leq \ddot{O}^*(\tilde{P}, \tilde{Q}) \vee \ddot{O}^*(\tilde{P}, \tilde{S})$ is obtained. $\square$

Theorem 3.4. Let $\tilde{P}$ be an IVIFM and $t, h \in \mathbb{N}^*$, then:

- (1) $H(\tilde{p}^h, \tilde{p}^t) = 0$,
- (2) $H^*(\tilde{p}^h, \tilde{p}^t) = 0$,
- (3) $\ddot{O}(\tilde{p}^h, \tilde{p}^t) = 0$,
- (4) $\ddot{O}^*(\tilde{p}^h, \tilde{p}^t) = 0$.

Proof. (1) For $h = 1$, $t = 1$, $H(\tilde{P}, \tilde{P}) = 0$.

For $h = 2$, $t = 1$, $H(\tilde{P}^2 \circ \tilde{P}, \tilde{P}) = H(\tilde{P}, \tilde{P}) \wedge H(\tilde{P}, \tilde{P}) = 0$.

For $h = n$, $t = 1$, $H(\tilde{P}^n, \tilde{P}) = 0$ holds. At the same time,

For $h = 1$, $t = 2$, $H(\tilde{P}, \tilde{P}^2 \circ \tilde{P}) = H(\tilde{P}, \tilde{P}) \wedge H(\tilde{P}, \tilde{P}) = 0$.

For $h = 1$, $t = n$, $H(\tilde{P}, \tilde{P}^n) = 0$.

From here $H(\tilde{p}^h, \tilde{p}^t) = 0$ is obtained.

The proof for (2), (3) and (4) is clearly seen. $\square$

4. SCALAR MULTIPLICATION WITH INTERVAL-VALUED INTUITIONISTIC FUZZY MATRIX

Definition 4.1. For $\lambda_i, \mu_i \in [0, 1]$, $1 \leq i \leq 2$ and $(\lambda, \mu) = ([\lambda_1, \lambda_2], [\mu_1, \mu_2])$,

$$(\lambda, \mu)((a, b)) = \begin{cases} (\lambda, \mu), & (\lambda, \mu) = (a, b), \\ 0, & \text{otherwise,} \end{cases}$$

where (λ, μ) is an interval-valued scalar, the scalar multiplication with an interval-valued matrix $\tilde{A}$ is defined as follows:

$$(\lambda, \mu) \circ \tilde{A} = \begin{bmatrix} (\lambda, \mu) & 0 & 0 \\ 0 & (\lambda, \mu) & 0 \\ 0 & 0 & (\lambda, \mu) \end{bmatrix} \circ A.$$

Theorem 4.1. *Let A be an IVIFSM and let X and Y be interval-valued scalars. Then,*

$$X \circ (Y \circ A) = (X \circ Y) \circ A.$$

Proof. Taking into consideration the associativity and distributivity properties of the minimum and maximum operations over each other, the proof can be easily constructed. $\square$

Example 4.1. Let

$$\begin{aligned} X &= ([0.1, 0.4], [0.3, 0.5]), & Y &= ([0.2, 0.3], [0.2, 0.7]), \\ A &= \begin{bmatrix} ([0.3, 0.5], [0.1, 0.2]) & ([0.1, 0.6], [0.3, 0.3]) & ([0.1, 0.6], [0.2, 0.3]) \\ ([0.4, 0.6], [0.3, 0.4]) & ([0.3, 0.5], [0.2, 0.4]) & ([0.4, 0.6], [0.1, 0.2]) \\ ([0.1, 0.6], [0.2, 0.3]) & ([0.3, 0.4], [0.3, 0.4]) & ([0.4, 0.6], [0, 0.1]) \end{bmatrix} \\ X \circ Y &= \begin{bmatrix} ([0.1, 0.3], [0.2, 0.5]) & ([0, 0], [0, 0]) & ([0, 0], [0, 0]) \\ ([0, 0], [0, 0]) & ([0.1, 0.3], [0.2, 0.5]) & ([0, 0], [0, 0]) \\ ([0, 0], [0, 0]) & ([0, 0], [0, 0]) & ([0.1, 0.3], [0.2, 0.5]) \end{bmatrix} \\ Y \circ A &= \begin{bmatrix} ([0.2, 0.3], [0.1, 0.2]) & ([0.1, 0.3], [0.2, 0.3]) & ([0.1, 0.3], [0.2, 0.3]) \\ ([0.2, 0.3], [0.2, 0.4]) & ([0.2, 0.3], [0.2, 0.4]) & ([0.2, 0.3], [0.1, 0.2]) \\ ([0.1, 0.3], [0.2, 0.3]) & ([0.2, 0.3], [0.2, 0.4]) & ([0.2, 0.3], [0, 0.1]) \end{bmatrix} \\ X \circ (Y \circ A) &= \begin{bmatrix} ([0.1, 0.3], [0.1, 0.2]) & ([0.1, 0.3], [0.2, 0.3]) & ([0.1, 0.3], [0.2, 0.3]) \\ ([0.1, 0.3], [0.2, 0.4]) & ([0.1, 0.3], [0.2, 0.4]) & ([0.1, 0.3], [0.1, 0.2]) \\ ([0.1, 0.3], [0.2, 0.3]) & ([0.1, 0.3], [0.2, 0.4]) & ([0.1, 0.3], [0, 0.1]) \end{bmatrix} \\ (X \circ Y) \circ A &= \begin{bmatrix} ([0.1, 0.3], [0.1, 0.2]) & ([0.1, 0.3], [0.2, 0.3]) & ([0.1, 0.3], [0.2, 0.3]) \\ ([0.1, 0.3], [0.2, 0.4]) & ([0.1, 0.3], [0.2, 0.4]) & ([0.1, 0.3], [0.1, 0.2]) \\ ([0.1, 0.3], [0.2, 0.3]) & ([0.1, 0.3], [0.2, 0.4]) & ([0.1, 0.3], [0, 0.1]) \end{bmatrix} \end{aligned}$$

Thus, the equality $X \circ (Y \circ A) = (X \circ Y) \circ A$ is satisfied.

Theorem 4.2. *Let A and B be IVIFSMs. For a scalar (λ, μ) , $((\lambda, \mu) \circ A) \circ B = (\lambda, \mu) \circ (A \circ B)$ holds.*

Proof. Let $A = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])$ and $B = ([m_{ij}, n_{ij}], [p_{ij}, r_{ij}])$ be IVIFSMs. For $\forall 1 \leq i, j \leq n$ and scalar $(\lambda, \mu) = ([\lambda_1, \lambda_2], [\mu_1, \mu_2])$,

$$\begin{aligned} (\lambda, \mu) \circ A &= \left(\left[\max_k \left\{ \min \left\{ \lambda_{ik}^{(1)}, a_{kj} \right\} \right\}, \max_k \left\{ \min \left\{ \lambda_{ik}^{(2)}, b_{kj} \right\} \right\} \right], \right. \\ &\quad \left. \left[\min_k \left\{ \max \left\{ \lambda_{ik}^{(1)}, c_{kj} \right\} \right\}, \min_k \left\{ \max \left\{ \lambda_{ik}^{(2)}, d_{kj} \right\} \right\} \right] \right) \\ ((\lambda, \mu) \circ A) \circ B &= \left(\left[\max_k \left\{ \min \left\{ \lambda_{ik}^{(1)}, m_{kj} \right\} \right\}, \max_k \left\{ \min \left\{ \lambda_{ik}^{(2)}, n_{kj} \right\} \right\} \right], \right. \\ &\quad \left. \left[\min_k \left\{ \max \left\{ \lambda_{ik}^{(1)}, p_{kj} \right\} \right\}, \min_k \left\{ \max \left\{ \lambda_{ik}^{(2)}, r_{kj} \right\} \right\} \right] \right) \\ &= (\lambda, \mu) \circ (A \circ B). \end{aligned}$$

Thus the equality $((\lambda, \mu) \circ A) \circ B = (\lambda, \mu) \circ (A \circ B)$ is obtained. $\square$

Example 4.2. Let $(\lambda, \mu) = ([0.5, 0.6], [0.1, 0.3])$,

$$A = \begin{bmatrix} ([0.1, 0.2], [0.1, 0.5]) & ([0.5, 0.6], [0.1, 0.3]) & ([0.2, 0.7], [0.1, 0.2]) \\ ([0.3, 0.6], [0.2, 0.4]) & ([0.2, 0.5], [0.5, 0.5]) & ([0.1, 0.5], [0.3, 0.5]) \\ ([0.7, 0.9], [0.1, 0.1]) & ([0.6, 0.8], [0.1, 0.2]) & ([0.7, 0.9], [0.1, 0.1]) \end{bmatrix}$$

$$B = \begin{bmatrix} ([0.2, 0.3], [0, 0.3]) & ([0.1, 0.4], [0, 0.2]) & ([0.1, 0.5], [0, 0.1]) \\ ([0.4, 0.7], [0.1, 0.3]) & ([0, 0.3], [0.2, 0.4]) & ([0, 0.4], [0.2, 0.4]) \\ ([0.1, 0.8], [0, 0]) & ([0.5, 0.6], [0, 0.1]) & ([0.6, 0.7], [0, 0]) \end{bmatrix}$$

The matrix $(\lambda, \mu) \circ A$ is

$$\begin{bmatrix} ([0.1, 0.2], [0.1, 0.3]) & ([0.5, 0.6], [0.1, 0.3]) & ([0.2, 0.6], [0.1, 0.2]) \\ ([0.3, 0.6], [0.1, 0.3]) & ([0.2, 0.5], [0.1, 0.3]) & ([0.1, 0.5], [0.1, 0.3]) \\ ([0.5, 0.6], [0.1, 0.1]) & ([0.5, 0.6], [0.1, 0.2]) & ([0.5, 0.6], [0.1, 0.1]) \end{bmatrix}$$

the matrix $((\lambda, \mu) \circ A) \circ B$ is

$$\begin{bmatrix} ([0.2, 0.3], [0.1, 0.2]) & ([0.1, 0.3], [0.2, 0.3]) & ([0.1, 0.3], [0.2, 0.3]) \\ ([0.2, 0.3], [0.2, 0.4]) & ([0.2, 0.3], [0.2, 0.4]) & ([0.2, 0.3], [0.1, 0.2]) \\ ([0.1, 0.3], [0.2, 0.3]) & ([0.2, 0.3], [0.2, 0.4]) & ([0.2, 0.3], [0, 0.1]) \end{bmatrix}$$

and the matrix $(\lambda, \mu) \circ (A \circ B)$ is

$$\begin{bmatrix} ([0.2, 0.3], [0.1, 0.2]) & ([0.1, 0.3], [0.2, 0.3]) & ([0.1, 0.3], [0.2, 0.3]) \\ ([0.2, 0.3], [0.2, 0.4]) & ([0.2, 0.3], [0.2, 0.4]) & ([0.2, 0.3], [0.1, 0.2]) \\ ([0.1, 0.3], [0.2, 0.3]) & ([0.2, 0.3], [0.2, 0.4]) & ([0.2, 0.3], [0, 0.1]) \end{bmatrix}$$

Hence $((\lambda, \mu) \circ A) \circ B = (\lambda, \mu) \circ (A \circ B)$ holds.

Theorem 4.3. Let A be an IVIFSM and let $(\lambda, \mu) = ([\lambda_1, \lambda_2], [\mu_1, \mu_2])$ and $(\lambda', \mu') = ([\lambda'_1, \lambda'_2], [\mu'_1, \mu'_2])$ be scalars. If $(\lambda, \mu) \leq (\lambda', \mu')$ then $(\lambda, \mu) \circ A \leq (\lambda', \mu') \circ A$ holds.

Proof. Let $A = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])$ be an IVIFSM. For $\forall 1 \leq i, j \leq n$ and scalar (λ, μ) where $\lambda_i \leq \lambda'_i$ and $\mu_i \leq \mu'_i$. Since $\lambda_{ik}^{(1)} \leq \lambda_{ik}^{(1)'}$, $\lambda_{ik}^{(2)} \leq \lambda_{ik}^{(2)'}$ and $\mu_{ik}^{(1)} \geq \mu_{ik}^{(1)'}$, $\mu_{ik}^{(2)} \geq \mu_{ik}^{(2)'}$,

$$\begin{aligned} (\lambda, \mu) \circ A &= \left(\left[\max_k \left\{ \min \left\{ \lambda_{ik}^{(1)}, a_{kj} \right\} \right\}, \max_k \left\{ \min \left\{ \lambda_{ik}^{(2)}, b_{kj} \right\} \right\} \right], \right. \\ &\quad \left. \left[\min_k \left\{ \max \left\{ \mu_{ik}^{(1)}, c_{kj} \right\} \right\}, \min_k \left\{ \max \left\{ \mu_{ik}^{(2)}, d_{kj} \right\} \right\} \right] \right) \\ &\leq \left(\left[\max_k \left\{ \min \left\{ \lambda_{ik}^{(1)'}, a_{kj} \right\} \right\}, \max_k \left\{ \min \left\{ \lambda_{ik}^{(2)'}, b_{kj} \right\} \right\} \right], \right. \\ &\quad \left. \left[\min_k \left\{ \max \left\{ \mu_{ik}^{(1)'}, c_{kj} \right\} \right\}, \min_k \left\{ \max \left\{ \mu_{ik}^{(2)'}, d_{kj} \right\} \right\} \right] \right) \\ &= (\lambda', \mu') \circ A \end{aligned}$$

is obtained. □

Example 4.3. Let

$$(\lambda, \mu) = ([0.5, 0.6], [0.1, 0.3]), \quad (\lambda', \mu') = ([0.1, 0.4], [0, 0.2]),$$

$$A = \begin{bmatrix} ([0.1, 0.2], [0.1, 0.5]) & ([0.5, 0.6], [0.1, 0.3]) & ([0.2, 0.7], [0.1, 0.2]) \\ ([0.3, 0.6], [0.2, 0.4]) & ([0.2, 0.5], [0.5, 0.5]) & ([0.1, 0.5], [0.3, 0.5]) \\ ([0.7, 0.9], [0.1, 0.1]) & ([0.6, 0.8], [0.1, 0.2]) & ([0.7, 0.9], [0.1, 0.1]) \end{bmatrix}$$

The matrix $(\lambda, \mu) \circ A$ is

$$\begin{bmatrix} ([0.1, 0.2], [0.1, 0.3]) & ([0.5, 0.6], [0.1, 0.3]) & ([0.2, 0.6], [0.1, 0.2]) \\ ([0.3, 0.6], [0.1, 0.3]) & ([0.2, 0.5], [0.1, 0.3]) & ([0.1, 0.5], [0.1, 0.3]) \\ ([0.5, 0.6], [0.1, 0.1]) & ([0.5, 0.6], [0.1, 0.2]) & ([0.5, 0.6], [0.1, 0.1]) \end{bmatrix}$$

and the matrix $(\lambda', \mu') \circ A$ is

$$\begin{bmatrix} ([0.1, 0.2], [0, 0.2]) & ([0.1, 0.4], [0, 0.2]) & ([0.1, 0.4], [0, 0.2]) \\ ([0.1, 0.4], [0, 0.2]) & ([0.1, 0.4], [0, 0.2]) & ([0.1, 0.4], [0, 0.2]) \\ ([0.1, 0.4], [0, 0.1]) & ([0.1, 0.4], [0, 0.2]) & ([0.1, 0.4], [0, 0.1]) \end{bmatrix}$$

Hence $(\lambda, \mu) \circ A \leq (\lambda', \mu') \circ A$ holds.

Theorem 4.4. Let A and B be IVIFSMs. For a scalar $(\lambda, \mu) = ([\lambda_1, \lambda_2], [\mu_1, \mu_2])$ and $A \leq B$, $(\lambda, \mu) \circ A \leq (\lambda, \mu) \circ B$ holds.

Proof. Let $A = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])$ and $B = ([m_{ij}, n_{ij}], [p_{ij}, r_{ij}])$ be IVIFSMs. For $\forall 1 \leq i, j \leq n$ and scalar $(\lambda, \mu) = ([\lambda_1, \lambda_2], [\mu_1, \mu_2])$,

$$\begin{aligned} (\lambda, \mu) \circ A &= \left(\left[\max_k \left\{ \min \left\{ \lambda_{ik}^{(1)}, a_{kj} \right\} \right\}, \max_k \left\{ \min \left\{ \lambda_{ik}^{(2)}, b_{kj} \right\} \right\} \right], \right. \\ &\quad \left. \left[\min_k \left\{ \max \left\{ \mu_{ik}^{(1)}, c_{kj} \right\} \right\}, \min_k \left\{ \max \left\{ \mu_{ik}^{(2)}, d_{kj} \right\} \right\} \right] \right) \\ &\leq \left(\left[\max_k \left\{ \min \left\{ \lambda_{ik}^{(1)}, m_{kj} \right\} \right\}, \max_k \left\{ \min \left\{ \lambda_{ik}^{(2)}, n_{kj} \right\} \right\} \right], \right. \\ &\quad \left. \left[\min_k \left\{ \max \left\{ \mu_{ik}^{(1)}, p_{kj} \right\} \right\}, \min_k \left\{ \max \left\{ \mu_{ik}^{(2)}, r_{kj} \right\} \right\} \right] \right) \\ &= (\lambda, \mu) \circ B \end{aligned}$$

Consequently the inequality $(\lambda, \mu) \circ A \leq (\lambda, \mu) \circ B$ is derived. $\square$

Example 4.4. Let $(\lambda, \mu) = ([0.5, 0.6], [0.1, 0.3])$,

$$A = \begin{bmatrix} ([0.1, 0.2], [0.1, 0.5]) & ([0.5, 0.6], [0.1, 0.3]) & ([0.2, 0.7], [0.1, 0.2]) \\ ([0.3, 0.6], [0.2, 0.4]) & ([0.2, 0.5], [0.5, 0.5]) & ([0.1, 0.5], [0.3, 0.5]) \\ ([0.7, 0.9], [0.1, 0.1]) & ([0.6, 0.8], [0.1, 0.2]) & ([0.7, 0.9], [0.1, 0.1]) \end{bmatrix}$$

$$B = \begin{bmatrix} ([0.2, 0.3], [0, 0.3]) & ([0.1, 0.4], [0, 0.2]) & ([0.1, 0.5], [0, 0.1]) \\ ([0.4, 0.7], [0.1, 0.3]) & ([0, 0.3], [0.2, 0.4]) & ([0, 0.4], [0.2, 0.4]) \\ ([0.1, 0.8], [0, 0]) & ([0.5, 0.6], [0, 0.1]) & ([0.6, 0.7], [0, 0]) \end{bmatrix}$$

The matrix $(\lambda, \mu) \circ A$ is

$$\begin{bmatrix} ([0.1, 0.2], [0.1, 0.3]) & ([0.5, 0.6], [0.1, 0.3]) & ([0.2, 0.6], [0.1, 0.2]) \\ ([0.3, 0.6], [0.1, 0.3]) & ([0.2, 0.5], [0.1, 0.3]) & ([0.1, 0.5], [0.1, 0.3]) \\ ([0.5, 0.6], [0.1, 0.1]) & ([0.5, 0.6], [0.1, 0.2]) & ([0.5, 0.6], [0.1, 0.1]) \end{bmatrix}$$

and the matrix $(\lambda, \mu) \circ B$ is

$$\begin{bmatrix} ([0.1, 0.2], [0.1, 0.3]) & ([0.5, 0.6], [0.1, 0.3]) & ([0.2, 0.6], [0.1, 0.2]) \\ ([0.3, 0.6], [0.1, 0.3]) & ([0.2, 0.5], [0.1, 0.3]) & ([0.1, 0.5], [0.1, 0.3]) \\ ([0.5, 0.6], [0.1, 0.1]) & ([0.5, 0.6], [0.1, 0.2]) & ([0.5, 0.6], [0.1, 0.1]) \end{bmatrix}$$

This yields $(\lambda, \mu) \circ A \leq (\lambda, \mu) \circ B$.

Theorem 4.5. Let A be an IVIFSM and $(\lambda, \mu), (\alpha, \beta)$ be scalars. Then, $((\lambda, \mu) \circ (\alpha, \beta)) \circ A = (\lambda, \mu) \circ ((\alpha, \beta) \circ A)$ is satisfied.

Proof. Let $A = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])$ be an IVIFSM. For $\forall 1 \leq i, j \leq n$ and scalars $(\lambda, \mu) = ([\lambda_1, \lambda_2], [\mu_1, \mu_2])$, $(\alpha, \beta) = ([\alpha_1, \alpha_2], [\beta_1, \beta_2])$,

$$(\lambda, \mu) \circ (\alpha, \beta) = \left(\left[\max_k \left\{ \min \left\{ \lambda_{ik}^{(1)}, \alpha_{kj} \right\} \right\}, \max_k \left\{ \min \left\{ \lambda_{ik}^{(2)}, \alpha_{kj} \right\} \right\} \right], \right. \\ \left. \left[\min_k \left\{ \max \left\{ \mu_{ik}^{(1)}, \beta_{kj} \right\} \right\}, \min_k \left\{ \max \left\{ \mu_{ik}^{(2)}, \beta_{kj} \right\} \right\} \right] \right)$$

is found. From this

$$((\lambda, \mu) \circ (\alpha, \beta)) \circ A = \left[\max_k \left\{ \min \left\{ \left(\max_k \left\{ \min \left\{ \lambda_{ik}^{(1)}, \alpha_{kj} \right\} \right\} \right), a_{ik} \right\} \right\}, \right. \\ \left. \left\{ \min_k \left\{ \left(\max_k \left\{ \min \left\{ \lambda_{ik}^{(2)}, \alpha_{kj} \right\} \right\} \right), b_{ik} \right\} \right\} \right], \\ \left[\min_k \left\{ \max \left\{ \left(\min_k \left\{ \max \left\{ \mu_{ik}^{(1)}, \beta_{kj} \right\} \right\} \right), c_{ik} \right\} \right\}, \right. \\ \left. \min_k \left\{ \max \left\{ \left(\min_k \left\{ \max \left\{ \mu_{ik}^{(2)}, \beta_{kj} \right\} \right\} \right), d_{ik} \right\} \right\} \right]$$

is established. $\square$

Theorem 4.6. *Let A be an IVIFSM and $(\lambda, \mu), (\alpha, \beta)$ be scalars. Then*

$$H(((\lambda, \mu) \circ (\alpha, \beta)) \circ A, (\lambda, \mu) \circ ((\alpha, \beta) \circ A)) = H((\lambda, \mu), (\lambda, \mu)) \wedge H((\lambda, \mu), (\alpha, \beta)) \\ \wedge H((\lambda, \mu), A) \wedge H((\alpha, \beta), (\lambda, \mu)) \wedge H((\alpha, \beta), (\alpha, \beta)) \wedge H((\alpha, \beta), A) \\ \wedge H(A, (\lambda, \mu)) \wedge H(A, (\alpha, \beta)) \wedge H(A, A)$$

holds true.

Proof. The proof follows straightforwardly from Theorem 4.5 and the properties of the Hamming distance measure. $\square$

Theorem 4.7. *Let A be an IVIFSM and (λ, μ) be a scalar. Then*

$$\ddot{O}(((\lambda, \mu) \circ \mu) \circ A, (\lambda, \mu) \circ (\mu \circ A)) = \ddot{O}((\lambda, \mu), (\lambda, \mu)) \wedge \ddot{O}((\lambda, \mu), \mu) \\ \wedge \ddot{O}((\lambda, \mu), A) \wedge \ddot{O}(\mu, (\lambda, \mu)) \wedge \ddot{O}(\mu, \mu) \wedge \ddot{O}(\mu, A) \\ \wedge \ddot{O}(A, (\lambda, \mu)) \wedge \ddot{O}(A, \mu) \wedge \ddot{O}(A, A)$$

is valid.

Proof. The proof is directly observed from Theorem 4.5 and the properties of the Euclidean distance measure. $\square$

Theorem 4.8. *Let A and B be IVIFSMs and let (λ, μ) be a scalar. Then $H((\lambda, \mu) \circ A, B) = H((\lambda, \mu), B) \wedge H(A, B)$ is derived.*

Proof. Let $A = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])$ and $B = ([m_{ij}, n_{ij}], [p_{ij}, r_{ij}])$ be IVIFSMs. For $\forall 1 \leq i, j \leq n$ and scalar $(\lambda, \mu) = ([\lambda_1, \lambda_2], [\mu_1, \mu_2])$,

$$(\lambda, \mu) \circ A = \left(\left[\max_k \left\{ \min \left\{ \lambda_{ik}^{(1)}, a_{kj} \right\} \right\}, \max_k \left\{ \min \left\{ \lambda_{ik}^{(2)}, b_{kj} \right\} \right\} \right], \right. \\ \left. \left[\min_k \left\{ \max \left\{ \mu_{ik}^{(1)}, c_{kj} \right\} \right\}, \min_k \left\{ \max \left\{ \mu_{ik}^{(2)}, d_{kj} \right\} \right\} \right] \right)$$

is formulated. Consequently, by the definition of the Hamming distance measure, $H((\lambda, \mu) \circ A, B) = H((\lambda, \mu), B) \wedge H(A, B)$ is obtained. $\square$

Theorem 4.9. *Let A and B be IVIFSMs and let (λ, μ) be a scalar. Then $\ddot{O}((\lambda, \mu) \circ A, B) = \ddot{O}((\lambda, \mu), B) \wedge \ddot{O}(A, B)$ is established.*

Proof. Let $A = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])$ and $B = ([m_{ij}, n_{ij}], [p_{ij}, r_{ij}])$ be IVIFSMs. For $\forall 1 \leq i, j \leq n$ and scalar $(\lambda, \mu) = ([\lambda_1, \lambda_2], [\mu_1, \mu_2])$,

$$(\lambda, \mu) \circ A = \left(\left[\max_k \left\{ \min \left\{ \lambda_{ik}^{(1)}, a_{kj} \right\} \right\}, \max_k \left\{ \min \left\{ \lambda_{ik}^{(2)}, b_{kj} \right\} \right\} \right], \right. \\ \left. \left[\min_k \left\{ \max \left\{ \mu_{ik}^{(1)}, c_{kj} \right\} \right\}, \min_k \left\{ \max \left\{ \mu_{ik}^{(2)}, d_{kj} \right\} \right\} \right] \right)$$

is formulated. By the properties of the Euclidean distance measure, $\ddot{O}((\lambda, \mu) \circ A, B) = \ddot{O}((\lambda, \mu), B) \wedge \ddot{O}(A, B)$ is derived. $\square$

Corollary 4.1. *Let A and B be IVIFSMs. The relation $H(A \circ B) = H(A) \wedge H(B)$ holds true.*

Proof. Taking the scalar $(\lambda, \mu) = ([1, 1], [0, 0])$ in Theorem 4.8, the proof is readily observed. $\square$

Corollary 4.2. *Let A and B be IVIFSMs. The equality $\ddot{O}(A \circ B) = \ddot{O}(A) \wedge \ddot{O}(B)$ is satisfied.*

Proof. Setting the scalar $(\lambda, \mu) = ([1, 1], [0, 0])$ in Theorem 4.8, the assertion follows directly. $\square$

Theorem 4.10. *Let A and B be IVIFSMs and let (λ, μ) be a scalar. Then $(\lambda, \mu) \circ (A \circ B) = ((\lambda, \mu) \circ A) \circ ((\lambda, \mu) \circ B)$ is valid.*

Proof. Let $A = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])$ and $B = ([m_{ij}, n_{ij}], [p_{ij}, r_{ij}])$ be IVIFSMs. For $\forall 1 \leq i, j \leq n$ and scalar $(\lambda, \mu) = ([\lambda_1, \lambda_2], [\mu_1, \mu_2])$, then

$$(\lambda, \mu) \circ A = \left(\left[\max_k \left\{ \min \left\{ \lambda_{ik}^{(1)}, a_{kj} \right\} \right\}, \max_k \left\{ \min \left\{ \lambda_{ik}^{(2)}, b_{kj} \right\} \right\} \right], \right. \\ \left. \left[\min_k \left\{ \max \left\{ \mu_{ik}^{(1)}, c_{kj} \right\} \right\}, \min_k \left\{ \max \left\{ \mu_{ik}^{(2)}, d_{kj} \right\} \right\} \right] \right) \\ (\lambda, \mu) \circ B = \left(\left[\max_k \left\{ \min \left\{ \lambda_{ik}^{(1)}, m_{kj} \right\} \right\}, \max_k \left\{ \min \left\{ \lambda_{ik}^{(2)}, n_{kj} \right\} \right\} \right], \right. \\ \left. \left[\min_k \left\{ \max \left\{ \mu_{ik}^{(1)}, p_{kj} \right\} \right\}, \min_k \left\{ \max \left\{ \mu_{ik}^{(2)}, r_{kj} \right\} \right\} \right] \right) \\ (\lambda, \mu) \circ (A \circ B) = \\ \left(\left[\max_k \left\{ \min \left\{ \lambda_{ik}^{(1)}, a_{kj} \right\} \right\}, \max_k \left\{ \min \left\{ \lambda_{ik}^{(2)}, b_{kj} \right\} \right\} \right], \right. \\ \left. \left[\min_k \left\{ \max \left\{ \mu_{ik}^{(1)}, c_{kj} \right\} \right\}, \min_k \left\{ \max \left\{ \mu_{ik}^{(2)}, d_{kj} \right\} \right\} \right] \right) \\ = ((\lambda, \mu) \circ A) \circ ((\lambda, \mu) \circ B)$$

is obtained. $\square$

Theorem 4.11. *Let A , B and C be IVIFSMs. The identity $H(A \circ (B \circ C)) = H((A \circ B) \circ C)$ is confirmed.*

Proof. Let

$A = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])$, $B = ([m_{ij}, n_{ij}], [p_{ij}, r_{ij}])$ and $C = ([s_{ij}, t_{ij}], [y_{ij}, z_{ij}])$ be IVIFSMs. For $\forall 1 \leq i, j \leq n$,

$$\begin{aligned} B \circ C &= \left(\left[\max_k \left\{ \min \left\{ m_{ik}^{(1)}, s_{kj}^{(2)} \right\} \right\}, \max_k \left\{ \min \left\{ n_{ik}^{(1)}, t_{kj}^{(2)} \right\} \right\} \right], \right. \\ &\quad \left. \left[\min_k \left\{ \max \left\{ p_{ik}^{(1)}, y_{kj}^{(2)} \right\} \right\}, \min_k \left\{ \max \left\{ r_{ik}^{(1)}, z_{kj}^{(2)} \right\} \right\} \right] \right) \\ A \circ B &= \left(\left[\max_k \left\{ \min \left\{ a_{ik}^{(1)}, m_{kj}^{(2)} \right\} \right\}, \max_k \left\{ \min \left\{ b_{ik}^{(1)}, n_{kj}^{(2)} \right\} \right\} \right], \right. \\ &\quad \left. \left[\min_k \left\{ \max \left\{ c_{ik}^{(1)}, p_{kj}^{(2)} \right\} \right\}, \min_k \left\{ \max \left\{ d_{ik}^{(1)}, r_{kj}^{(2)} \right\} \right\} \right] \right) \\ A \circ (B \circ C) &= \left[\max_k \left\{ \min \left\{ a_{ik}, \left(\max_k \left\{ \min \left\{ m_{ik}^{(1)}, s_{kj}^{(2)} \right\} \right\} \right) \right\} \right\}, \right. \\ &\quad \left. \max_k \left\{ \min \left\{ b_{ik}, \left(\max_k \left\{ \min \left\{ n_{ik}^{(1)}, t_{kj}^{(2)} \right\} \right\} \right) \right\} \right\} \right], \\ &\quad \left[\min_k \left\{ \max \left\{ c_{ik}, \left(\min_k \left\{ \max \left\{ p_{ik}^{(1)}, y_{kj}^{(2)} \right\} \right\} \right) \right\} \right\}, \right. \\ &\quad \left. \min_k \left\{ \max \left\{ d_{ik}, \left(\min_k \left\{ \max \left\{ r_{ik}^{(1)}, z_{kj}^{(2)} \right\} \right\} \right) \right\} \right\} \right] \\ (A \circ B) \circ C &= \left[\max_k \left\{ \min \left\{ \left(\max_k \left\{ \min \left\{ a_{ik}^{(1)}, m_{kj}^{(2)} \right\} \right\} \right), s_{ik} \right\} \right\}, \right. \\ &\quad \left. \max_k \left\{ \min \left\{ \left(\max_k \left\{ \min \left\{ b_{ik}^{(1)}, n_{kj}^{(2)} \right\} \right\} \right), t_{ik} \right\} \right\} \right], \\ &\quad \left[\min_k \left\{ \max \left\{ \left(\min_k \left\{ \max \left\{ c_{ik}^{(1)}, p_{kj}^{(2)} \right\} \right\} \right), y_{ik} \right\} \right\}, \right. \\ &\quad \left. \min_k \left\{ \max \left\{ \left(\min_k \left\{ \max \left\{ d_{ik}^{(1)}, r_{kj}^{(2)} \right\} \right\} \right), z_{ik} \right\} \right\} \right] \end{aligned}$$

Consequently, $H(A \circ (B \circ C)) = H((A \circ B) \circ C)$ is obtained. $\square$

Corollary 4.3. *Let A , B and C be IVIFSMs. The relation*

$$\begin{aligned} H(A \circ (B \circ C), (A \circ B) \circ C) &= H(A, A) \wedge H(A, B) \wedge H(A, C) \wedge H(B, A) \\ &\quad \wedge H(B, B) \wedge H(B, C) \wedge H(C, A) \wedge H(C, B) \wedge H(C, C) \end{aligned}$$

is established.

Corollary 4.4. *Let A , B and C be IVIFSMs. The identity*

$$\begin{aligned} \ddot{O}(A \circ (B \circ C), (A \circ B) \circ C) &= \ddot{O}(A, A) \wedge \ddot{O}(A, B) \wedge \ddot{O}(A, C) \wedge \ddot{O}(B, A) \\ &\quad \wedge \ddot{O}(B, B) \wedge \ddot{O}(B, C) \wedge \ddot{O}(C, A) \wedge \ddot{O}(C, B) \wedge \ddot{O}(C, C) \end{aligned}$$

holds true.

5. CONCLUSION

The main contributions of this study are twofold. First, four measures—Hamming, Generalized Hamming, Euclidean, and Generalized Euclidean—are formulated for interval-valued intuitionistic fuzzy matrices, and their fundamental properties are established, including invariance under transposition. Second, a scalar multiplication operation based on interval-valued scalars is introduced, and its algebraic properties, including associativity and distributive laws, are investigated. The theoretical results are supported by illustrative examples. Overall, this study extends the algebraic framework of interval-valued intuitionistic fuzzy matrices and provides a foundation for further theoretical developments and applications in uncertainty modeling and decision-making.

6. ACKNOWLEDGMENTS

The authors would like to thank the reviewers and editors of Journal of Universal Mathematics.

Author's Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Availability of Data and Materials

Not applicable.

Funding

The authors declare that they have received no financial support for this study.

The Declaration of Conflict of Interest/ Common Interest

The authors declared that no conflict of interest or common interest.

The Declaration of Ethics Committee Approval

This study does not be necessary ethical committee permission or any special permission.

The Declaration of Research and Publication Ethics

The authors declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the authors declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

REFERENCES

- [1] K. T. Atanassov, *Intuitionistic fuzzy sets*, VII ITKR's Session, Sofia, June 1983 (Deposed in Central Sci.-Techn. Library of Bulg. Acad. of Sci., No. 1697/84) (in Bulgarian), (1983).
- [2] K. T. Atanassov, *Intuitionistic fuzzy sets and interval valued intuitionistic fuzzy sets*, Advanced Studies in Contemporary Mathematics, Vol.28, No.2, pp.167-176 (2018).

- [3] K. T. Atanassov, G. Gargov, *Interval valued intuitionistic fuzzy sets*, Fuzzy Sets and Systems, Vol.31, No.3, pp.343-349 (1989).
- [4] Y. Wang, Y. J. Lei, *A technique for constructing intuitionistic fuzzy entropy*, Control and Decision, Vol.22, No.12, pp.1390-1394 (2007).
- [5] C. P. Wei et al., *Entropy, similarity measure of interval-valued intuitionistic fuzzy sets and their applications*, Information Sciences, Vol.181, pp.4273-4286 (2011).
- [6] Z. S. Xu, *A method based on distance measure for interval-valued intuitionistic fuzzy group decision making*, Information Sciences, Vol.180, pp.181-190 (2010).
- [7] L. A. Zadeh, *Fuzzy sets*, Information and Control, Vol.8, pp.338-353 (1965).
- [8] Q. Zhang et al., *Some new entropy measures for interval-valued intuitionistic fuzzy sets based on distances and their relationships with similarity and inclusion measures*, Information Sciences, Vol.283, pp.55-69 (2014).

MERSIN UNIVERSITY, FACULTY OF ARTS AND SCIENCES, DEPARTMENT OF MATHEMATICS,
MERSIN, TÜRKİYE

Email address: gcuvalcioglu@mersin.edu.tr

MERSIN UNIVERSITY SCIENCE INSTITUTE, MERSIN, TÜRKİYE

Email address: gizemartn@gmail.com