ABEL STATISTICAL DELTA QUASI CAUCHY SEQUENCES OF REAL NUMBERS

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Abstract. In this paper, we investigate the concept of Abel statistical delta quasi Cauchy sequences. A real function $f$ is called Abel statistically delta ward continuous if it preserves Abel statistical delta quasi Cauchy sequences, where a sequence $(\alpha_k)$ of points in $\mathbb{R}$ is called Abel statistically delta quasi Cauchy if $\lim_{x \to 1^-} (1 - x) \sum_{k|\Delta^2 \alpha_k| \geq \varepsilon} x^k = 0$ for every $\varepsilon > 0$, where $\Delta^2 \alpha_k = \alpha_{k+2} - 2\alpha_{k+1} + \alpha_k$ for every $k \in \mathbb{N}$. Some other types of continuities are also studied and interesting results are obtained.

1. Introduction

Throughout this paper, $\mathbb{N}$, and $\mathbb{R}$ will denote the set of positive integers, and the set of real numbers, respectively. The boldface letters such as $\alpha$, $\beta$, $\zeta$ will be used for sequences $\alpha = (\alpha_n)$, $\beta = (\beta_n)$, $\zeta = (\zeta_n)$, ... of points in $\mathbb{R}$. A real function $f$ is continuous if and only if it preserves Abel statistical convergence, i.e. for each point $\ell$ in the domain, $\text{Abel}_{st} \lim_{n \to \infty} f(\alpha_n) = f(\ell)$ whenever $\text{Abel}_{st} \lim_{n \to \infty} \alpha_n = \ell$.

Using the idea of continuity of a real function in this manner, many kinds of continuities were introduced and investigated, not all but some of them we recall in the following: ward continuity [12], $p$-ward continuity [19], $\delta$-ward continuity [15], $\delta^2$-ward continuity [4], statistical ward continuity, [16], $\lambda$-statistical ward continuity [29], $\rho$-statistical ward continuity [6], [21], slowly oscillating continuity [10], [44], [28], quasi-slowly oscillating continuity [31], $\Delta$-quasi-slowly oscillating continuity [33], upward and downward statistical continuities [20], lacunary statistical ward continuity [7], [47], and [48], lacunary statistical $\delta$ ward continuity [25], lacunary statistical $\delta^2$ ward continuity [10], $N_\delta$-ward continuity [18], [24], [36], [8], [20], [37], and $N_\delta$-$\delta$-ward continuity [8], which enabled some authors to obtain interesting results.

The purpose of this paper is to introduce and investigate the concept of Abel statistical $\delta$-ward continuity of a real function, and prove interesting theorems.

2010 Mathematics Subject Classification. Primary: 40A05 ; Secondaries: 26A15, 40A30.

Key words and phrases. Abel statistical convergence, summability, quasi-Cauchy sequences, continuity.

2. Abel statistical δ quasi Cauchy sequences

A sequence \((\alpha_k)\) is called statistically convergent to an element \(\ell\) of \(\mathbb{R}\) if \(\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : |\alpha_k - \ell| \geq \varepsilon\}| = 0\) for each \(\varepsilon > 0\) (see [34], [14], [21], and [26]).

A sequence \((\alpha_k)\) of real numbers is called Abel convergent (or Abel summable) to \(\ell\) if the series \(\sum_{k=0}^{\infty} \alpha_k x^k\) is convergent for \(0 \leq x < 1\) and \(\lim_{x \to 1} (1-x) \sum_{k=0}^{\infty} \alpha_k x^k = \ell\) (1, 3, and 35). In this case, we write \(\text{Abel}\lim \alpha_k = \ell\).

A sequence \((\alpha_k)\) of points in \(\mathbb{R}\) is called Abel quasi-Cauchy if \(\lim_{x \to 1} (1-x) \sum_{k=0}^{\infty} \Delta \alpha_k x^k = 0\) for every \(x > 0\) (30).

Recently the concept of Abel statistical convergence of a sequence is investigated in [33] in the sense that a sequence \((\alpha_k)\) is called Abel statistically convergent to a real number \(L\) if \(4 \lim_{x \to 1} (1-x) \sum_{k:|\alpha_k - L| \geq \varepsilon} x^k = 0\) for every \(\varepsilon > 0\), and denoted by \(\text{Abelst}\lim \alpha_k = L\).

A sequence \((\alpha_k)\) of points in \(\mathbb{R}\) is called Abel statistically quasi Cauchy if

\[
\lim_{x \to 1} (1-x) \sum_{k:|\Delta^2 \alpha_k| \geq \varepsilon} x^k = 0
\]

for every \(x > 0\) (30).

Now we introduce the concept of Abel statistically \(\delta\) quasi Cauchyness in the following:

**Definition 2.1.** A sequence of points in a subset \(A\) of \(\mathbb{R}\) is called Abel statistically \(\delta\) quasi Cauchy if

\[
\lim_{x \to 1} (1-x) \sum_{k:|\Delta^2 \alpha_k| \geq \varepsilon} x^k = 0
\]

for every \(x > \varepsilon > 0\), where \(\Delta^2 \alpha_k = \alpha_{k+2} - 2\alpha_{k+1} + \alpha_k\) for every \(k \in \mathbb{N}\).

Any Abel statistically quasi-Cauchy sequence is Abel statistically \(\delta\) quasi Cauchy, but the converse is not always true. Any quasi-Cauchy sequence is Abel statistically \(\delta\) quasi Cauchy, but the converse is not always true. Any Abel statistically convergent sequence is Abel statistically \(\delta\) quasi Cauchy. There are Abel statistically \(\delta\) quasi Cauchy sequences which are not Abel statistically quasi Cauchy. Since the set of all convergent sequences \(c\) is a proper subset of \(Abel_{st}^\delta\), and \(Abel_{st}\) is a proper subset of \(Abel_{st}^\delta\), the set of Abel statistical \(\delta\) quasi Cauchy sequences, one can easily find that \(c \subset \Delta \subset Abel_{st}^\delta \subset Abel_{st}^\delta\), where \(c, \Delta, \Delta Abel_{st},\) and \(\Delta^2 Abel_{st}\),
denote the set of convergent sequences, the set of quasi Cauchy sequences, the set of Abel statistically quasi Cauchy sequences, and the set of Abel statistically \( \delta \) quasi Cauchy sequences.

**Theorem 2.1.** The sum of two Abel statistical \( \delta \) quasi-Cauchy sequences is Abel statistical \( \delta \) quasi-Cauchy.

**Proof.** Let \( (\alpha_k) \) and \( (\beta_k) \) be Abel statistical \( \delta \) quasi-Cauchy sequences of of points in \( A \). Then \( \lim_{x \to 1^-} (1-x) \sum_{k:|\Delta^2\alpha_k| \geq \varepsilon} x^k = 0 \) and \( \lim_{x \to 1^-} (1-x) \sum_{k:|\Delta^2\beta_k| \geq \varepsilon} x^k = 0 \) for every \( \varepsilon > 0 \). Then \( \lim_{x \to 1^-} (1-x) \sum_{k:|\Delta^2(\alpha_k+\beta_k)| \geq \varepsilon} x^k \leq \lim_{x \to 1^-} (1-x) \sum_{k:|\Delta^2\beta_k| \geq \varepsilon} x^k + \lim_{x \to 1^-} (1-x) \sum_{k:|\Delta^2\beta_k| \geq \varepsilon} x^k \). This completes the proof of the theorem. \( \square \)

Now we give the definition of Abel statistical \( \delta \) ward compactness.

**Definition 2.2.** A subset \( A \) of \( \mathbb{R} \) is called Abel statistically \( \delta \) ward compact if any sequence of points in \( A \) has an Abel statistically \( \delta \) quasi-Cauchy subsequence.

First, we note that any finite subset of \( \mathbb{R} \) is Abel statistically \( \delta \) ward compact, the union of two Abel statistically \( \delta \) ward compact subsets of \( \mathbb{R} \) is Abel statistically \( \delta \) ward compact and the intersection of any family of Abel statistically \( \delta \) ward compact subsets of \( \mathbb{R} \) is Abel statistically \( \delta \) ward compact. Any \( G \)-sequentially compact subset of \( \mathbb{R} \) is Abel statistically \( \delta \) ward compact for a regular subsequential method \( G \) (see [11], [17]). Furthermore any subset of an Abel statistically \( \delta \) ward compact set is Abel statistically \( \delta \) ward compact, any bounded subset of \( \mathbb{R} \) is Abel statistically \( \delta \) ward compact, any slowly oscillating compact subset of \( \mathbb{R} \) is Abel statistically \( \delta \) ward compact (see [10] for the definition of slowly oscillating compactness).

**Theorem 2.2.** If a function \( f \) is uniformly continuous on a subset \( A \) of \( \mathbb{R} \), then \( (f(\alpha_k)) \) is Abel statistical \( \delta \) quasi-Cauchy whenever \( (\alpha_k) \) is a quasi-Cauchy sequence of points in \( A \).

**Proof.** Take any quasi-Cauchy sequence \( (\alpha_k) \) of points in \( A \), and let \( \varepsilon \) be any positive real number. By uniform continuity of \( f \), there exists a \( \delta > 0 \) such that \( |f(\alpha) - f(\beta)| < \varepsilon \) whenever \( |\alpha - \beta| < \delta \) and \( \alpha, \beta \in E \). Since \( (\alpha_k) \) is a quasi-Cauchy sequence, there exists a positive integer \( k_0 \) such that \( |\alpha_{k+1} - \alpha_k| < \delta \) for \( k \geq k_0 \). Thus

\[
\lim_{x \to 1^-} (1-x) \sum_{k:|\Delta^2\alpha_k| \geq \varepsilon} x^k = 0.
\]

This completes the proof of the theorem. \( \square \)

**Definition 2.3.** A function defined on a subset \( A \) of \( \mathbb{R} \) is called Abel statistically \( \delta \) ward continuous if it preserves Abel statistical \( \delta \) quasi-Cauchy sequences, i.e. \( (f(\alpha_n)) \) is an Abel statistical \( \delta \) quasi-Cauchy sequence whenever \( (\alpha_n) \) is.

We note that Abel statistical \( \delta \) ward continuity cannot be obtained by any subsequential method \( G \) ( [9], [17] ). The composition of two Abel statistical \( \delta \) ward continuous functions is Abel statistical \( \delta \) ward continuous.

**Theorem 2.3.** If \( f \) is Abel statistically \( \delta \) ward continuous on a subset \( A \) of \( \mathbb{R} \), then it is Abel statistically ward continuous on \( A \).
Proof. Let \((a_n)\) be any sequence with \(\text{Abel}_{st} \lim_{k \to \infty} \Delta a_k = 0\). Then the sequence
\[(a_1, a_1, a_2, a_2, \ldots, a_n, a_n, \ldots)\]
is Abel statistical \(\delta\) quasi-Cauchy hence, by the hypothesis, the sequence
\[(f(a_1), f(a_1), f(a_2), f(a_2), \ldots, f(a_n), f(a_n), \ldots)\]
is Abel statistical \(\delta\) quasi-Cauchy. It follows from this that
\[(f(a_1), f(a_2), \ldots, f(a_n), \ldots)\]
is Abel statistical quasi-Cauchy. This completes the proof of the theorem. \(\square\)

Corollary 2.4. Any Abel statistically \(\delta\) ward continuous on a subset \(A\) of \(\mathbb{R}\) is ordinary continuous on \(A\).

Theorem 2.5. The sum of two Abel statistical \(\delta\) ward continuous functions is Abel statistical \(\delta\) ward continuous.

Proof. The proof of this theorem follows easily, so is omitted. \(\square\)

If \(c\) is a constant real number and \(f\) is an Abel statistically \(\delta\) ward continuous function, then \(cf\) is Abel statistically \(\delta\) ward continuous. Thus the set of Abel statistical \(\delta\) ward continuous functions is a vector subspace of the vector space of continuous functions. Maximum of two Abel statistical \(\delta\) ward continuous functions is Abel statistical \(\delta\) ward continuous, and minimum of two Abel statistical \(\delta\) ward continuous functions is Abel statistical \(\delta\) ward continuous, which follow from
\[\max\{f, g\} = \frac{1}{2}(f + g + |f - g|)\]
and
\[\min\{f, g\} = \frac{1}{2}(f + g - |f - g|),\]
respectively.

Theorem 2.6. Abel statistically \(\delta\) ward continuous image of any Abel statistically \(\delta\) ward compact subset of \(\mathbb{R}\) is Abel statistically \(\delta\) ward compact.

Proof. Assume that \(f\) is a Abel statistically \(\delta\) ward continuous function on a subset \(A\) of \(\mathbb{R}\), and \(B\) is an Abel statistically \(\delta\) ward compact subset of \(A\). Let \((\beta_n)\) be any sequence of points in \(f(B)\). Write \(\beta_n = f(\alpha_n)\) where \(\alpha_n \in A\) for each positive integer \(n\). Abel statistically \(\delta\) ward compactness of \(B\) implies that there is a subsequence \((\gamma_k) = (\alpha_{n_k})\) of \((\alpha_n)\) with Abel\_st \(t \lim_{k \to \infty} \Delta^2 \gamma_k = 0\). Write \((t_k) = (f(\gamma_k))\). As \(f\) is Abel statistically \(\delta\) ward continuous, \((f(\gamma_k))\) is Abel statistically \(\delta\) quasi-Cauchy. Thus \(f(B)\) is Abel statistically \(\delta\) ward compact. This completes the proof of the theorem. \(\square\)

Corollary 2.7. Abel statistically \(\delta\) ward continuous image of any compact subset of \(\mathbb{R}\) is Abel statistically \(\delta\) ward compact.

Corollary 2.8. Abel statistically \(\delta\) ward continuous image of a \(G\)-sequentially compact subset of \(\mathbb{R}\) is Abel statistically \(\delta\) ward compact for any subsequential regular method \(G\).

3. Conclusion

In this paper, we obtain results related to Abel statistically \(\delta\) ward continuity, Abel statistically \(\delta\) ward compactness, ward continuity, continuity, and uniform continuity. We suggest to investigate Abel statistically \(\delta\) quasi-Cauchy sequences of fuzzy points or soft points (see \([23, 38]\) for the definitions and related concepts in fuzzy setting, and see \([24, 33]\) for the soft setting). We also suggest to investigate Abel statistically \(\delta\) quasi-Cauchy double sequences (see for example \([27,\)
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[32], and [40] for the definitions and related concepts in the double sequences case). For another further study, we suggest to investigate Abel statistically Cauchy sequences of points in an abstract metric space ([39], [45], [44], [22], [41], and [28]).

4. ACKNOWLEDGMENTS

The author acknowledges that some of the results were presented at the 2nd International Conference of Mathematical Sciences, 31 July 2018-6 August 2018, (ICMS 2018) Maltepe University, Istanbul, Turkey, and the statements of some results in this paper will be appeared in AIP Conference Proceeding of 2nd International Conference of Mathematical Sciences, (ICMS 2018) Maltepe University, Istanbul, Turkey ([42]).

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