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On generalized derivations of incline algebras

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Abstract

In this paper, as a generalization of derivation of an incline algebra, the notion of generalized derivation in an incline algebra is introduced and some of its properties are investigated in an incline and integral incline algebra.

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1. Introduction

The notion of derivation on a ring has an important role for the characterization of rings. As generalizations of derivations, α-derivations, generalized derivations on prime and semi-prime rings are studied by a lot of researchers [10], [6]. In [11] the notion of derivation on a lattice was defined and some of its related properties were examined firstly by Szasz , G. In [5] the problems initiated by Szasz are pursued and completed by Luca Ferrari. Later, derivations, f-derivations, symmetric bi-derivations, symmetric f-biderivations on a lattice and some properties related with these derivations were discussed by Xin, X. L. , Li, T.Y. and Lu, J. H. , and Öztürk, M.A. and Çeven, Y.and A. Firat and S. Ayar Özbal in [12], [4],[4],[8] respectively.

In [2] the notion of inclines is introduced and their applications are studied by Cao, Z. Q. , Kim, K. H. , and Roush, F. W. . The notion of derivation for an incline algebra is introduced by Al-Shehri, N. O. in [1] and he discussed some of its properties. We introduced the notion of f-derivation in an incline algebra and studied it some properties in [9]. In this paper, as a generalization of derivation of an incline algebra, the notion of generalized derivation in an incline algebra is introduced and, some related properties are investigated.

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2. Preliminaries

2.1. Definition. [2] An incline algebra is a non-empty set R with binary operations denoted by $+$ and $*$ satisfying the following axioms for all $x, y, z \in R$:

(RI) $x + y = y + x$,

(RII) $x + (y + z) = (x + y) + z$,

(RIII) $x * (y * z) = (x * y) * z$,

(RIV) $x * (y + z) = (x * y) + (x * z)$,

(RV) $(y + z) * x = (y * x) + (z * x)$,

(RVI) $x + x = x$,

(RVII) $x + (x * y) = x$,

(RVIII) $y + (x * y) = y$.

Furthermore, an incline algebra R is said to be commutative if $x * y = y * x$ for all $x, y \in R$.

For convenience, we pronounce " $+$ " (resp." $*$ ") as addition (resp. multiplication). Every distributive lattice is an incline. An incline is a distributive lattice (as a semiring) if and only if $x * x = x$ for all $x \in R$ ([3, Proposition (1.1.1)]). A subincline of an incline R is a nonempty subset M of R which is closed under addition and multiplication. An ideal in an incline R is a subincline $M \subseteq R$ such that if $x \in M$ and $y \leq x$ then $y \in M$. An element 0 in an incline algebra R is a zero element if $x + 0 = x = 0 + x$ and $x * 0 = 0 * x = 0$ for any $x \in R$. An element 1 ($\neq$ zero element) in an incline algebra R is called a multiplicative identity if for any $x \in R$, $x * 1 = 1 * x = x$. A non-zero element a in an incline algebra R with zero element is said to be a left (resp. right) zero divisor if there exists a non-zero element $b \in R$ such that $a * b = 0(resp.\ b * a = 0)$. A zero divisor is an element of R which is both a left zero divisor and a right zero divisor. An incline algebra R with multiplicative identity 1 and zero element 0 is called an integral incline if it has no zero divisors.

Note that $x \leq y$ if and only if $x + y = y$ for all $x, y \in R$. It is easy to see that $\leq$ is a partial order on R and that for any $x, y \in R$, the element $x + y$ is the least upper bound of $\{x, y\}$. We say that $\leq$ is induced by operation $+$. It follows that

(1) $x * y \leq x$ and $y * x \leq x$ for all $x, y \in R$.

(2) $y \leq z$ implies $x * y \leq x$ and $y * x \leq z * x$ for any $x, y, z \in R$.

(3) If $x \leq y$, $a \leq b$, then $x + a \leq y + b$, $x * a \leq y * b$.

2.2. Definition. [1] Let R be an incline and $d : R \to R$ be a function. d is called as a derivation of R if it satisfies the following condition

$$d(x * y) \quad = \quad (d(x) * y) + (x * d(y))$$

for all $x, y \in R$.

2.3. Proposition. [1] Let R be an incline with a zero element and d be a derivation of R. Then $d0 = 0$.

3. The Generalized Derivations of Incline Algebras

The following definition introduces the notion of a generalized derivation for an incline algebra.

3.1. Definition. Let R be an incline algebra. A function $D : R \to R$ is called a generalized derivation of R, if there exists a derivation d of R such that

$$D(x * y) \quad = \quad (D(x) * \ y) + (x * \quad d(y))$$

for all $x, y \in R$.

3.1. Example. Let $R = \{0, a, b, c, d, 1\}$, and the sum " $+$ " and product " $*$ " be defined on R as follows:

+	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	1	1	1	1
b	b	1	b	1	b	1
c	c	1	1	c	1	1
d	d	1	b	c	d	1
1	1	1	1	1	1	1

*	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	0
b	0	0	d	d	d	d
c	0	0	d	d	d	d
d	0	0	d	d	d	d
1	0	0	d	d	d	d

Then $(R, +, *)$ is an incline but not a distributive lattice [7].
We define a function $D : R \to R$ for all x in R by

$$D(x) = \begin{cases} 0, & \text{x=0,c,1} \\ a, & \text{x=a} \\ d, & \text{b,d} \end{cases}$$

Then D is not a derivation of R since $D(1 * \ c) = D(d) = d$, but $(D(1) * c) + (1 * D(c)) = (0 * c) + (1 * 0) = 0 + 0 = 0$ and thus $D(1 * \quad c) \quad \neq (D(1) * \ c) + \ (1 * \ D(c))$.

But if we define d as a derivation of R as

$$d(x) = \begin{cases} 0, & \text{x=0,a} \\ d, & \text{otherwise} \end{cases}$$

then D satisfies the equation in Definition 3.1, i.e. D is a generalized derivation of R.

3.2. Proposition. Let R be an incline algebra and D be a generalized derivation of R associated with a derivation d of R. Then the followings hold for all x, y in R:

(i) $D(x * y) \leq D(x) + d(y)$,

(ii) If $x \leq y$ and then $D(x * y) \leq y$.

Proof:

(i) Let $x, y \in R$. We know that from (1) we have $D(x) * y \leq D(x)$ and $x * d(y) \leq d(y)$. Then by using (3) we get $D(x * y) = (D(x) * y) + (x * d(y)) \leq D(x) + d(y)$.
Hence we find $D(x * y) \leq D(x) + d(y)$.

(ii) Let $x \leq y$. Then by using (3) and (1) we get $x * d(y) \leq y * d(y) \leq y$. Similarly, we can get $D(x) * y \leq y$. Then we obtain, $D(x * y) = (D(x) * y) + (x * d(y)) \leq y + y = y$. Hence we have $D(x * y) \leq y$.

3.3. Proposition. Let R be an incline algebra with a zero element and D be a generalized derivation of R associated with a derivation d of R. Then $D(0) = 0$.

Proof: Since R is an incline algebra with a zero element we have $x * 0 = 0 * x = 0$ for all $x \in R$ then we can write
$D(0) = D(x*0) = (D(x)*0) + (x*d(0)) = x*d(0)$. By Proposition 2.3 we have $d(0) = 0$. Therefore $D(0) = 0$.

3.4. Proposition. Let R be an incline algebra with a multiplicative identity element and D be a generalized derivation of R associated with a derivation d of R. Then the followings hold for all $x \in R$:

(i) $x * d(1) \leq D(x)$,

(ii) If $d(1) = 1$, then $x \leq D(x)$.

Proof:

(i) Since R is an incline algebra with a multiplicative identity element we have $x * 1 = 1 * x = x$ for all $x \in R$, then we can write $D(x) = D(x * 1) = (D(x) * 1) + (x * d(1))$. Then we have $D(x) = D(x) + (x * d(1))$. Therefore we get, $x * d(1) \leq D(x)$.

(ii) It can be derived from (i).

3.5. Proposition. Let R be an integral incline and D be a generalized derivation of R associated with a derivation d of R and a be an element of R. Then for all $x \in R$ we have:

(i) $a * D(x) = 0$ implies that $a = 0$ or $d = 0$,

(ii) $d(x) * a = 0$ and $D(x) * a = 0$ imply that $a = 0$ or $D = 0$.

Proof:

(i) Let $a * D(x) = 0$ for all $x \in R$. If we replace x by $x * y$ for $y \in R$ we get

$$\begin{aligned} 0 = a * D(x) = a * D(x * y) &= a * [(D(x) * y) + (x * d(y))] \\ &= (a * (D(x) * y)) + (a * (x * d(y))) \\ &= a * (x * d(y)) \end{aligned}$$

In this equation by taking $x = 1$ we get $a * d(y) = 0$. Since R is an integral incline we have $a = 0$ or $d = 0$.

(ii) Let $d(x) * a = 0$ and $D(x) * a = 0$ for all $x \in R$.If we replace x by $x * y$ for $y \in R$ in $D(x) * a = 0$ we get

$$\begin{aligned} 0 = D(x) * a = D(x * y) * a &= [(D(x) * y) + (x * d(y))] * a \\ &= ((D(x) * y) * a) + ((x * d(y)) * a) \\ &= (D(x) * y) * a \end{aligned}$$

In this equation by taking $y = 1$ we get $D(x) * a = 0$. Since R is an integral incline we have $a = 0$ or $D = 0$.

3.6. Theorem. Let M be a nonzero ideal of an integral incline R. If D is a nonzero generalized derivation of R associated with a nonzero derivation d of R, then D is nonzero on M.

Proof: Assume that D is a nonzero generalized derivation of R associated with a nonzero derivation d of R but D is zero generalized derivation on M $x \in M$. Then we have $D(x) = 0$. Let $y \in R$. By (1) $x * y \leq x$ and since M is an ideal of R, we have $D(x * y) = 0$, so we can write that

$0 = D(x * y) = (D(x) * y) + (x * d(y)) = x * d(y)$.

We know by our assumption that R has no zero divisors, so we have $x = 0$ for all $x \in M$ or $d(y) = 0$ for all $y \in R$. Since M is a nonzero ideal of R, we get that $d(y) = 0$ for all $y \in R$. This contradicts with our assumption that is d is a nonzero derivation on R. Hence, D is nonzero on M.

3.7. Theorem. Let D be a nonzero generalized derivation of an integral incline R associated with a nonzero derivation d on M. If M is a nonzero ideal of R, and $a \in R$ such that $a * D(M) = 0$, then $a = 0$.

Proof: By Theorem 3.6 we know that there is an element m in M such that $D(m) \neq 0$. Let M be a nonzero ideal of R, and $a \in R$ such that $a * d(M) = 0$. Then for $m, n \in M$ we can write

$$\begin{aligned} 0 = a * D(m * n) = a * (D(m) * n + m * d(n)) &= a * D(m) * n + a * m * d(n) \\ &= a * m * d(n). \end{aligned}$$

Since R is an integral incline, d is a nonzero derivation on M and M is a nonzero ideal we have $a = 0$.

3.8. Definition. Let D be a generalized derivation of an incline algebra R associated with a derivation of R. If $x \leq y$ implies that $D(x) \leq D(y)$ for all $x, y \in R$, then D is called an isotone generalized derivation.

3.9. Proposition. Let D be a generalized derivation of an incline algebra R associated with a derivation d of R. If for all $x, y \in R$ we have $D(x + y) = D(x) + D(y)$, then the followings hold for all $x, y \in R$:

(i) $D(x * y) \leq D(x)$,

(ii)$D(x * y) \leq D(y)$,

(iii) D is an isotone derivation.

Proof:

(i) Let $x, y \in R$. By using $(R7)$ we have $D(x) = D(x + x * y) = D(x) + D(x * y)$. Therefore we get that $D(x * y) \leq D(y)$.

(ii) Proof can be get similarly with the previous one.

(iii) Let $x \leq y$, then we have $x + y = y$, therefore we have $D(y) = D(x + y) = D(x) + D(y)$. Hence $D(x) \leq D(y)$.

3.10. Theorem. Let D_1, D_2 be generalized derivations of an incline algebra R associated with derivations d_1, d_2 derivations of R respectively. Let $D_1(x+y) = D_1(x) + D_1(y)$ and $D_2(x+y) = D_2(x) + D_2(y)$ for all $x, y \in R$. Define $D_1 D_2(x) = D_1(D_2(x))$ for all x in R. If $D_1 D_2 = 0$ and $d_1 d_2 = 0$ then $D_2 D_1$ is a generalized derivation of R associated with a derivation $d_2 d_1$ derivation of R.

Proof: Let $D_1 D_2 = 0$. Then we have

$$\begin{aligned} D_1 D_2(x * y) &= D_1(D_2(x) * y + x * d_2(y) \\ &= D_1(D_2(x)) * y + D_2(x) * d_1(y) + D_1(x) * d_2(y) + x * d_1 d_2(y). \end{aligned}$$

Therefore we get $D_2(x) * d_1(y) + D_1(x) * d_2(y) = 0$.

$$\begin{aligned} D_2 D_1(x * y) &= D_2(D_1(x) * y + x * d_1(y) \\ &= D_2(D_1(x)) * y + D_1(x) * d_2(y) + D_2(x) * d_1(y) + x * d_2 d_1(y) \\ &= D_2(D_1(x)) * y + x * d_2 d_1(y). \end{aligned}$$

Hence $D_2 D_1$ is a generalized derivation of R associated with a derivation $d_2 d_1$ derivation of R.

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