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A new method for solving nonlinear fractional differential equations

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Abstract: In this paper, a new extended Kudryashov method for solving fractional nonlinear differential equations is proposed. The fractional derivative in this paper is considered in the sense of modified Riemann-Liouville. We also handle the time-fractional fifth$-$order Sawada$-$Kotera equation and the time-fractional generalized Hirota$-$Satsuma coupled KdV equation to illustrate the the simplicity and the effectiveness of this method. Solutions of these equations are obtained in analytical traveling wave solution form including hyperbolic and trigonometric functions.

Keywords: Time-fractional fifth-order Sawada-Kotera equation, time-fractional generalized Hirota-Satsuma coupled KdV equation, extended Kudryashov method.

1 Introduction

The development of theory of fractional calculus has afforded additional perspectives for the theory of fractional calculus, especially in modeling dynamical processes in fluids and porous structures [25,27]. Fractional derivatives also appear in the theory of control of dynamical systems, when the controlled system is delineated by fractional differential equations. Studies, have shown that a fractional order controller can provide better performance than an integer order controller. The mathematical modeling and simulation of systems and processes naturally leads to differential equations of fractional order and to requirement to solve such equations [2,23].

Numerical methods for solving fractional differential equations have an intense period from both theoretical and the viewpoint of applications in physics, chemistry, fluid mechanics, quantum mechanics and other fields of science. In literature, exact solutions of fractional differential equations have attracted the attention of researcher from different fields. Several research works have proposed techniques for solving fractional differential equations, such as $G'/G-$expansion method [3,28,30] , Exp$-$function method [12,29], first integral method [1,8,19,22], sub$-$equation method [13,24,32], Jacobi elliptic funtion method [9,11,26,31] , modified Kudryashov method [4,5,6,17,20], extended tanh method [7],modified simple equation method [18] and others.

In recent years, the time-fractional fifth-order Sawada$-$Kotera equation and the time-fractional Hirota-Satsuma coupled KdV equation appear in mathematical modeling of physical phenomena such as dispersive media. Moreover, traveling wave solutions of these equations have been studied in [10,21,26,30,28,32], the symmetrical Fibonacci function solutions and hyperbolic function solutions have been obtained by classical Kudryashov method in [6].

In this paper we propose a new extended Kudryashov method for fractional differential equations based on homogenous balancing principle by means of traveling wave transformation. In this method, by using the transformation $\xi = \frac{kx^{\beta}}{\Gamma(1+\beta)} + \frac{ny^{\gamma}}{\Gamma(1+\gamma)} + \frac{mz^{\delta}}{\Gamma(1+\delta)} + \cdots + \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}$, a given fractional differential equation turn into fractional ordinary differential equation whose solutions are in the form $u(\xi) = \sum_{i=0}^{N} a_i Q^i(\xi)$, where $Q(\xi)$ satisfies the fractional Riccati

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equation $D_\xi^\alpha Q = Q^3 - Q$. In addition, we handle the time-fractional fifth-order Sawada-Kotera equation and the time-fractional Hirota-Satsuma coupled KdV equation, then the feedback is implemented in Mathematica. Figures and results show the effectiveness of the proposed method in comparision with the classical method. The paper is organized as follows. Section 2 gives an overview of Jumarie's fractional derivative [14,15]. Section 3 presents the extended Kudryashov method procedure. Illustrative examples are given in Section 4 to attest the effectiveness of the proposed method. We finish with Section 5 providing conclusions.

2 Preliminaries

Definition 1. *A real function $f(t)$, $t > 0$, is said to be in the space C_κ, $\kappa \in R$, if there exists areal number $p > \kappa$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0,\infty)$, and it is said to be in the space C_κ^m if $f^m \in C_\kappa, m \in N$ [26,27].*

Definition 2. *The modified Riemann-Liouville derivative is defined as [26,27]:*

$$D_x^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x (x-\xi)^{-\alpha} [f(\xi) - f(0)]\, d\xi, & 0 < \alpha \leq 1, \\ (f^{(n)}(x))^{\alpha - n}, & n \leq \alpha < n+1, \;\; n \geq 1. \end{cases} \tag{1}$$

where

$$D_x^\alpha f(x) := \lim_{h \downarrow 0} h^{-\alpha} \sum_{k=0}^{\infty} (-1)^k f[x + (\alpha - k)h]. \tag{2}$$

Moreover, some properties for the proposed modified Riemann-Liouville derivative are given in [16] as follows:

$$D_t^\alpha t^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}, \quad \gamma > 0, \tag{3}$$

$$D_t^\alpha c = 0, \tag{4}$$

$$D_t^\alpha (c_1 f(t) + c_2 g(t)) = c_1 D_t^\alpha f(t) + c_2 D_t^\alpha g(t), \tag{5}$$

where c, c_1, c_2 are constants and (3),(4),(5) are direct results of the equality $D^\alpha x(t) = \Gamma(1+\alpha) D x(t)$ which holds for non-differentiable functions.

3 The extended Kudryashov method

We present the main steps of the extended Kudryashov method as follows.

For a given nonlinear FDEs for a function u of independent variables, $X = (x, y, z, \ldots, t)$:

$$F\left(u, u_t, u_x, u_y, u_z, \ldots, D_t^\alpha u, D_x^\alpha u, D_y^\alpha u, D_z^\alpha u, \ldots\right) = 0. \tag{6}$$

where $D_t^\alpha u, D_x^\alpha u, D_y^\alpha u$ and $D_z^\alpha u$ are the modified Riemann-Liouville derivatives of u with respect to t, x, y and z. F is a polynomial in $u = u(x, y, z, \ldots, t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

Step 1. We investigate the traveling wave solutions of Eq.(6) by making the transformations in the form:

$$u(x, y, z, \ldots, t) = u(\xi), \quad \xi = \frac{kx^\beta}{\Gamma(1+\beta)} + \frac{ny^\gamma}{\Gamma(1+\gamma)} + \frac{mz^\delta}{\Gamma(1+\delta)} + \cdots + \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}, \tag{7}$$

where k,n,m and λ are arbitrary constants. Then Eq.(6) reduces to a nonlinear ordinary differential equation of the form:

$$G=(u,u_{\xi},u_{\xi\xi},u_{\xi\xi\xi},\ldots)=0. \tag{8}$$

Step 2. We suppose that the reduced equation admits the following solution:

$$u(\xi)=\sum_{i=0}^{N}a_iQ^i(\xi) \tag{9}$$

where $Q(\xi)=\frac{\pm 1}{\sqrt{1\pm e^{2\xi}}}$ and the function Q is the solution of equation

$$Q_{\xi}(\xi)=Q^3(\xi)-Q(\xi). \tag{10}$$

Step 3. According to the method, we assume that the solution of Eq.(8) can be expressed in the form

$$u(\xi)=a_NQ^N+\cdots. \tag{11}$$

In order to determine the value of the pole order N, we balance the highest order nonlinear terms in Eq.(8) analogously as in the classical Kudryashov method. Supposing $u^l(\xi)u^{(s)}(\xi)$ and $(u^{(p)}(\xi))^r$ are the highest order nonlinear terms of Eq.(2.8) and balancing the highest order nonlinear terms we have:

$$N=\frac{2(s-rp)}{r-l-1}. \tag{12}$$

Step 4. Substituting Eq.(9) into Eq.(8) and equating the coefficients of Q^i to zero, we get a system of algebraic equations. By solving this system, we obtain the exact solutions of Eq.(6). And the obtained solutions can depend on hyperbolic functions.

4 Examples

4.1 Time-Fractional Fifth-Order Sawada-Kotera Equation

We consider the following the time-fractional fifth-order Sawada-Kotera equation

$$D_t^{\alpha}u+u_{xxxxx}+45u_xu^2+15(u_xu_{xx}+uu_{xxx})=0. \tag{13}$$

where $t>0, \quad 0<\alpha\leq 1$ is an important unidirectional nonlinear evolution equation which belongs to many set of conservation rule in physics and modeling the waves that disperse opposite directions.
By considering the traveling wave transformation

$$u(x,t)=u(\xi), \quad \xi=kx+\frac{ct^{\alpha}}{\Gamma(1+\alpha)}.$$

where $k,c\neq 0$ are constants. Equation (13) can be reduced to the following ordinary differential equation:

$$cu'+k^5u^{(5)}+45kuu'+15k^3(u'u''+uu''')=0. \tag{14}$$

Also we take

$$u(\xi)=a_0+a_1Q+\cdots+a_NQ^N \tag{15}$$

where $Q(\xi) = \frac{\pm 1}{\sqrt{1 \pm e^{2\xi}}}$. We note that the function Q is the solution of $Q'(\xi) = Q^3(\xi) - Q(\xi)$. Balancing the the linear term of the highest order with the highest order nonlinear term in Eq.(14), we compute

$$N = 4. \tag{16}$$

Thus, we have

$$u(\xi) = a_0 + a_1 Q(\xi) + a_2 Q^2(\xi) + a_4 Q^3(\xi) + a_4 Q^4(\xi) \tag{17}$$

and substituting derivatives of $u(\xi)$ with respect to ξ in Eq.(14) we obtain

$$u'(\xi) = 4a_4 Q^6(\xi) + 3a_3 Q^5(\xi) + (2a_2 - 4a_4) Q^4(\xi) + (a_1 - 3a_3) Q^3(\xi) - 2a_2 Q^2(\xi) - a_1 Q(\xi), \tag{18}$$

$$\begin{aligned} u''(\xi) &= 24a_4 Q^8(\xi) + 15a_3 Q^7(\xi) + (8a_2 - 40a_4) Q^6(\xi) + (3a_1 - 24a_3) Q^5(\xi) \\ &+ (16a_4(\xi) - 12a_2 Q^4(\xi) + (9a_3 - 4a_1) Q^3(\xi) + 4a_2 Q^2(\xi) + a_1 Q(\xi), \end{aligned} \tag{19}$$

$$\begin{aligned} u'''(\xi) &= 192a_4 Q^{10}(\xi) + 105a_3 Q^9(\xi) + (48a_2 - 432a_4) Q^8(\xi) \\ &+ (15a_1 - 225a_3) Q^7(\xi) + (304a_4 - 96a_2) Q^6(\xi) \\ &+ (147a_3 - 27a_1) Q^5(\xi) + (56a_2 - 64a_4) Q^4(\xi) + (13a_1 - 27a_3) Q^3(\xi) \\ &- 8a_2) Q^2(\xi) - a_1 Q(\xi), \end{aligned} \tag{20}$$

$$\begin{aligned} u^{(iv)}(\xi) &= 1920a_4 Q^{12} + 945a_3 Q^{11}(\xi) + (384a_2 - 53760) Q^{10}(\xi) \\ &+ (105a_1 - 2520a_3) Q^9(\xi) + (5280a_4 - 960a_2) Q^8(\xi) \\ &+ (2310a_3 - 240a_1) Q^7(\xi) + (800a_2 - 2080a_4) Q^6(\xi) \\ &+ (174a_1 - 816a_3) Q^5(\xi) + (256a_4 - 240a_2) Q^4(\xi) + (81a_3 - 40a_1) Q^3(\xi) \\ &+ 16a_2 Q^2(\xi) + a_1 Q(\xi), \end{aligned} \tag{21}$$

$$\begin{aligned} u^{(v)}(\xi) &= 23040a_4 Q^{14}(\xi) + 10395a_3 Q^{13}(\xi) + (3840a_2 - 76800a_4) Q^{12}(\xi) \\ &+ (945a_1 - 33075a_3) Q^{11}(\xi) + (96000a_4 - 11520a_2) Q^{10}(\xi) \\ &+ (38850a_3 - 2625a_1) Q^9(\xi) + (12480a_2 - 54720a_4) Q^8(\xi) \\ &+ (2550a_1 - 20250a_3) Q^7(\xi) + (13504a_4 - 5760a_2) Q^6(\xi) \\ &+ (4323a_3 - 990a_1) Q^5(\xi) + (992a_2 - 1024a_4) Q^4(\xi) \\ &+ (121a_1 - 243a_3) Q^3(\xi) - 32a_2 Q^2(\xi) - a_1 Q(\xi). \end{aligned} \tag{22}$$

Substituting Eqs.(18–4.10) into Eq.(14) and collecting the coefficient of each power of Q^i, setting each of coefficient to zero, solving the resulting system of algebraic equations we obtain the following solutions.

Case 1.

$$\begin{aligned} &a_0 = -1, \quad a_1 = 0, \quad a_2 = 16, \quad a_3 = 0 \\ &a_4 = -16, \quad k = -\sqrt{3}, \quad c = -81\sqrt{3}. \end{aligned} \tag{23}$$

Inserting Eq.(23) into Eq.(17), we obtain the following solutions of Eq.(13)

$$\begin{aligned} u_1(x,t) &= -1 + \frac{4}{cosh^2\left[-2\sqrt{3}\left(x + \frac{81t^\alpha}{\Gamma(1+\alpha)}\right)\right]}, \\ u_2(x,t) &= -1 + \frac{4}{sinh^2\left[-2\sqrt{3}\left(x + \frac{81t^\alpha}{\Gamma(1+\alpha)}\right)\right]}. \end{aligned}$$

Case 2.

$$\begin{aligned} &a_0 = -1, \quad a_1 = 0, \quad a_2 = 16, \quad a_3 = 0 \\ &a_4 = -16, \quad k = \sqrt{3}, \quad c = 81\sqrt{3}. \end{aligned} \tag{24}$$

Inserting Eq.(24) into Eq.(17), we obtain the following solutions of Eq.(13):

$$u_3(x,t) = -1 + \frac{4}{cosh^2\left[2\sqrt{3}\left(x+\frac{81t^\alpha}{\Gamma(1+\alpha)}\right)\right]},$$
$$u_4(x,t) = -1 + \frac{4}{sinh^2\left[2\sqrt{3}\left(x+\frac{81t^\alpha}{\Gamma(1+\alpha)}\right)\right]}.$$

Case 3.

$$a_0 = \frac{1}{9}(3-4k^2), \quad a_1 = 0, \quad a_2 = \frac{16k^2}{3}, \quad a_3 = 0$$
$$a_4 = -\frac{16k^2}{3}, \quad k = k, \quad c = \frac{1}{3}(32k^5 - 45k). \tag{25}$$

Inserting Eq.(25) into Eq.(17), we obtain the following solutions of Eq.(13)

$$u_5(x,t) = \frac{1}{9}(3-4k^2) + \frac{4k^2}{3cosh^2\left[2k\left(x+\frac{(32k^4-45)t^\alpha}{3\Gamma(1+\alpha)}\right)\right]},$$
$$u_6(x,t) = \frac{1}{9}(3-4k^2) + \frac{4k^2}{3cosh^2\left[2k\left(x+\frac{(32k^4-45)t^\alpha}{3\Gamma(1+\alpha)}\right)\right]}.$$

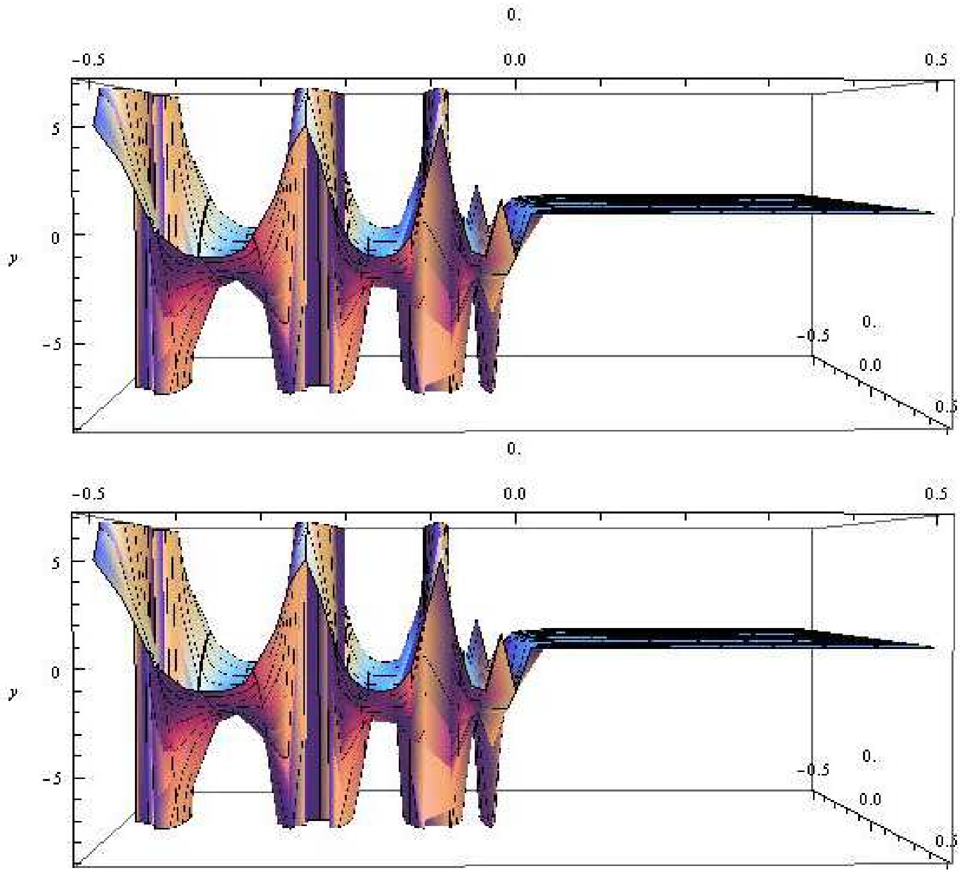

Fig. 1: Solution of (13) at $k = -\sqrt{3}$, $c = -81\sqrt{3}$ and $\alpha = 0.5$.

Figure 1 shows hyperbolic wave structure of the solution of time-fractional fifth-order Sawada-Kotera equation.

Remark 1. Whereas two solutions were obtained by classical Kudryashov method in [6], six solutions are obtained by this method. In case 3, polynomial coefficients and parameters are in the same equivalence class with the solutions ontained by classical method. Moreover, increase in k parameter effects the wavelenght and speed of the wave. Thus, the wave is achieved faster progress.

4.2 Time-fractional Hirota-Satsuma coupled KdV equation

We apply the above method to the space-time fractional Hirota-Satsuma-coupled KdV equation:

$$\begin{aligned} D_t^{\alpha}u &= \frac{1}{4}u_{xxx} + 3uu_x + 3(-v^2+w)_x, \\ D_t^{\alpha}v &= -\frac{1}{2}v_{xxx} - 3uv_x, \\ D_t^{\alpha}w &= -\frac{1}{2}w_{xxx} - 3uw_x \end{aligned}$$

where $u=u(x,t)$, $v=v(x,t)$ and $w=w(x,t)$, $\quad t>0, \quad 0<\alpha\leq 1$. This system models the interaction between two long waves that have distinct dispersion relation.
For our purpose, we use the transformations

$$u(x,t) = \tfrac{1}{\lambda}u^2(\xi), \quad v(x,t) = -\lambda + u(\xi), \quad w(x,t) = 2\lambda^2 - 2\lambda u(\xi),$$

where $\xi = x - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}$ then Eqs.(26) reduced to the ordinary differential equation as follows:

$$\lambda u'' + 2u^3 - 2\lambda^2 u = 0. \tag{26}$$

Also we take

$$g(\xi) = u(\xi) = \sum_{i=0}^{N} a_i Q^i \tag{27}$$

where $Q(\xi) = \pm\frac{1}{(1\pm e^{2\xi})^{1/2}}$. We note that the function Q is the solution of $Q'(\xi) = Q^3(\xi) - Q(\xi)$. Balancing u'' and u^3 in Eq.(26), we compute

$$N = 2. \tag{28}$$

Thus, we have

$$u(\xi) = a_0 + a_1 Q(\xi) + a_2 Q^2(\xi) \tag{29}$$

and taking derivatives of $u(\xi)$ with respect to ξ, we obtain

$$u'(\xi) = 2a_2Q^4(\xi) + a_1Q^3(\xi) - 2a_2Q^2(\xi) - a_1Q(\xi), \tag{30}$$

$$u''(\xi) = 8a_2Q^6(\xi) + 3a_1Q^5(\xi) - 12a_2Q^4(\xi) - 4a_1Q^3(\xi) + 4a_2Q^2(\xi) + a_1Q(\xi) \tag{31}$$

Substituting Eq.(30) and Eq.(31) into Eq.(26) and collecting the coefficient of each power of Q^i, setting each of coefficient to zero, solving the resulting system of algebraic equations we obtain the following solutions.

Case 1.

$$a_0 = -1, \quad a_1 = 0, \quad a_2 = 2 \quad \lambda = -1. \tag{32}$$

Inserting Eq.(32) into Eq.(29), we obtain the following solutions of Eqs.(26)

$$\begin{aligned} u_1(x,t) &= -tanh^2\left[2x + \tfrac{2t^{\alpha}}{\Gamma(1+\alpha)}\right], \\ v_1(x,t) &= 1 - tanh\left[2x + \tfrac{2t^{\alpha}}{\Gamma(1+\alpha)}\right], \\ w_1(x,t) &= 2 - 2tanh\left[2x + \tfrac{2t^{\alpha}}{\Gamma(1+\alpha)}\right]. \end{aligned}$$

$$
\begin{aligned}
u_2(x,t) &= -coth^2\left[2x+\frac{2t^\alpha}{\Gamma(1+\alpha)}\right],\\
v_2(x,t) &= 1-coth\left[2x+\frac{2t^\alpha}{\Gamma(1+\alpha)}\right],\\
w_2(x,t) &= 2-2coth\left[2x+\frac{2t^\alpha}{\Gamma(1+\alpha)}\right].
\end{aligned}
$$

.

Case 2.

$$
a_0=1, \quad a_1=0, \quad a_2=-2 \quad \lambda=-1. \tag{33}
$$

Inserting Eq.(4.2) into Eq.(29), we obtain the following solutions of Eqs.(26)

$$
\begin{aligned}
u_3(x,t) &= -tanh^2\left[2x+\frac{2t^\alpha}{\Gamma(1+\alpha)}\right],\\
v_3(x,t) &= 1+tanh\left[2x+\frac{2t^\alpha}{\Gamma(1+\alpha)}\right],\\
w_3(x,t) &= 2+2tanh\left[2x+\frac{2t^\alpha}{\Gamma(1+\alpha)}\right].
\end{aligned}
$$

$$
\begin{aligned}
u_3(x,t) &= -coth^2\left[2x+\frac{2t^\alpha}{\Gamma(1+\alpha)}\right],\\
v_3(x,t) &= 1+coth\left[2x+\frac{2t^\alpha}{\Gamma(1+\alpha)}\right], \quad .\\
w_3(x,t) &= 2+2coth\left[2x+\frac{2t^\alpha}{\Gamma(1+\alpha)}\right].
\end{aligned}
$$

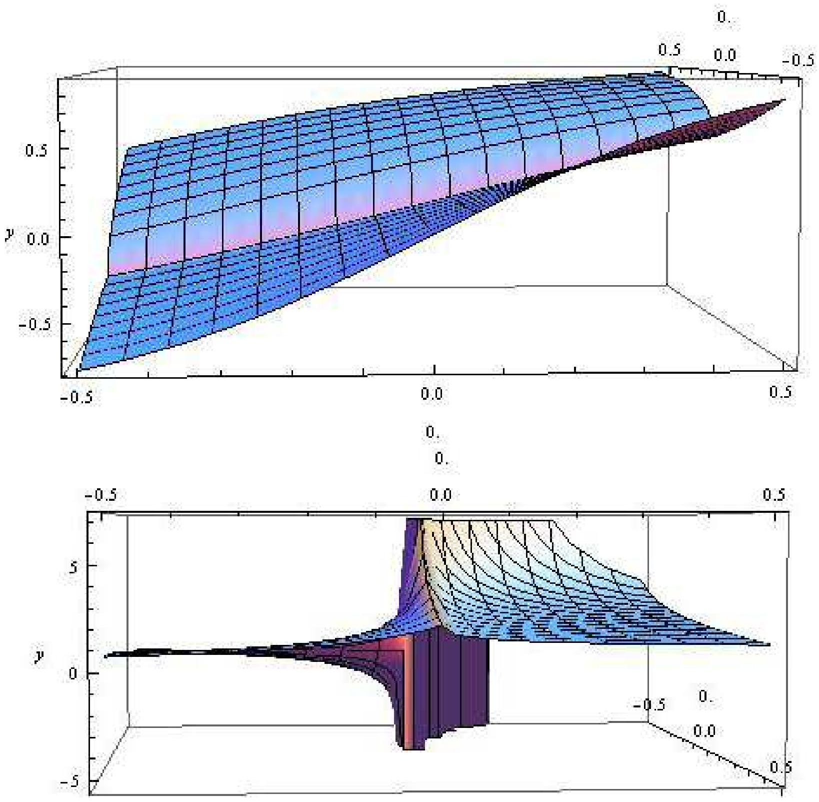

Fig. 2: Solution of u(x,t) in (26) at $\lambda=-1$ and $\alpha=0.5$.

Figure 2 shows the hyperbolic wave structure of the space-time fractional Hirota-Satsuma-coupled KdV equation.

Remark 2. Although the obtained solutions of space-time fractional Hirota-Satsuma-coupled KdV equation are in the similar structure and the same equivalence class of the solutions obtained classical Kudryashov method, coefficients of polynomial and λ parameter that determines the speed of wave, are increased by four times. Altering in λ parameter provides the increase in speed of the wave.

5 Conclusion

In this work, we have proposed a new extended Kudryashov method to solve nonlinear fractional differential equations with the help of Mathematica. By this way, degree of the auxilary polynomials are increased and more solutions are provided an opportunity for some models. The time-fractional fifth-order Sawada-Kotera equation and the space-time fractional Hirota-Satsuma-coupled KdV equation are handled to demonstrate the effectiveness of the proposed method. In comparision with the classical Kudryashov method, more traveling wave solutions are obtained. In addition, change in the parameters that determine the speed of the wave effects both the wavelenght and the speed of the solutions. Consequently, the method is effective and convenient for solving other type of space-time fractional differential equations in which the homogenous balance principle is satisfied.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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