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ABEL STATISTICAL DELTA QUASI CAUCHY SEQUENCES OF REAL NUMBERS

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Abstract. In this paper, we investigate the concept of Abel statistical delta quasi Cauchy sequences. A real function f is called Abel statistically delta ward continuous it preserves Abel statistical delta quasi Cauchy sequences, where a sequence (α_k) of points in $\mathbb{R}$ is called Abel statistically delta quasi Cauchy if $\lim_{x\to 1^-}(1-x)\sum_{k:|\Delta^2\alpha_k|\geq\varepsilon} x^k = 0$ for every $\varepsilon > 0$, where $\Delta^2\alpha_k = \alpha_{k+2} - 2\alpha_{k+1} + \alpha_k$ for every $k \in \mathbb{N}$. Some other types of continuities are also studied and interesting results are obtained.

1. Introduction

Throughout this paper, $\mathbb{N}$, and $\mathbb{R}$ will denote the set of positive integers, and the set of real numbers, respectively. The boldface letters such as $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\zeta}$ will be used for sequences $\boldsymbol{\alpha} = (\alpha_n)$, $\boldsymbol{\beta} = (\beta_n)$, $\boldsymbol{\zeta} = (\zeta_n)$, ... of points in $\mathbb{R}$. A real function f is continuous if and only if it preserves Abel statistical convergence, i.e. for each point ℓ in the domain, $Abel_{st} - \lim_{n\to\infty} f(\alpha_n) = f(\ell)$ whenever $Abel_{st} - \lim_{n\to\infty} \alpha_n = \ell$.

Using the idea of continuity of a real function in this manner, many kinds of continuities were introduced and investigated, not all but some of them we recall in the following: ward continuity ([12], [5]), p-ward continuity ([19]), δ-ward continuity ([15]), δ^2-ward continuity ([4]), statistical ward continuity, ([16]), λ-statistical ward continuity ([29]), ρ-statistical ward continuity ([6], [21]), slowly oscillating continuity ([10, 44, 28]), quasi-slowly oscillating continuity ([31]), Δ-quasi-slowly oscillating continuity ([13]), upward and downward statistical continuities ([20]), lacunary statistical ward continuity ([7], [47], and [48]), lacunary statistical δ ward continuity ([25]), lacunary statistical δ^2 ward continuity ([46]), N_θ-ward continuity ([18], [24], [36], [8], [36], [37]), and N_θ-δ-ward continuity ([8]), which enabled some authors to obtain interesting results.

The purpose of this paper is to introduce and investigate the concept of Abel statistical δ-ward continuity of a real function, and prove interesting theorems.

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2. Abel statistical δ quasi Cauchy sequences

A sequence (α_k) is called statistically convergent to an element ℓ of $\mathbb{R}$ if $\lim_{n\to\infty}\frac{1}{n}|\{k \leq n : |\alpha_k - \ell| \geq \varepsilon\}| = 0$ for each $\varepsilon > 0$ (see [34], [14], [21], and [26]).

A sequence (α_k) of real numbers is called Abel convergent (or Abel summable) to ℓ if the series

$$\Sigma_{k=0}^{\infty}\alpha_k x^k$$

is convergent for $0 \leq x < 1$ and $\lim_{x\to 1^-}(1-x)\sum_{k=0}^{\infty}\alpha_k x^k = \ell$ ([1], [3], and [35]). In this case, we write $Abel-\lim \alpha_k = \ell$. The concept of a Cauchy sequence involves far more than that the distance between successive terms is tending to 0 and specially speaking, than that the distance between successive terms is Abel convergent to zero. Nevertheless, sequences which satisfy this weaker property, i.e. Abel quasi Cauchy sequences satisfying $Abel - lim\Delta\ \alpha_k = 0$, are interesting in their own right. In other words, a sequence (α_k) of points in $\mathbb{R}$ is called Abel quasi-Cauchy if $(\Delta\alpha_k)$ is Abel convergent to 0, i.e. the series

$$\sum_{k=0}^{\infty}\Delta\alpha_k x^k$$

is convergent for $0 \leq x < 1$ and

$$\lim_{x\to 1^-}(1-x)\sum_{k=0}^{\infty}\Delta\alpha_k x^k = 0$$

where $\Delta\alpha_k = \alpha_{k+1} - \alpha_k$.

Recently the concept of Abel statistical convergence of a sequence is investigated in [43] in the sense that a sequence (α_k) is called Abel statistically convergent to a real number L if $4\lim_{x\to 1^-}(1-x)\sum_{k:|\alpha_k - L|\geq\varepsilon} x^k = 04$ for every $\varepsilon > 0$, and denoted by $Abel_{st} - lim\ \alpha_k = L$.

A sequence (α_k) of points in $\mathbb{R}$ is called Abel statistically quasi Cauchy if

$$\lim_{x\to 1^-}(1-x)\sum_{k:|\Delta\alpha_k|\geq\varepsilon} x^k = 0$$

for every $\varepsilon > 0$ ([30]).

Now we introduce the concept of Abel statistically δ quasi Cauchyness in the following:

Definition 2.1. *A sequence of points in a subset A of $\mathbb{R}$ is called Abel statistically δ quasi Cauchy if*

$$\lim_{x\to 1^-}(1-x)\sum_{k:|\Delta^2\alpha_k|\geq\varepsilon} x^k = 0$$

for every $\varepsilon > 0$, where $\Delta^2\alpha_k = \alpha_{k+2} - 2\alpha_{k+1} + \alpha_k$ for every $k \in \mathbb{N}$.

Any Abel statistically quasi-Cauchy sequence is Abel statistically δ quasi Cauchy, but the converse is not always true. Any quasi-Cauchy sequence is Abel statistically δ quasi Cauchy, but the converse is not always true. Any Abel statistically convergent sequence is Abel statistically δ quasi Cauchy. There are Abel statistically δ quasi Cauchy sequences which are not Abel statistically quasi Cauchy. Since the set of all convergent sequences c is a proper subset of $Abel_{st}^{\delta}$, and $Abel_{st}$ is a proper subset of $Abel_{st}^{\delta^2}$, the set of Abel statistical δ quasi Cauchy sequences, one can easily find that $c \subset \Delta \subset Abel_{st}^{\delta} \subset Abel_{st}^{\delta^2}$, where c, Δ, $\Delta Abel_{st}$, and $\Delta^2 Abel_{st}$, denote the set of convergent sequences, the set of quasi Cauchy sequences, the set of Abel statistically quasi Cauchy sequences, and the set of Abel statistically δ quasi Cauchy sequences.

Theorem 2.1. *The sum of two Abel statistical δ quasi-Cauchy sequences is Abel statistical δ quasi-Cauchy.*

Proof. Let (α_k) and (β_k) be Abel statistical δ quasi-Cauchy sequences of of points in A. Then $\lim_{x\to 1^-}(1-x)\sum_{k:|\Delta^2\alpha_k|\geq\varepsilon} x^k = 0$ and $\lim_{x\to 1^-}(1-x)\sum_{k:|\Delta^2\beta_k|\geq\varepsilon} x^k = 0$ for every $\varepsilon > 0$. Then $\lim_{x\to 1^-}(1-x)\sum_{k:|\Delta^2(\alpha_k+\beta_k)|\geq\varepsilon} x^k \leq \lim_{x\to 1^-}(1-x)\sum_{k:|\Delta^2\alpha_k|\geq\varepsilon} x^k + \lim_{x\to 1^-}(1-x)\sum_{k:|\Delta^2\beta_k|\geq\varepsilon} x^k$. This completes the proof of the theorem. □

Now we give the definition of Abel statistical δ ward compactness.

Definition 2.2. *A subset A of $\mathbb{R}$ is called Abel statistically δ ward compact if any sequence of points in A has an Abel statistical δ quasi-Cauchy subsequence.*

First, we note that any finite subset of $\mathbb{R}$ is Abel statistically δ ward compact, the union of two Abel statistically δ ward compact subsets of $\mathbb{R}$ is Abel statistically δ ward compact and the intersection of any family of Abel statistically δ ward compact subsets of $\mathbb{R}$ is Abel statistically δ ward compact. Any G-sequentially compact subset of $\mathbb{R}$ is Abel statistically δ ward compact for a regular subsequential method G (see [11], [17]). Furthermore any subset of an Abel statistically δ ward compact set is Abel statistically δ ward compact, any bounded subset of $\mathbb{R}$ is Abel statistically δ ward compact, any slowly oscillating compact subset of $\mathbb{R}$ is Abel statistically δ ward compact (see [10] for the definition of slowly oscillating compactness).

Theorem 2.2. *If a function f is uniformly continuous on a subset A of $\mathbb{R}$, then $(f(\alpha_k))$ is Abel statistical δ quasi-Cauchy whenever (α_k) is a quasi-Cauchy sequence of points in A.*

Proof. Take any quasi-Cauchy sequence (α_k) of points in A, and let ε be any positive real number. By uniform continuity of f, there exists a $\delta > 0$ such that $|f(\alpha) - f(\beta)| < \varepsilon$ whenever $|\alpha - \beta| < \delta$ and $\alpha, \beta \in E$. Since (α_k) is a quasi-Cauchy sequence, there exists a positive integer k_0 such that $|\alpha_{k+1} - \alpha_k| < \delta$ for $k \geq k_0$. Thus

$$\lim_{x\to 1^-}(1-x)\sum_{k:|\Delta^2\alpha_k|\geq\varepsilon} x^k = 0.$$

This completes the proof of the theorem. □

Definition 2.3. *A function defined on a subset A of $\mathbb{R}$ is called Abel statistically δ ward continuous if it preserves Abel statistical δ quasi-Cauchy sequences, i.e. $(f(\alpha_n))$ is an Abel statistical δ quasi-Cauchy sequence whenever (α_n) is.*

We note that Abel statistical δ ward continuity cannot be obtained by any sequential method G ([9], [17]). The composition of two Abel statistical δ ward continuous functions is Abel statistical δ ward continuous.

Theorem 2.3. *If f is Abel statistically δ ward continuous on a subset A of $\mathbb{R}$, then it is Abel statistically ward continuous on A.*

Proof. Let (α_n) be any sequence with $Abel_{st}-\lim_{k\to\infty}\Delta\alpha_k = 0$. Then the sequence

$$(\alpha_1, \alpha_1, \alpha_2, \alpha_2, ..., \alpha_n, \alpha_n, ...)$$

is Abel statistical δ quasi-Cauchy hence, by the hypothesis, the sequence

$$(f(\alpha_1), f(\alpha_1), f(\alpha_2), f(\alpha_2), ..., f(\alpha_n), f(\alpha_n), ...)$$

is Abel statistical δ quasi-Cauchy . It follows from this that

$$(f(\alpha_1), f(\alpha_2), ..., f(\alpha_n), ...)$$

is Abel statistical quasi-Cauchy. This completes the proof of the theorem. □

Corollary 2.4. *Any Abel statistically δ ward continuous on a subset A of $\mathbb{R}$ is ordinary continuous on A.*

Theorem 2.5. *The sum of two Abel statistical δ ward continuous functions is Abel statistical δ ward continuous.*

Proof. The proof of this theorem follows easily, so is omitted. □

If c is a constant real number and f is an Abel statistically δ ward continuous function, then cf is Abel statistically δ ward continuous. Thus the set of Abel statistical δ ward continuous functions is a vector subspace of the vector space of continuous functions. Maximum of two Abel statistical δ ward continuous functions is Abel statistical δ ward continuous, and minimum of two Abel statistical δ ward continuous functions is Abel statistical δ ward continuous, which follow from $max\{f, g\} = \frac{1}{2}(f + g + |f - g|)$ and $min\{f, g\} = \frac{1}{2}(f + g - |f - g|)$, respectively.

Theorem 2.6. *Abel statistically δ ward continuous image of any Abel statistically δ ward compact subset of $\mathbb{R}$ is Abel statistically δ ward compact.*

Proof. Assume that f is a Abel statistically δ ward continuous function on a subset A of $\mathbb{R}$, and B is an Abel statistically δ ward compact subset of A. Let (β_n) be any sequence of points in $f(B)$. Write $\beta_n = f(\alpha_n)$ where $\alpha_n \in A$ for each positive integer n. Abel statistically δ ward compactness of B implies that there is a subsequence $(\gamma_k) = (\alpha_{n_k})$ of (α_n) with $Abel_st - \lim_{k\to\infty}\Delta^2\gamma_k = 0$. Write $(t_k) = (f(\gamma_k))$. As f is Abel statistically δ ward continuous, $(f(\gamma_k))$ is Abel statistically δ quasi-Cauchy. Thus $f(B)$ is Abel statistically δ ward compact. This completes the proof of the theorem. □

Corollary 2.7. *Abel statistically δ ward continuous image of any compact subset of $\mathbb{R}$ is Abel statistically δ ward compact.*

Corollary 2.8. *Abel statistically δ ward continuous image of a G-sequentially compact subset of $\mathbb{R}$ is Abel statistically δ ward compact for any subsequential regular method G.*

3. Conclusion

In this paper, we obtain results related to Abel statistically δ ward continuity, Abel statistically δ ward compactness, ward continuity, continuity, and uniform continuity. We suggest to investigate Abel statistically δ quasi-Cauchy sequences of fuzzy points or soft points (see [23], [38] for the definitions and related concepts in fuzzy setting, and see [2], and [33] for the soft setting). We also suggest to investigate Abel statistically δ quasi-Cauchy double sequences (see for example [27],

[32], and [40] for the definitions and related concepts in the double sequences case). For another further study, we suggest to investigate Abel statistically δ Cauchy sequences of points in an abstract metric space ([39], [45], [44], [22], [41], and [28]).

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