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n-coherent rings in terms of complexes

Lu Bo * and Liu Zhongkui †

Abstract

The aim of this paper is to investigate n-coherent rings using complexes. To this end, the concepts of n-injective complexes and n-flat complexes are introduced and studied.

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1. Introduction. The notion of coherent rings was first appear in Chase's paper [[3]] without being mentioned by name. The term coherent was first used by Bourbaki in [[2]]. Since then, coherent rings have became a vigorously active area of research, see [[13]].

Coherent rings have been characterized in various ways using modules by many authors such as Chase, Cheatham, Ding, Stone, Stenström and Vasconcelos (see [[3, 4, 8, 14, 16]]). For example, a ring R is left coherent if and only if the direct product of any flat right R-modules is flat if and only if the direct limit of FP-injective left R-modules is FP-injective [[3, 14]]. In [[4], Theorem 1], Cheatham and Stone characterized coherent rings using the notion of character module as follows:
The following statements are equivalent:
(1) R is a left coherent ring;
(2) A left R-module M is injective if and only if M^+ is flat;
(2) A left R-module M is injective if and only if M^{++} is injective;
(2) A right R-module M is flat if and only if M^{++} is flat.

The homological theory of complexes of modules has been studied by many authors such as Christensen, Enochs, Foxby, Garcĺa Rozas, Holm, Liu and Wang. Several characterizations of coherent rings also have been done in various ways using complexes. For instance, a ring R is right coherent if and only if any complex of left R-modules has a flat

*College of Mathematics and Computer Science, Northwest University for Nationalities Lanzhou 730124, Gansu, China.
Email: lubo55@126.com Corresponding Author.
†Department of Mathematics, Northwest Normal University Lanzhou 730070, Gansu, China.
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preenvelope [[12], Theorem 5.2.2]; a ring R is right coherent if and only if the direct limit of FP-injective complexes of left R-modules is FP-injective [[17], Proposition 2.30].

The concept of n-coherent rings was introduced by Costa in [[7]]. In [[1]], Bennis introduced the notion of n-$\mathscr{X}$-coherent rings and gave some characterizations of it using n-$\mathscr{X}$-injective and n-$\mathscr{X}$-flat modules for a class of R-modules $\mathscr{X}$.

Motivated by the above work, the object of this paper is to characterize left n-coherent rings using complexes. To this end, we firstly introduce and study n-injective and n-flat complexes for a fixed positive integer n. We show the following results as our main results in this note (cf. Theorem 4.11).

1.1. Theorem. *Let R be a ring and n a fixed positive integer. Then the following are equivalent*:
(1) *R is left n-coherent*;
(2) *Every direct product of n-flat complexes of right R-modules is n-flat*;
(3) *Every direct limit of n-injective complexes of left R-modules is n-injective*;
(4) $\mathrm{Ext}^n(A, \lim\limits_{\rightarrow} C^i) \cong \lim\limits_{\rightarrow} \mathrm{Ext}^n(A, C^i)$ *for every n-presented complex A of left R-modules and direct system $\{C^i\}_{i\in I}$ of complexes of left R-modules*;
(5) $\overline{\mathrm{Tor}}_n(\prod\limits_{\alpha\in I} D^\alpha, A) \cong \prod\limits_{\alpha\in I} \overline{\mathrm{Tor}}_n(D^\alpha, A)$ *for any family $\{D^\alpha\}_{\alpha\in\Lambda}$ of complexes and any n-presented complex A of left R-modules*;
(6) *A complex C of left R-modules is n-injective if and only if C^+ is n-flat*;
(7) *A complex C of left R-modules is n-injective if and only if C^{++} is n-injective*;
(8) *A complex C of right R-modules is n-flat if and only if C^{++} is n-flat*;
(9) *For any ring S, $\underline{\mathrm{Hom}}(\underline{\mathrm{Ext}}^n(A,B), D) \cong \overline{\mathrm{Tor}}_n(\underline{\mathrm{Hom}}(B,D), A)$ for any n-presented complex A of left R-modules, any complex B of (R,S)-bimodules, any injective complex D of right S-modules.*

The paper is organized as follows:

In section 2 of this article, some notations are given.

In section 3, some isomorphisms are established which will be used to prove the main results of this paper.

In section 4, we firstly introduce and study n-injective and n-flat complexes for a fixed positive integer n. We give various equivalent conditions for a ring to be left n-coherent using n-injective and n-flat complexes.

2. Preliminaries. Throughout this paper, R denotes a ring with unity, R-Mod denotes the category of R-modules and $\mathscr{C}(R)$ denotes the abelian category of complexes of R-modules. A complex

$$\cdots \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} C_{-1} \xrightarrow{\delta_{-1}} \cdots$$

of R-modules will be denoted by (C,δ) or C.

We will use superscripts to distinguish complexes. So if $\{C^i\}_{i\in I}$ is a family of complexes, C^i will be

$$\cdots \xrightarrow{\delta_2} C_1^i \xrightarrow{\delta_1} C_0^i \xrightarrow{\delta_0} C_{-1}^i \xrightarrow{\delta_{-1}} \cdots .$$

Given a left R-module M, we use $D^m(M)$ to denote the complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{id} M \longrightarrow 0 \longrightarrow \cdots$$

with M in the mth and $(m-1)$th positions and set $\overline{M} = D^0(M)$. We also use $S^m(M)$ to denote the complex with M in the mth place and 0 in the other places and set $\underline{M} = S^0(M)$.

Given a complex C and an integer m, $\sum^m C$ denotes the complex such that $(\sum^m C)_l = C_{(l-m)}$, and whose boundary operators are $(-1)^m\delta_{l-m}$. The mth homology module of C

is the module $\mathrm{H}_m(C) = \mathrm{Z}_m(C)/\mathrm{B}_m(C)$ where $\mathrm{Z}_m(C) = \mathrm{Ker}(\delta_m^C)$ and $\mathrm{B}_m(C) = \mathrm{Im}(\delta_{m+1}^C)$. We set $\mathrm{H}_m(C) = \mathrm{H}^{-m}(C)$.

Let C be a complex of left R-modules (resp., of right R-modules), and let D be a complex of left R-modules. We will denote by $\mathrm{Hom}^{\cdot}(C, D)$ (resp., $C \otimes^{\cdot} D$) the usual homomorphism complex (resp., tensor product) of the complexes C and D.

Given two complexes C and D, let $\underline{\mathrm{Hom}}(C, D) = \mathrm{Z}(\mathrm{Hom}^{\cdot}(C, D))$. We then see that $\underline{\mathrm{Hom}}(C, D)$ can be made into a complex with $\mathrm{Hom}(C, D)_m$ the abelian group of morphisms from C to $\sum^{-m} D$ and with boundary operator given by $f \in \mathrm{Hom}(C, D)_m$, then $\delta_m(f) : C \to \sum^{-(m-1)} D$ with $\delta_m(f)_l = (-1)^m \delta^D f_l$ for any $l \in \mathbb{Z}$. For any complex C, $C^+ = \underline{\mathrm{Hom}}(C, \overline{\mathbb{Q}/\mathbb{Z}})$. Let C be a complex of right R-modules and D be a complex of left R-modules. We define $C \overline{\otimes} D$ to be $\frac{(C \otimes^{\cdot} D)}{\mathrm{B}(C \otimes^{\cdot} D)}$. Then with the maps

$$\frac{(C \otimes^{\cdot} D)_m}{\mathrm{B}_m(C \otimes^{\cdot} D)} \to \frac{(C \otimes^{\cdot} D)_{m-1}}{\mathrm{B}_{m-1}(C \otimes^{\cdot} D)},\ x \otimes y \longmapsto \delta^C(x) \otimes y,$$

where $x \otimes y$ is used to denote the coset in $\frac{(C \otimes^{\cdot} D)_m}{\mathrm{B}_m(C \otimes^{\cdot} D)}$, we get a complex. We note that the new functor $\underline{\mathrm{Hom}}(C, D)$ will have right derived functors whose values will be complexes. These values should certainly be denoted $\underline{\mathrm{Ext}}^i(C, D)$. It is not hard to see that $\underline{\mathrm{Ext}}^i(C, D)$ is the complex

$$\cdots \to \mathrm{Ext}^i(C, \Sigma^{-(m+1)} D) \to \mathrm{Ext}^i(C, \Sigma^{-m} D) \to \mathrm{Ext}^i(C, \Sigma^{-(m-1)} D) \to \cdots$$

with boundary operator induced by the boundary operator of D. For a complex C of left R-modules we have two functors $-\overline{\otimes} C : \mathscr{C}_R \to \mathscr{C}_{\mathbb{Z}}$ and $\underline{\mathrm{Hom}}(C, -) : {}_R\mathscr{C} \to \mathscr{C}_{\mathbb{Z}}$, where $\mathscr{C}_R$ (resp., ${}_R\mathscr{C}$) denotes the category of complexes of right R-modules (resp., left R-modules). Since $-\overline{\otimes} C : \mathscr{C}_R \to \mathscr{C}_{\mathbb{Z}}$ is a right exact functor, we can construct left derived functors, which we denote by $\overline{\mathrm{Tor}}_1(-, C)$.

3. n-Presented complexes and some isomorphisms. In this section, we first introduce and study the concept of n-presented complexes. Moreover, some isomorphisms which are used to prove the following results are shown.

3.1. Definition ([[10]]). A complex C is called finitely generated if, in the case where we can write C $= \sum_{i \in I} D^i$ with $D^i \in \mathscr{C}(R)$ subcomplexes of C, there exists a finite subset $J \subseteq I$ such that C $= \sum_{i \in J} D^i$.

A complex C is called finitely presented if C is finitely generated and for every exact sequence of complexes $0 \to K \to L \to C \to 0$ with L finitely generated, K is also finitely generated.

3.2. Lemma ([[10]]). *A complex C is finitely generated if and only if C is bounded and C_m is finitely generated in R-*Mod *for all $m \in \mathbb{Z}$.*

*A complex C is finitely presented if and only if C is bounded and C_m is finitely presented in R-*Mod *for all $m \in \mathbb{Z}$.*

It is clear that we have the following results:

3.3. Lemma. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of complexes. Then the following statements hold:*
(1) *If A is finitely generated and B is finitely presented, then C is finitely presented*;
(2) *If A and C are finitely presented, then so is B*;
(3) *If R is left coherent ring, and B, C are finitely presented, then so is C.*

3.4. Lemma. *Let C be a complex. Then the following statements are equivalent:*
(1) *C is finitely presented*;
(2) *There exists an exact sequence $0 \to L \to P \to C \to 0$ of complexes, where P is finitely generated projective, and L is finitely generated*;

(3) *There exists an exact sequence* $P^1 \to P^0 \to C \to 0$ *of complexes, where* P^0, P^1 *are finitely generated projective, and* P^0_m, P^1_m *are free for all* $m \in \mathbb{Z}$.

An R-module M is called n-presented if it has a finite n-presentation, i.e., there is an exact sequence

$$F_n \to F_{n-1} \to \ldots \to F_1 \to F_0 \to M \to 0$$

in which each F_i is finitely generated free.

Now, we extend the notion of n-presented modules to that of complexes and characterize such complexes.

3.5. Definition. Let $n \geq 0$ be an integer. A complex C is said to be n-presented if there is an exact sequence $P^n \to P^{n-1} \to \cdots \to P^1 \to P^0 \to C \to 0$ of complexes, where P^i is finitely generated projective, and P^i_m is free for $i = 0, 1, \cdots, n$ and all $m \in \mathbb{Z}$.

3.6. Remark. (1) A complex C is n-presented if and only if C is bounded and C_m is n-presented in R-Mod for all $m \in \mathbb{Z}$;
(2) A complex C is n-presented if and only if there is an exact sequence of complexes

$$0 \to K^n \to P^{n-1} \to \cdots \to P^1 \to P^0 \to C \to 0$$

where P^i is finitely generated projective, P^i_m is free for $i = 0, 1, \cdots, n-1$ and all $m \in \mathbb{Z}$, K^n is finitely generated;
(3) A complex C is n-presented ($n \geq 1$) if and only if there is an exact sequence of complexes

$$0 \to K \to P \to C \to 0,$$

where K is $(n-1)$-presented and P is finitely generated projective.

3.7. Lemma. *Let* $n \geq 1$ *be an integer and* $0 \to K \to P \to C \to 0$ *an exact sequence of complexes. Then*

(1) *If* P *is n-presented and* K *is* $(n-1)$*-presented, then* C *is n-presented*;
(2) *If* K *and* C *are n-presented, then so is* P;
(3) *If* C *is n-presented and* P *is* $(n-1)$*-presented, then* K *is* $(n-1)$*-presented.*

Proof. It is similar to the proof of [[13], Theorem 2.1.2] by Remark 3.6 (1). □

Let I be a set. An R-module M is called I-graded if there exists a family $\{M_i\}_{i\in I}$ of submodules of M such that $M = \bigoplus_{i\in I} M_i$. A $\mathbb{Z}$-graded module is simply called a graded module. General background about graded modules can be found in [[6]].

3.8. Lemma. *Let* $\{C^i\}_{i\in I}$ *be a family of complexes,* D *a finitely generated complex. Then* $\underline{\mathrm{Hom}}(D, \bigoplus_{i\in I} C^i) \cong \bigoplus_{i\in I} \underline{\mathrm{Hom}}(D, C^i)$ *as complexes.*

Proof. Firstly,

$$\alpha : \bigoplus_{i\in I} \mathrm{Hom}^{\cdot}(D, C^i) \to \mathrm{Hom}^{\cdot}(D, \bigoplus_{i\in I} C^i)$$

is an isomorphism by $x = (x^i)_{i\in I} \longmapsto \sum_{i\in I} \mathrm{Hom}^{\cdot}(D, \varepsilon^i)(x^i) = \sum_{i\in I} \varepsilon^i x^i$, where $x = (x^i)_{i\in I} \in (\bigoplus_{i\in I} \mathrm{Hom}^{\cdot}(D, C^i))_l = \bigoplus_{i\in I}(\mathrm{Hom}^{\cdot}(D, C^i))_l$ with $x^i \in \mathrm{Hom}^{\cdot}(D, C^i)_l$ and $\varepsilon^j : C^j \longmapsto \bigoplus_{i\in I} C^i$ is the natural embedding (see [[6],Proposition 2.5.16]).

Secondly, we will show that $\underline{\mathrm{Hom}}(D, \bigoplus_{i\in I} C^i) \cong \bigoplus_{i\in I} \underline{\mathrm{Hom}}(D, C^i)$. We define a morphism

$$\gamma = \alpha|_{\bigoplus_{i\in I}\underline{\mathrm{Hom}}(D,C^i)} : \bigoplus_{i\in I}\underline{\mathrm{Hom}}(D,C^i) \to \underline{\mathrm{Hom}}(D,\bigoplus_{i\in I}C^i).$$

Then γ is a graded isomorphism of graded modules with degree$\gamma = 0$. On the other hand, for any $(x^i)_{i\in I} \in (\bigoplus_{i\in I}\underline{\mathrm{Hom}}(D,C^i))_l$,

$$\gamma\delta^{\bigoplus_{i\in I}\underline{\mathrm{Hom}}(D,C^i)}(x^i)_{i\in I} = \alpha\delta^{\bigoplus_{i\in I}\underline{\mathrm{Hom}}(D,C^i)}(x^i)_{i\in I} = \alpha(\delta^{\underline{\mathrm{Hom}}(D,C^i)}(x^i))_{i\in I} =$$

$$\sum_{i\in I}\mathrm{Hom}(D,\varepsilon^i)\delta^{\underline{\mathrm{Hom}}(D,C^i)}(x^i) = \sum_{i\in I}\varepsilon^i(-1)^l\delta^{C^i}(x^i) = (-1)^l\sum_{i\in I}\varepsilon^i\delta^{C^i}(x^i),$$

and

$$\delta^{\underline{\mathrm{Hom}}(D,\bigoplus_{i\in I}C^i)}\gamma(x^i)_{i\in I} = \delta^{\underline{\mathrm{Hom}}(D,\bigoplus_{i\in I}C^i)}\alpha(x^i)_{i\in I} = \delta^{\underline{\mathrm{Hom}}(D,\bigoplus_{i\in I}C^i)}(\sum_{i\in I}\varepsilon^i x^i)$$

$$= (-1)^l\delta^{\bigoplus_{i\in I}C^i}(\sum_{i\in I}\varepsilon^i x^i) = (-1)^l\sum_{i\in I}\delta^{\bigoplus_{i\in I}C^i}\varepsilon^i x^i = (-1)^l\sum_{i\in I}\varepsilon^i\delta^{C^i}(x^i).$$

Thus γ is an isomorphism of complexes, and hence $\underline{\mathrm{Hom}}(D,\bigoplus_{i\in I}C^i) \cong \bigoplus_{i\in I}\underline{\mathrm{Hom}}(D,C^i)$. □

3.9. Lemma. *Let $\{C^i\}_{i\in I}$ be any direct system of complexes. Then a finitely generated complex D is finitely presented if and only if* $\mathrm{Hom}(D,\lim\limits_{\to}C^i) \cong \lim\limits_{\to}\mathrm{Hom}(D,C^i)$.

Proof. ($\Rightarrow$) It follows from Stenström [[15], Chap. V, Proposition 3.4].

($\Leftarrow$) Let $\{M_i\}_{i\in I}$ be a family of R-modules. Then $\sum^{-n}\overline{M_i}$ is a complex for all $n \in \mathbb{Z}$ and $i \in I$. Hence $\mathrm{Hom}(D,\lim\limits_{\to}\sum^{-n}\overline{M_i}) \cong \lim\limits_{\to}\mathrm{Hom}(D,\sum^{-n}\overline{M_i})$ for all $n \in \mathbb{Z}$, which implies that

$$\underline{\mathrm{Hom}}(D,\lim_{\to}\overline{M_i}) \cong \lim_{\to}\underline{\mathrm{Hom}}(D,\overline{M_i}).$$

Since $\underline{\mathrm{Hom}}(D,\lim\limits_{\to}\overline{M_i}) \cong \underline{\mathrm{Hom}}(D,\overline{\lim\limits_{\to}M_i}) \cong \mathrm{Hom}_R(D,\lim\limits_{\to}M_i)$ and $\underline{\mathrm{Hom}}(D,\overline{M_i}) \cong \mathrm{Hom}_R(D,M_i)$, we have that $\mathrm{Hom}_R(D^k,\lim\limits_{\to}M_i) \cong \lim\limits_{\to}\mathrm{Hom}_R(D^k,M_i)$ for all $k \in \mathbb{Z}$, then D^k is a finitely presented R-module. Therefore, D is finitely presented. □

3.10. Lemma. *Let $\{C^i\}_{i\in I}$ be a family of complexes, D a finitely presented complex. Then $D\overline{\otimes}\prod_{i\in I}C^i \cong \prod_{i\in I}(D\overline{\otimes}C^i)$ as complexes.*

Proof. Firstly,

$$\alpha : D\otimes^{\cdot}\prod_{i\in I}C^i \longrightarrow \prod_{i\in I}(D\otimes^{\cdot}C^i)$$

is an isomorphism by $x \longmapsto ((D\otimes^{\cdot}\pi^i)(x))_{i\in I}$, where $x = d\otimes c \in (D\otimes^{\cdot}\prod_{i\in I}C^i)_l$ and $\pi^j : \prod_{i\in I}C^i \longrightarrow C^j$ is the natural projection (see [[6], Proposition 2.5.17]).

Secondly, we will show that $D\overline{\otimes}\prod_{i\in I}C^i \cong \prod_{i\in I}(D\overline{\otimes}C^i)$. Since we have the following commutative diagram:

$$\begin{array}{ccccc}
(D\otimes^{\cdot}\prod\limits_{i\in I}C^i)_l & \longrightarrow & \frac{(D\otimes^{\cdot}\prod\limits_{i\in I}C^i)_l}{B_l(D\otimes^{\cdot}\prod\limits_{i\in I}C^i)} & \longrightarrow & 0\\
\alpha_l\big\downarrow & & \beta_l\big\downarrow & & \\
(\prod\limits_{i\in I}D\otimes^{\cdot}C^i)_l & \longrightarrow & \frac{(D\otimes^{\cdot}\prod\limits_{i\in I}C^i)_l}{B_l(\prod\limits_{i\in I}D\otimes^{\cdot}C^i)} & \longrightarrow & 0,
\end{array}$$

where $\beta : \frac{(D\otimes^{\cdot}\prod_{i\in I}C^i)_l}{B_l(D\otimes^{\cdot}\prod_{i\in I}C^i)} \longrightarrow \frac{(D\otimes^{\cdot}\prod_{i\in I}C^i)_l}{B_l(\prod_{i\in I}D\otimes^{\cdot}C^i)}$ is given by the assignment

$$d\otimes c + B(D\otimes^{\cdot}\prod_{i\in I}C^i) \longrightarrow \alpha(d\otimes c) + B(\prod_{i\in I}D\otimes^{\cdot}C^i)$$

for any $d\otimes c \in (D\otimes^{\cdot}\prod_{i\in I}C^i)_l$. Thus β is a graded isomorphism of graded modules with degree 0. Moreover,

$$\beta\delta^{D\overline{\otimes}\prod_{i\in I}C^i}(d\otimes c + B(D\otimes^{\cdot}\prod_{i\in I}C^i))$$

$$= \beta(\delta^D(d)\otimes c) = \alpha(\delta^D(d)\otimes c) = (\delta^D(d)\otimes\pi^i(c))_{i\in I}$$

and

$$\delta^{\prod_{i\in I}(D\overline{\otimes}C^i)}\beta(d\otimes c + B(D\otimes^{\cdot}\prod_{i\in I}C^i))$$

$$= \delta^{\prod_{i\in I}(D\overline{\otimes}C^i)}(\alpha(d\otimes c) + B(D\otimes^{\cdot}\prod_{i\in I}C^i))$$

$$= \delta^{\prod_{i\in I}(D\overline{\otimes}C^i)}\alpha(d\otimes c) = (\delta^{D\overline{\otimes}C^i}\alpha(d\otimes c))_{i\in I} = (\delta^D(d)\otimes\pi^i(c))_{i\in I}.$$

Therefore, β is an isomorphism of complexes. □

3.11. Lemma. *Let $n\geq 1$ be an integer, D an n-presented complex and $\{C^i\}_{i\in I}$ a direct system of complexes. Then* $\operatorname{Ext}^{n-1}(D,\varinjlim C^i)\cong\varinjlim\operatorname{Ext}^{n-1}(D,C^i)$.

Proof. We do an induction on n. If $n=1$, then the result follows from Lemma 3.9.

Let $n=2$ and D be an 2-presented complex. Then there exists an exact sequence of complexes $0\to L\to P\to D\to 0$ with P finitely generated projective and L finitely presented. Thus there is a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\operatorname{Hom}(P,\varinjlim C^i) & \longrightarrow & \operatorname{Hom}(L,\varinjlim C^i) & \longrightarrow & \operatorname{Ext}^1(D,\varinjlim C^i) & \longrightarrow & 0\\
\cong\big\downarrow & & \cong\big\downarrow & & \big\downarrow & & \\
\varinjlim\operatorname{Hom}(P,C^i) & \longrightarrow & \varinjlim\operatorname{Hom}(L,C^i) & \longrightarrow & \varinjlim\operatorname{Ext}^1(D,C^i) & \longrightarrow & 0.
\end{array}$$

Since $\operatorname{Hom}(P,\varinjlim C^i)\cong\varinjlim\operatorname{Hom}(P,C^i)$ and $\operatorname{Hom}(L,\varinjlim C^i)\cong\varinjlim\operatorname{Hom}(L,C^i)$ by Lemma 3.9, we have $\operatorname{Ext}^1(D,\varinjlim C^i)\cong\varinjlim\operatorname{Ext}^1(D,C^i)$.

If $n>2$, then it follows from the standard homological method. Therefore, $\operatorname{Ext}^{n-1}(D,\varinjlim C^i)\cong\varinjlim\operatorname{Ext}^{n-1}(D,C^i)$. □

3.12. Lemma. *Let $n \geq 1$ be an integer, D an n-presented complex and $\{N^\alpha\}_{\alpha \in I}$ a family of complexes. Then $\overline{\mathrm{Tor}}_{n-1}(\prod_{\alpha\in I} N^\alpha, D) \cong \prod_{\alpha \in I} \overline{\mathrm{Tor}}_{n-1}(N^\alpha, D)$.*

Proof. We do an induction on n. If $n = 1$, then the result follows from Lemma 3.10.

Let $n = 2$ and D be an 2-presented complex. Then there exists an exact sequence of complexes $0 \to L \to P \to D \to 0$ with P finitely generated projective and L finitely presented. Thus there is a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \overline{\mathrm{Tor}}_1(\prod_{\alpha\in I} N^\alpha, D) & \longrightarrow & (\prod_{\alpha\in I} N^\alpha)\overline{\otimes}L & \longrightarrow & (\prod_{\alpha\in I} N^\alpha)\overline{\otimes}P \\
 & & \downarrow & & \cong\downarrow & & \cong\downarrow \\
0 & \longrightarrow & \prod_{\alpha\in I}\overline{\mathrm{Tor}}_1(N^\alpha, D) & \longrightarrow & \prod_{\alpha\in I}(N^\alpha\overline{\otimes}L) & \longrightarrow & \prod_{\alpha\in I}(N^\alpha\overline{\otimes}P).
\end{array}$$

Since $(\prod_{\alpha\in I} N^\alpha)\overline{\otimes}L \cong \prod_{\alpha\in I}(N^\alpha\overline{\otimes}L)$ and $(\prod_{\alpha\in I} N^\alpha)\overline{\otimes}P \cong \prod_{\alpha\in I}(N^\alpha\overline{\otimes}P)$ by Lemma 3.10, we have $\overline{\mathrm{Tor}}_1(\prod_{\alpha\in I} N^\alpha, D) \cong \prod_{\alpha\in I}\overline{\mathrm{Tor}}_1(N^\alpha, D)$.

If $n > 2$, then it follows from the standard homological method. Therefore, $\overline{\mathrm{Tor}}_{n-1}(\prod_{\alpha\in I} N^\alpha, D) \cong \prod_{\alpha\in I}\overline{\mathrm{Tor}}_{n-1}(N^\alpha, D)$. □

3.13. Lemma ([[12]]). *Let R and S be rings, L a complex of right S-modules, K a complex of (R,S)-bimodules and P a complex of left R-modules. Suppose that P is finitely presented and L is injective as complexes of right S-modules. Then $\underline{\mathrm{Hom}}(K, L)\overline{\otimes}P \cong \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(P, K), L)$ as complexes. This isomorphism is functorial in P, K and L.*

3.14. Lemma. (1) *Let R and S be rings, n a fixed positive integer, A an n-presented complex of left R-modules, B a complex of (R,S)-bimodules, C an injective complex of right S-modules. Then $\underline{\mathrm{Hom}}(\underline{\mathrm{Ext}}^{n-1}(A, B), C) \cong \overline{\mathrm{Tor}}_{n-1}(\underline{\mathrm{Hom}}(B, C), A)$.*
(2) *Let R and S be rings, n a fixed positive integer, A a complex of left R-modules, B a complex of right (R,S)-bimodules, C an injective complex of right S-modules. Then $\underline{\mathrm{Ext}}^n(A, \underline{\mathrm{Hom}}(B, C) \cong \underline{\mathrm{Hom}}(\overline{\mathrm{Tor}}_n(B, A), C)$.*

Proof. (1) We do an induction on n. If $n = 1$, then the result follows from Lemma 3.13.

Let $n = 2$ and A be an 2-presented complex. Then there exists an exact sequence of complexes $0 \to K \to P \to A \to 0$ with P finitely generated projective and K finitely presented in $\mathscr{C}(R)$. Thus we have the commutative diagram with exact rows by Lemma 3.13:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \underline{\mathrm{Hom}}(\underline{\mathrm{Ext}}^1(A, B), C) & \longrightarrow & \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(K, B), C) & \longrightarrow & \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(P, B), C) \\
 & & \downarrow & & \cong\downarrow & & \cong\downarrow \\
0 & \longrightarrow & \underline{\mathrm{Tor}}_1(\underline{\mathrm{Hom}}(B, C), A) & \longrightarrow & \underline{\mathrm{Hom}}(B, C) \otimes K & \longrightarrow & \underline{\mathrm{Hom}}(B, C) \otimes P.
\end{array}$$

Hence, $\underline{\mathrm{Hom}}(\underline{\mathrm{Ext}}^1(A, B), C) \cong \overline{\mathrm{Tor}}_1(\underline{\mathrm{Hom}}(B, C), A)$.

If $n > 2$, then it follows from the standard homological method. Therefore, $\underline{\mathrm{Hom}}(\underline{\mathrm{Ext}}^{n-1}(A, B), C) \cong \overline{\mathrm{Tor}}_{n-1}(\underline{\mathrm{Hom}}(B, C), A)$.

(2) It follows by similar arguments since $\underline{\mathrm{Hom}}(A\overline{\otimes}B, C) \cong \underline{\mathrm{Hom}}(A, \underline{\mathrm{Hom}}(B, C))$ for any complex A, B and C. □

3.15. Remark. It is not hard to see that

$$\underline{\mathrm{Hom}}(D, \prod_{i\in I} C^i) \cong \prod_{i\in I}\underline{\mathrm{Hom}}(D, C^i),$$

$$D\overline{\otimes}\bigoplus_{i\in I}C^i \cong \bigoplus_{i\in I}(D\overline{\otimes}C^i),$$

$$\underline{\mathrm{Ext}}^n(D, \prod_{i\in I}C^i) \cong \prod_{i\in I}\underline{\mathrm{Ext}}^n(D, C^i),$$

and

$$\overline{\mathrm{Tor}}_n(\bigoplus_{\alpha\in I}N^\alpha, D) \cong \bigoplus_{\alpha\in I}\overline{\mathrm{Tor}}_n(N^\alpha, D)$$

for a fixed positive integer n, any complex D and any family $\{C^i\}_{i\in I}$ of complexes by analogy with the proof of the results above.

4. n-Injective complexes and n-flat complexes. In what follows, if A is n-presented, i.e., there is a finite n-presentation $F^n \to F^{n-1} \to \cdots \to F^1 \to F^0 \to A \to 0$, we will write $K^0 = A, K^1$=Ker$(F^0 \to A)$, K^i=Ker$(F^{i-1} \to F^{i-2})$ for $2 \le i \le n$. Clearly, each K^i is $(n-i)$-presented for $0 \le i < n$.

A complex E is said to be FP-injective if $\mathrm{Ext}^1(P, C) = 0$ for any finitely presented complex P, if and only if $\underline{\mathrm{Ext}}^1(P, C) = 0$ for any finitely presented complex P [[17]]. A complex F is flat if and only if $\overline{\mathrm{Tor}}_1(F, C) = 0$ ($\overline{\mathrm{Tor}}_i(F, C) = 0$ for any $i \ge 1$) for any complex C if and only if $\overline{\mathrm{Tor}}_1(F, P) = 0$ ($\overline{\mathrm{Tor}}_i(F, P) = 0$ for any $i \ge 1$) for any finitely presented complex P [[9]].

To characterize left n-coherent rings for a fixed positive integer n, we introduce the following definitions.

4.1. Definition. (1) A complex C is called n-injective if $\underline{\mathrm{Ext}}^n(D, C) = 0$ for any n-presented complex D;
(2) A complex C is called n-flat if $\overline{\mathrm{Tor}}_n(C, D) = 0$ for any n-presented complex D.

4.2. Remark. (1) It is obvious that a complex D is 1-injective (resp. 1-flat) if and only if D is FP-injective (resp. flat); and any n-injective (resp. n-flat) complex is $n+1$-injective (resp. $n+1$-flat). However, the converse is not true in general (see Example 4.12).
(2) It is clear that the class of all n-injective complexes and the class of all n-flat complexes are closed under extensions and summands.
(3) A complex C is n-injective if and only if Ext^n(D,C)=0 for any n-presented complex D.
(4) If R is a left coherent ring and C is an n-flat (resp. n-injective) complex, then $\overline{\mathrm{Tor}}_i(C, F) = 0$ (resp. $\underline{\mathrm{Ext}}^i(F, C) = 0$) for each n-presented complex F and $i \ge 1$.

4.3. Proposition. *Let $\{C^i\}_{i\in I}$ be a family of complexes of R-modules. Then*

(1) $\prod_{i\in I} C^i$ *is n-injective if and only if each* C^i *is n-injective*;
(2) $\bigoplus_{i\in I} B^i$ *is n-flat if and only if each* B^i *is n-flat.*

Proof. (1) It follows from the isomorphism $\underline{\mathrm{Ext}}^n(N, \prod_{i\in I}C^i) \cong \prod_{i\in I}\underline{\mathrm{Ext}}^n(N, C^i)$, where N is a complex of R-modules.

(2) It follows from the isomorphism $\overline{\mathrm{Tor}}_n(\bigoplus_{i\in I}B^i, N) \cong \bigoplus_{i\in I}\overline{\mathrm{Tor}}_n(B^i, N)$, where N is a complex of R-modules. □

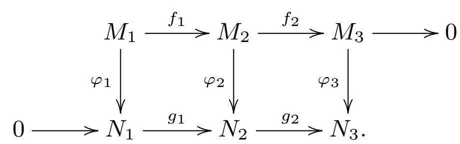
4.4. Proposition. *Let C be a complex of right R-modules and n a fixed positive integer. Then C is n-flat if and only if C^+ is n-injective.*

Proof. It follows from the isomorphism $\underline{\mathrm{Ext}}^n(D, C^+) \cong \overline{\mathrm{Tor}}_n(C, D)^+$ for any complex D. □

4.5. Lemma. *A complex C is n-injective if and only if, for every n-presentation $F^n \to F^{n-1} \to \cdots \to F^1 \to F^0 \to A \to 0$ of a complex A, every $f : K^n \to C$ can be extended a map $g : F^{n-1} \to C$.*

Proof. We have an exact sequence of complexes $0 \to K^n \to F^{n-1} \to K^{n-1} \to 0$, and an isomorphism $\underline{\mathrm{Ext}}^n(A, C) \cong \underline{\mathrm{Ext}}^1(K^{n-1}, C)$ for any complex C. Therefore, the result follows by definition of n-injective complexes. □

4.6. Lemma. *Consider the commutative diagram with exact rows in $\mathscr{C}(R)$:*



Then the following assertions are equivalent:

(a) there exists $\alpha : M_3 \to N_2$ with $\alpha g_2 = \varphi_3$;

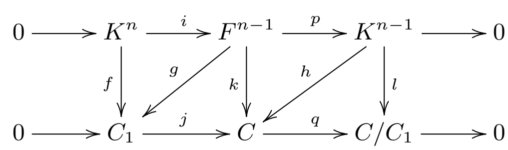
(b) there exists $\beta : M_2 \to N_1$ with $f_1\beta = \varphi_1$.

Proof. (b)⇒(a) If $\beta : M_2 \to N_1$ has the given propery, then $g_1\beta f_1 = g_1\varphi_1 = \varphi_2 f_1$, i.e. $(\varphi_2 g_1 \beta) f_1 = 0$. Since f_2 is the cokernel of f_1, there exists $\alpha : M_3 \to N_2$ with $\alpha f_2 = \varphi_2 - g_1\beta$. This implies $g_2\alpha f_2 = g_2\varphi_2 - g_2 g_1 \beta = g_2\varphi_2 = \varphi_3 f_2$. f_2 being epic we conclude $g_2\alpha = \varphi_3$.

(a)⇒(b) is obtained similarly. □

4.7. Proposition. *The class of all n-injective complexes and the class of all n-flat complexes are closed under pure subcomplexes.*

Proof. Let C_1 be a pure subcomplex of an n-injective complex C. For any finite n-presentation $F^n \to F^{n-1} \to \cdots \to F^1 \to F^0 \to A \to 0$ of A and any map $f : K^n \to C_1$, by Lemma 4.5 and Lemma 4.6, we get the following diagram with exact rows:



where i and j are inclusion maps. So C_1 is n-injective by Lemma 4.5 again.

Let S be a pure subcomplex of an n-flat complex C. Then the pure exact sequence $0 \to S \to C \to C/S \to 0$ induces the split exact sequence $0 \to (C/S)^+ \to C^+ \to S^+ \to 0$. Thus S^+ is n-injective since C^+ is n-injective by Proposition 4.4. So S is n-flat by Proposition 4.4 again. □

4.8. Lemma. *Let $\{C^i\}_{i\in I}$ be a family of complexes. Then*

(1) $\bigoplus_{i\in I} C^i$ *is a pure subcomplex of* $\prod_{i\in I} C^i$;

(2) $\prod_{i\in I} C^i$ *is a pure subcomplex of* $\prod_{i\in I} (C^i)^{++}$.

Proof. (1) Since for any finitely presented complex P, we have $(\prod_{i\in I} C^i)\overline{\otimes}P \cong \prod_{i\in I}(C^i\overline{\otimes}P)$ by Lemma 3.10. Thus we get the following commutative diagram:

$$\begin{array}{ccccc} & & (\bigoplus_{i\in I} C^i)\overline{\otimes}P & \longrightarrow & (\prod_{i\in I} C^i)\overline{\otimes}P \\ & & \downarrow\cong & & \downarrow\cong \\ 0 & \longrightarrow & \bigoplus_{i\in I}(C^i\overline{\otimes}P) & \longrightarrow & \prod_{i\in I}(C^i\overline{\otimes}P). \end{array}$$

Hence, $\bigoplus_{i\in I} C^i$ is a pure subcomplex of $\prod_{i\in I} C^i$.

(2) It is similar to the proof of (1) since C^i is a pure subcomplex of $(C^i)^{++}$ for each $i \in I$. □

4.9. Lemma. *The following are equivalent for a bounded complex C of right R-modules*:
(1) *C is finitely generated*;
(2) $C\overline{\otimes}\prod_\Lambda A^\lambda \to \prod_\Lambda(C\overline{\otimes}A^\lambda)$ *is an epimorphism for every family $\{A^\lambda\}_\Lambda$ of complexes of left R-modules.*

Proof. (1) ⇒ (2) Let $0 \to K \to F \to C \to 0$ be an exact sequence of complexes with F finitely generated projective. Then we have the following commutative diagram:

$$\begin{array}{ccccccc} K\overline{\otimes}\prod_\Lambda A^\lambda & \longrightarrow & F\overline{\otimes}\prod_\Lambda A^\lambda & \longrightarrow & C\overline{\otimes}\prod_\Lambda A^\lambda & \longrightarrow & 0 \\ \downarrow\tau_K & & \downarrow\tau_F & & \downarrow\tau_C & & \\ \prod_\Lambda(K\overline{\otimes}A^\lambda) & \longrightarrow & \prod_\Lambda(F\overline{\otimes}A^\lambda) & \longrightarrow & \prod_\Lambda(C\overline{\otimes}A^\lambda) & \longrightarrow & 0 \end{array}$$

with exact rows. But τ_F is isomorphism by Lemma 3.10. So τ_C is onto.

(2) ⇒ (1) Since C is bounded, we can assume that C has the following form:

$$\cdots \longrightarrow 0 \longrightarrow C_m \longrightarrow C_{m-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0 \longrightarrow \cdots.$$

It is enough to prove that C_j is finitely generated in R-Mod for $j = 1, \cdots, m$. Let $\{M_i\}_{i\in I}$ be a family of left R-modules. Then

$$(C \otimes^{\cdot} \prod_{i\in I} \underline{M_i})_m = \bigoplus_{t\in\mathbb{Z}} C_t \otimes (\prod_{i\in I} \underline{M_i})_{m-t} = C_m \otimes \prod_{i\in I} M_i$$

and

$$(\prod_{i\in I} C \otimes^{\cdot} \underline{M_i})_m = \prod_{i\in I}\bigoplus_{t\in\mathbb{Z}} C_t \otimes (\underline{M_i})_{m-t} = \prod_{i\in I} C_m \otimes M_i.$$

$$C \otimes^{\cdot} \prod_{i\in I} \underline{M_i}: \quad \cdots \to 0 \xrightarrow{\delta_{m+1}} C_m \otimes \prod_{i\in I} M_i \xrightarrow{\delta_m} \cdots \xrightarrow{\delta_2} C_1 \otimes \prod_{i\in I} M_i \xrightarrow{\delta_1} 0 \to \cdots.$$

$$\prod_{i\in I} C \otimes^{\cdot} \underline{M_i}: \quad \cdots \to 0 \xrightarrow{\sigma_{m+1}} \prod_{i\in I} C_m \otimes M_i \xrightarrow{\sigma_m} \cdots \xrightarrow{\sigma_2} \prod_{i\in I} C_1 \otimes M_i \xrightarrow{\sigma_1} 0 \to \cdots.$$

Hence $C\overline{\otimes}\prod_{i\in I}\underline{M_i}$ and $\prod_{i\in I}(C\overline{\otimes}\underline{M_i})$ have the following form:

$C\overline{\otimes}\prod_{i\in I}\underline{M_i}$:

$$\cdots \to 0 \longrightarrow C_m \otimes \prod_{i\in I} M_i \longrightarrow \frac{C_{m-1}\otimes\prod_{i\in I} M_i}{\mathrm{Im}\delta_m} \longrightarrow \cdots \longrightarrow \frac{C_1\otimes\prod_{i\in I} M_i}{\mathrm{Im}\delta_2} \longrightarrow 0 \to \cdots.$$

$\prod_{i\in I}(C\overline{\otimes}\underline{M_i})$:

$$\cdots \to 0 \longrightarrow \prod_{i\in I}(C_m \otimes M_i) \longrightarrow \frac{\prod_{i\in I}(C_{m-1}\otimes M_i)}{\mathrm{Im}\sigma_m} \longrightarrow \cdots \longrightarrow \frac{\prod_{i\in I}(C_1\otimes M_i)}{\mathrm{Im}\sigma_2} \longrightarrow 0 \to \cdots .$$

Since $C\overline{\otimes}\prod_{i\in I}\underline{M_i} \to \prod_{i\in I}(C\overline{\otimes}\underline{M_i})$ is epic, $C_m \otimes \prod_{i\in I} M_i \to \prod_{i\in I}(C_m \otimes M_i)$ is epic, then C_m is finitely generated in R-Mod by [[11], Lemma 3.2.21]. If we replace the complex $\underline{M_i}$ with $\overline{M_i}$, we have C_{m-1} is finitely generated in R-Mod. If we replace $\underline{M_i}$ with $\cdots \to 0 \to M_0 \to M_1 \to M_2 \to 0 \to \cdots$, we have C_{m-2} is finitely generated in R-Mod. We continue the process, we can get C_j is finitely generated in R-Mod for $j = 1, \cdots, m$ by [[11], Lemma 3.2.21]. □

4.10. Lemma. *The following are equivalent for a bounded complex C of right R-modules*:
(1) *C is finitely presented*;
(2) $C\overline{\otimes}\prod_\Lambda A^\lambda \to \prod_\Lambda(C\overline{\otimes}A^\lambda)$ *is an isomorphism for every family $\{A^\lambda\}_\Lambda$ of complexes of left R-modules.*

Proof. (1) ⇒ (2) It follows by Lemma 3.10.

(2) ⇒ (1) C is finitely generated by the Lemma 4.9 above. So let $0 \to K \to F \to C \to 0$ be exact with F finitely generated projective. It now suffices to show that K is finitely generated. But for any Λ, we have a commutative diagram:

$$\begin{array}{ccccccc} K\overline{\otimes}\prod_\Lambda A^\lambda & \longrightarrow & F\overline{\otimes}\prod_\Lambda A^\lambda & \longrightarrow & C\overline{\otimes}\prod_\Lambda A^\lambda & \longrightarrow & 0 \\ \downarrow{\scriptstyle \tau_K} & & \downarrow{\scriptstyle \tau_F} & & \downarrow{\scriptstyle \tau_C} & & \\ \prod_\Lambda(K\overline{\otimes}A^\lambda) & \longrightarrow & \prod_\Lambda(F\overline{\otimes}A^\lambda) & \longrightarrow & \prod_\Lambda(C\overline{\otimes}A^\lambda) & \longrightarrow & 0 \end{array}$$

with exact rows where τ_F and τ_C are isomorphisms. So τ_K is onto and hence K is finitely generated by Lemma 4.9.

A ring R is left coherent if and only if the direct limit of FP-injective complexes of left R-modules is FP-injective [[17]]. Now we will give some characterizations of n-coherent rings using the results above.

4.11. Theorem. *Let R be a ring and n a fixed positive integer. Then the following are equivalent*:
(1) *R is left n-coherent*;
(2) *Every direct product of n-flat complexes of right R-modules is n-flat*;
(3) *Every direct limit of n-injective complexes of left R-modules is n-injective*;
(4) $\mathrm{Ext}^n(A, \varinjlim C^i) \cong \varinjlim \mathrm{Ext}^n(A, C^i)$ *for every n-presented complex A of left R-modules and direct system $\{C^i\}_{i\in I}$ of complexes of left R-modules*;
(5) $\overline{\mathrm{Tor}}_n(\prod_{\alpha\in I} D^\alpha, A) \cong \prod_{\alpha\in I}\overline{\mathrm{Tor}}_n(D^\alpha, A)$ *for any family $\{D^\alpha\}_{\alpha\in\Lambda}$ of complexes and any n-presented complex A of left R-modules*;
(6) *A complex C of left R-modules is n-injective if and only if C^+ is n-flat*;
(7) *A complex C of left R-modules is n-injective if and only if C^{++} is n-injective*;
(8) *A complex C of right R-modules is n-flat if and only if C^{++} is n-flat*;
(9) *For any ring S, $\underline{\mathrm{Hom}}(\underline{\mathrm{Ext}}^n(A, B), D) \cong \overline{\mathrm{Tor}}_n(\underline{\mathrm{Hom}}(B, D), A)$ for any n-presented complex A of left R-modules, any complex B of (R,S)-bimodules, any injective complex D of right S-modules.*

Proof. (1) ⇒ (4) It follows by Lemma 3.11.

(4) ⇒ (3) It is trivial.

(3) ⇒ (1) Let A be an n-presented complex of left R-modules. It is sufficient to show that K^n is finitely presented. Let $\{C^i\}_{i\in I}$ be a family of n-injective complexes of

left R-modules, where I is a directed set. Then $\varinjlim C^i$ is n-injective by (3), and hence $\mathrm{Ext}^1(K^{n-1}, \varinjlim C^i) = \mathrm{Ext}^n(A, \varinjlim C^i) = 0$.

Thus there is a commutative diagram with exact rows:

$$\begin{array}{ccc}
\mathrm{Hom}(K^{n-1}, \varinjlim C^i) & \xrightarrow{f_1} & \varinjlim \mathrm{Hom}(K^{n-1}, C^i) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(F^{n-1}, \varinjlim C^i) & \xrightarrow{f_2} & \varinjlim \mathrm{Hom}(F^{n-1}, C^i) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(K^{n}, \varinjlim C^i) & \xrightarrow{f_3} & \varinjlim \mathrm{Hom}(K^{n}, C^i) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}.$$

Since both K^{n-1} and F^{n-1} are finitely presented, f_1 and f_2 are isomorphisms by Lemma 3.9. Hence f_3 is an isomorphism. K^n is finitely generated, so K^n is finitely presented by Lemma 3.9. Thus A is $(n+1)$-presented. Therefore, R is left n-coherent.

(1) $\Rightarrow$ (5) It holds by Lemma 3.12.

(5) $\Rightarrow$ (2) It is obvious.

(2) $\Rightarrow$ (1) Let A be an n-presented complex of left R-modules. We will show that K^{n-1} is 2-presented. For any family $\{A^i\}_{i\in I}$ of n-flat complexes of right R-modules, $\prod_{i\in I} A^i$ is an n-flat complex. Thus the exact sequence of complexes $0 \to K^n \to F^{n-1} \to K^{n-1} \to 0$ gives rise to the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\prod_{i\in I} A^i)\overline{\otimes} K^n & \longrightarrow & (\prod_{i\in I} A^i)\overline{\otimes} F^{n-1} & \longrightarrow & (\prod_{i\in I} A^i)\overline{\otimes} K^{n-1} & \longrightarrow & 0 \\
 & & \phi_1 \downarrow & & \phi_2 \downarrow & & \phi_3 \downarrow & & \\
0 & \longrightarrow & \prod_{i\in I}(A^i\overline{\otimes} K^n) & \longrightarrow & \prod_{i\in I}(A^i\overline{\otimes} F^{n-1}) & \longrightarrow & \prod_{i\in I}(A^i\overline{\otimes} K^{n-1}) & \longrightarrow & 0.
\end{array}$$

By Lemma 4.10, ϕ_2 and ϕ_3 are isomorphisms, and hence ϕ_1 is an isomorphism. Thus K^n is finitely presented, and so K^{n-1} is 2-presented, hence A is $n+1$-presented.

(6) $\Rightarrow$ (7) Let C be a complex of left R-modules. If C is n-injective, then C^+ is n-flat by (6), and so C^{++} is n-injective by Proposition 4.4. Conversely, if C^{++} is n-injective, then C is a pure subcomplex of C^{++} (see [[12], Proposition 5.1.4]). So C is n-injective by Proposition 4.7.

(7) $\Rightarrow$ (8) If C is an n-flat complex of right R-modules, then C^+ is an n-injective complex of left R-modules by Proposition 4.4. Hence C^{+++} is n-injective by (7). Thus C^{++} is n-flat by Proposition 4.4. Conversely, if C^{++} is n-flat, then C is n-flat by Proposition 4.7.

(8) $\Rightarrow$ (2) Let $\{C^i\}_{i\in I}$ be a family of n-flat complexes of right R-modules. By Proposition 4.3, $\bigoplus_{i\in I} C^i$ is n-flat, so $(\bigoplus_{i\in I} C^i)^{++} \cong (\prod_{i\in I} {C^i}^+)^+$ is n-flat by (8). But $\bigoplus_{i\in I}(C^i)^+$ is a pure subcomplex of $\prod_{i\in I}(C^i)^+$ by Lemma 4.8, and so $(\prod_{i\in I}(C^i)^+)^+ \to (\bigoplus_{i\in I}(C^i)^+)^+ \to 0$ splits. Thus $\prod_{i\in I}(C^i)^{++} \cong (\bigoplus_{i\in I}(C^i)^+)^+$ is n-flat. Since $\prod_{i\in I} C^i$ is a pure subcomplex of $\prod_{i\in I}(C^i)^{++}$ by Lemma 4.8, $\prod_{i\in I} C^i$ is n-flat by Proposition 4.7.

(1) $\Rightarrow$ (9) It follows from Lemma 3.14.

(9) ⇒ (6) Let $S = \mathbb{Z}$, $D = \overline{\mathbb{Q}/\mathbb{Z}}$ and $B = C$. Then

$$\overline{\mathrm{Tor}}_n(C^+, A) \cong \underline{\mathrm{Ext}}^n(A, C)^+$$

for all n-presented complexes A of left R-modules by (9), and hence (6) holds. □

4.12. Example. If R is $n+1$-coherent but not n-coherent, then we can form a direct limit $\varinjlim C^i$ of n-injective complexes $\{C^i\}_{i\in I}$, which is not n-injective but is necessary $n+1$-injective; we can also form a direct product $\prod_{\alpha\in I} C^i$ of n-flat complexes $\{C^i\}_{i\in I}$, which is not n-flat but is necessary $n+1$-flat.

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References

[1] D. Bennis, n-$\mathscr{X}$-coherent rings, Int. Electron. J. Algebra. 7 (2010) 128-139.
[2] N. Bourbaki, Algèbre Commutative, Chapitre 1-2, Masson, Paris, 1961.
[3] S. U. Chase, Direct products of modules, Trans. Amer. Math. Soc. 97 (1960) 457-473.
[4] T. J. Cheatham, D. R. Stone, Flat and projective character modules, Proc. Amer. Math. Soc. 81 (1981) 175-177.
[5] J. L. Chen and N. Q. Ding, On n-coherent rings, Comm. Algebra. 24(10) (1996) 3211-3216.
[6] L. W. Christensen, H. B. Foxby and H. Holm, Derived Category Methods in Commutative Algebra, 2011.
[7] D. L. Costa, Parameterizing families of non-noetherian rings, Comm. Algebra 22 (1994) 3997-4011.
[8] N. Q. Ding and L. X. Mao, On a new characterization of coherent rings, Publ. Math. (Debreen), 71 (2007) 67-82
[9] E. E. Enochs, J. R. Garcßa Rozas, Tensor products of complexes, Math. J. Okayama Univ. 39 (1997) 17-39.
[10] E. E. Enochs, J. R. Garcßa Rozas, Flat covers of complexes, J. Algebra 210 (1998) 86-102.
[11] E. E. Enochs and O. M. G. Jenda, Relative Homological Algebra, (Walter de Gruyter, 2000).
[12] J. R. Garcßa Rozas, Covers and envelopes in the category of complexes of modules, Boca Raton London New York Washington, D.C. (1999).
[13] S. Glaz, Commutative Coherent Rings, Lect. Notes Math. 1371, Springer-Verlag, Berlin, 1989. Publishing. (1993) 172-181.
[14] B. Stenström, Coherent rings and FP-injective modules, J. London Math. Soc. 2 (1970) 323-329.
[15] S. Stenström, Rings of Quotients, New York: Springer-Verlag, 1975.
[16] W. V. Vasconcelos, The Rings of Dimension Two, Marcel Dekker, New York, 1976.
[17] Z. P. Wang and Z. K. Liu, FP-injective complexes and FP-injective dimension of complexes, J. Aust. Math. Soc. (2011) 1-25.