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Integral representations and new generating functions of Chebyshev polynomials

Clemente Cesarano *

Abstract

In this paper we use the two-variable Hermite polynomials and their operational rules to derive integral representations of Chebyshev polynomials. The concepts and the formalism of the Hermite polynomials $H_n(x, y)$ are a powerful tool to obtain most of the properties of the Chebyshev polynomials. By using these results, we also show how it is possible to introduce relevant generalizations of these classes of polynomials and we derive for them new identities and integral representations. In particular we state new generating functions for the first and second kind Chebyshev polynomials.

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1. Introduction

The Hermite polynomials [1] can be introduced by using the concept and the formalism of the generating function and related operational rules. In the following we recall the main definitions and properties.

1.1. Definition. The two-variable Hermite Polynomials $H_m^{(2)}(x, y)$ of Kampé de Fériet form [2, 3] are defined by the following formula

$$H_m^{(2)}(x, y) = \sum_{n=0}^{\left[\frac{m}{2}\right]} \frac{m!}{n!(m-2n)!} y^n x^{m-2n} \tag{1.1}$$

*Faculty of Engineering, International Telematic University UNINETTUNO Corso Vittorio Emanuele II, 39 – 00186 – Roma (Italy)
Email: `c.cesarano@uninettunouniversity.net`

We will indicate the two-variable Hermite polynomials of Kampé de Fériet form by using the symbol $H_m(x,y)$ instead than $H_m^{(2)}(x,y)$.

The two-variable Hermite polynomials $H_m(x,y)$ are linked to the ordinary Hermite polynomials by the following relations

$$H_m\left(x,-\frac{1}{2}\right) = He_m(x),$$

where

$$He_m(x) = m!\sum_{r=0}^{\left[\frac{m}{2}\right]} \frac{(-1)^r x^{n-2r}}{r!(n-2r)!2^r}$$

and

$$H_m\left(2x,-1\right) = H_m(x),$$

where

$$H_m(x) = m!\sum_{r=0}^{\left[\frac{m}{2}\right]} \frac{(-1)^r (2x)^{n-2r}}{r!(n-2r)!}$$

and it is also important to note that the Hermite polynomials $H_m(x,y)$ satisfy the relation

$$H_m(x,0) = x^m. \tag{1.2}$$

1.2. Proposition. *The polynomials $H_m(x,y)$ solve the following partial differential equation:*

$$\frac{\partial^2}{\partial x^2}H_m(x,y) = \frac{\partial}{\partial y}H_m(x,y). \tag{1.3}$$

Proof. By deriving, separately with respect to x and to y, in the (1), we obtain

$$\begin{aligned} \frac{\partial}{\partial x}H_m(x,y) &= mH_{m-1}(x,y) \\ \frac{\partial}{\partial y}H_m(x,y) &= H_{m-2}(x,y). \end{aligned}$$

From the first of the above relation, by deriving again with respect to x and by noting the second identity, we end up with the (7). □

The *Proposition 1* help us to derive an important operational rule for the Hermite polynomials $H_m(x,y)$. In fact, by considering the differential equation (7) as linear ordinary in the variable y and by remanding the (6) we can immediately state the following relation:

$$H_m(x,y) = e^{y\frac{\partial^2}{\partial x^2}}x^m. \tag{1.4}$$

The generating function of the above Hermite polynomials can be state in many ways, we have in fact:

1.3. Proposition. *The polynomials $H_m(x,y)$ satisfy the following differential difference equation:*

$$\begin{aligned} \frac{d}{dz}Y_n(z) &= anY_{n-1}(z)+bn(n-1)Y_{n-2}(z) \\ Y_n(0) &= \delta_{n,0} \end{aligned} \tag{1.5}$$

where a and b are real numbers.

Proof. By using the generating function method, by putting:

$$G(z;t)=\sum_{n=0}^{+\infty}\frac{t^n}{n!}Y_n(z),$$

with t continuous variable,we can rewrite the (9) in the form

$$\begin{aligned} \frac{d}{dz}G(z;t) &= \left(at+bt^2\right)G(z;t) \\ G(0;t) &= 1 \end{aligned}$$

that is a linear ordinary differential equation and then its solution reads

$$G(z;t)=\exp\left(xt+yt^2\right)$$

where we have putted $az=x$ and $bz=y$. Finally, by exploiting the r.h.s of the previous relation we find the thesis and also the relation linking the Hermite polynomials and their generating function

$$\exp\left(xt+yt^2\right)=\sum_{n=0}^{+\infty}\frac{t^n}{n!}H_n(x,y). \tag{1.6}$$

□

The use of operational identities, may significantly simplify the study of Hermite generating functions and the discovery of new relations, hardly achievable by conventional means.

By remanding that the following identity

$$e^{-\frac{1}{4}\frac{d^2}{dx^2}}(2x)^n=\left(2x-\frac{d}{dx}\right)^n(1) \tag{1.7}$$

is linked to the standard Burchnall identity [4], we can immediately state the following relation.

1.4. Proposition. *The operational definition of the polynomials $H_n(x)$ reads:*

$$e^{-\frac{1}{4}\frac{d^2}{dx^2}}(2x)^n=H_n(x). \tag{1.8}$$

Proof. By exploiting the r.h.s of the (13), we immediately obtain the Burchnall identity

$$\left(2x-\frac{d}{dx}\right)^n=n!\sum_{s=0}^{n}(-1)^s\frac{1}{(n-s)!s!}H_{n-s}(x)\frac{d^s}{dx^s} \tag{1.9}$$

after using the decoupling Weyl identity [4, 5, 6], since the commutator of the operators of l.h.s. is not zero. The derivative operator of the (15) gives a not trivial contribution only in the case $s = 0$ and then we can conclude with

$$\left(2x - \frac{d}{dx}\right)^n (1) = H_n(x)$$

which prove the statement. □

The Burchnall identity can be also inverted to give another important relation for the Hermite polynomials $H_n(x)$. We find in fact:

1.5. Proposition. *The polynomials $H_n(x)$ satisfy the following operational identity:*

$$(1.10)\qquad H_n\left(x + \frac{1}{2}\frac{d}{dx}\right) = \sum_{s=0}^{n} \binom{n}{s} (2x)^{n-s} \frac{d^s}{dx^s}.$$

Proof. By multiplying the l.h.s. of the above relation by $\frac{t^n}{n!}$ and then summing up, we obtain:

$$\sum_{n=0}^{+\infty} \frac{t^n}{n!} H_n\left(x + \frac{1}{2}\frac{d}{dx}\right) = e^{2\left(x+\frac{1}{2}\right)\left(\frac{d}{dx}\right)t - t^2}.$$

By using the Weyl identity, the r.h.s. of the above equation reads:

$$e^{2\left(x+\frac{1}{2}\right)\left(\frac{d}{dx}\right)t - t^2} = e^{2xt} e^{t\frac{d}{dx}}$$

and from which (17) immediately follows, after expanding the r.h.s and by equating the like $t-$powers. □

The previous results can be used to derive some addition and multiplication relations for the Hermite polynomials.

1.6. Proposition. *The polynomials $H_n(x)$ satisfy the following identity, $\forall\ n, m \in N$:*

$$(1.11)\qquad H_{n+m}(x) = \sum_{s=0}^{\min(n,m)} (-2)^s \binom{n}{s}\binom{m}{s} s! H_{n-s}(x) H_{m-s}(x).$$

Proof. By using the *Proposition 3*, we can write:

$$H_{n+m}(x) = \left(2x - \frac{d}{dx}\right)^n \left(2x - \frac{d}{dx}\right)^m = \left(2x - \frac{d}{dx}\right)^n H_m(x)$$

and by exploiting the r.h.s. of the above relation, we find:

$$H_{n+m}(x) = \sum_{s=0}^{n} (-1)^s \binom{n}{s} H_{n-s}(x) \frac{d^s}{dx^s} H_m(x).$$

After noting that the following operational identity holds:

$$\frac{d^s}{dx^s} H_m(x) = \frac{2^s m!}{(m-s)!} H_{m-s}(x)$$

we obtain immediately the statement. □

From the above proposition we can immediately derive as a particular case, the following identity:

$$(1.12)\qquad H_{2n}(x) = (-1)^n 2^n (n!)^2 \sum_{s=0}^{n} \frac{(-1)^s \left[H_s(x)\right]^2}{2^s (s!)^2 (n-s)!}.$$

The use of the identity (17), stated in *Proposition 4*, can be exploited to obtain the inverse of relation contained in (24). We have indeed:

1.7. Proposition. *Given the Hermite polynomial $H_n(x)$, the square $\left[H_n(x)\right]^2$ can be written as:*

$$(1.13)\qquad H_n(x)H_n(x) = \left[H_n(x)\right]^2 = 2^n (n!)^2 \sum_{s=0}^{n} \frac{H_{2n}(x)}{2^s (s!)^2 (n-s)!}.$$

Proof. We can write:

$$\left[H_n(x)\right]^2 = e^{-\frac{1}{4}\frac{d^2}{dx^2}} \left[H_n\left(x + \frac{1}{2}\frac{d}{dx}\right) H_n\left(x + \frac{1}{2}\frac{d}{dx}\right)\right],$$

by using the relation (17), we find, after manipulating the r.h.s.:

$$\left[H_n(x)\right]^2 = e^{-\frac{1}{4}\frac{d^2}{dx^2}} \left[2^n (n!)^2 \sum_{s=0}^{n} \frac{(2x)^{2n}}{2^s (s!)^2 (n-s)!}\right]$$

and then, from the Burchnall identity (16), the thesis. □

A generalization of the identities stated for the one variable Hermite polynomials can be easily done for the polynomials $H_n(x, y)$.

We have in fact:

1.8. Proposition. *The following identity holds*

$$(1.14)\qquad \left(x + 2y\frac{\partial}{\partial x}\right)^n (1) = \sum_{s=0}^{n} (2y)^s \binom{n}{s} H_n(x, y) \frac{\partial^s}{\partial x^s}(1).$$

Proof. By multiplying the l.h.s. of the above equation by $\frac{t^n}{n!}$ and then summing up, we find

$$\sum_{n=0}^{+\infty} \frac{t^n}{n!} \left(x + 2y\frac{\partial}{\partial x}\right)^n = e^{t\left(x + 2y\frac{\partial}{\partial x}\right)}(1).$$

By noting that the commutator of the two operators of the r.h.s. is

$$\left[tx, t2y\frac{\partial}{\partial x}\right] = -2t^2 y$$

we obtain

$$(1.15)\qquad \sum_{n=0}^{+\infty} \frac{t^n}{n!} \left(x + 2y\frac{\partial}{\partial x}\right)^n = e^{xt + yt^2} e^{2ty\frac{\partial}{\partial x}}(1).$$

After expanding and manipulating the r.h.s. of the previous relation and by equating the like t powers we find immediately the (28). □

By using the *Proposition 7* and the definition of polynomials $H_n(x,y)$, we can derive a generalization of the Burchnall-type identity

$$e^{y\frac{\partial^2}{\partial x^2}}x^n = \left(x + 2y\frac{\partial}{\partial x}\right)^n \tag{1.16}$$

and the related inverse

$$H_n\left(x - 2y\frac{\partial}{\partial x}, y\right) = \sum_{s=0}^{n}(-2y)^s\binom{n}{s}x^{n-s}\frac{\partial^s}{\partial x^s}. \tag{1.17}$$

We can also generalize the multiplication rules obtained for the Hermite polynomials $H_n(x)$, stated in *Proposition 5*.

1.9. Proposition. *Given the Kampé de Fériet Hermite polynomials $H_n(x,y)$, we have*

$$H_{n+m}(x,y) = m!n!\sum_{s=0}^{\min(n,m)}(2y)^s\frac{H_{n-s}(x,y)H_{m-s}(x,y)}{(n-s)!(m-s)!s!}. \tag{1.18}$$

Proof. By using the relations stated in the (28) and (32), we can write

$$H_{n+m}(x,y) = \left(x + 2y\frac{\partial}{\partial x}\right)^n H_m(x,y)$$

and then

$$H_{n+m}(x,y) = \sum_{s=0}^{n}(2y)^s\binom{n}{s}H_n(x,y)\frac{\partial^s}{\partial x^s}H_m(x,y). \tag{1.19}$$

By noting that

$$\frac{\partial^s}{\partial x^s}x^m = \frac{m!}{(m-2s)!}x^{m-2s}$$

we obtain

$$\frac{\partial^s}{\partial x^s}H_m(x,y) = \frac{m!}{(m-s)!}H_{m-s}(x,y).$$

After substituting the above relation in the (36) and rearranging the terms we immediately obtain the thesis. □

From the previous results, it also immediately follows:

$$H_n(x,y)H_m(x,y) = n!m!\sum_{s=0}^{\min(n,m)}(-2y)^s\frac{H_{n+m-2s}(x,y)}{(n-s)!(m-s)!s!}. \tag{1.20}$$

The previous identity and the equation (34) can be easily used to derive the particular case for $n = m$. We have in fact

(1.21) $$H_{2n}(x,y) = 2^n (n!)^2 \sum_{s=0}^{n} \frac{[H_s(x,y)]^2}{(s)!^2 (n-s)! 2^s}$$

(1.22) $$[H_n(x,y)]^2 = (-2y)^n (n!)^2 \sum_{s=0}^{n} \frac{(-1)^s H_{2s}(x,y)}{(n-s)!(s!)^2 2^s}.$$

Before concluding this section we want prove two other important relations satisfied by the Hermite polynomials $H_n(x,y)$.

1.10. Proposition. *The Hermite polynomials $H_n(x,y)$ solve the following differential equation:*

(1.23) $$2y \frac{\partial^2}{\partial x^2} H_n(x,y) + x \frac{\partial}{\partial x} H_n(x,y) = n H_n(x,y)$$

Proof. By using the results derived from the *Proposition 7*, we can easily write that:

$$\left(x + 2y \frac{\partial}{\partial x} \right) H_n(x,y) = H_{n+1}(x,y)$$

and from the previous recurrence relations:

$$\frac{\partial}{\partial x} H_n(x,y) = n H_{n-1}(x,y)$$

we have

$$\left(x + 2y \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial x} \right) H_n(x,y) = n H_n(x,y)$$

which is the thesis.

□

From this statement can be also derived an important recurrence relation. In fact, by noting that:

(1.24) $$H_{n+1}(x,y) = x H_n(x,y) + 2y \frac{\partial}{\partial x} H_n(x,y)$$

and then we can conclude with:

(1.25) $$H_{n+1}(x,y) = x H_n(x,y) + 2ny H_{n-1}(x,y).$$

2. Integral representations of Chebyshev polynomials

In this section we will introduce new representations of Chebyshev polynomials [7, 8, 9, 10, 11], by using the Hermite polynomials and the method of the generating function. Since the second kind Chebyshev polynomials $U_n(x)$ reads

(2.1) $$U_n(x) = \frac{\sin\left[(n+1)\arccos(x)\right]}{\sqrt{1-x^2}},$$

by exploiting the right hand side of the above relation, we can immediately get the following explicit form

$$U_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k (n-k)!(2x)^{n-2k}}{k!(n-2k)!}. \tag{2.2}$$

2.1. Proposition. *The second kind Chebyshev polynomials satisfy the following integral representation* [9]*:*

$$U_n(x) = \frac{1}{n!} \int_0^{+\infty} e^{-t} t^n H_n\left(2x, -\frac{1}{t}\right) dt. \tag{2.3}$$

Proof. By noting that

$$n! = \int_0^{+\infty} e^{-t} t^n dt$$

we can write

$$(n-k)! = \int_0^{+\infty} e^{-t} t^{n-k} dt. \tag{2.4}$$

From the explicit form of the Chebyshev polynomials $U_n(x)$, given in the (49), and by recalling the standard form of the two-variable Hermite polynomials:

$$H_n(x,y) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{y^k x^{n-2k}}{k!(n-2k)!}$$

we can immediately write:

$$U_n(x) = \int_0^{+\infty} e^{-t} t^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k t^{-k} (2x)^{n-2k}}{k!(n-2k)!} dt$$

and then the thesis. □

By following the same procedure, we can also obtain an analogous integral representation for the Chebyshev polynomials of first kind $T_n(x)$. Since their explicit form is given by:

$$T_n(x) = \frac{n}{2} \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k (n-k-1)!(2x)^{n-2k}}{k!(n-2k)!}, \tag{2.5}$$

by using the same relations written in the previous proposition, we easily obtain:

$$T_n(x) = \frac{1}{2(n-1)!} \int_0^{+\infty} e^{-t} t^{n-1} H_n\left(2x, -\frac{1}{t}\right) dt. \tag{2.6}$$

These results can be useful in several physics and engineering problems, for instance in electromagnetic field problems and particle accelerators analysis [12, 13, 14] In the previous Section we have stated some useful operational results regarding the two-variable Hermite polynomials; in particular we have derived their fundamental recurrence relations. These relations can be used to state important results linking the Chebyshev polynomials of the first and second kind [7, 9].

2.2. Theorem. *The Chebyshev polynomials $T_n(x)$ and $U_n(x)$ satisfy the following recurrence relations:*

$$\begin{aligned} \frac{d}{dx}U_n(x) &= nW_{n-1}(x) \\ U_{n+1}(x) &= xW_n(x) - \frac{n}{n+1}W_{n-1}(x) \end{aligned} \tag{2.7}$$

and

$$T_{n+1}(x) = xU_n(x) - U_{n-1}(x) \tag{2.8}$$

where

$$W_n(x) = \frac{2}{(n+1)!}\int_0^{+\infty} e^{-t}t^{n+1}H_n\left(2x, -\frac{1}{t}\right)dt.$$

Proof. The recurrence relations for the standard Hermite polynomials $H_n(x,y)$ stated in the first Section, can be costumed in the form

$$\begin{aligned} \left[(2x) + \left(-\frac{1}{t}\right)\frac{\partial}{\partial x}\right]H_n\left(2x, -\frac{1}{t}\right) &= H_{n+1}\left(2x, -\frac{1}{t}\right) \\ \frac{1}{2}\frac{\partial}{\partial x}H_n\left(2x, -\frac{1}{t}\right) &= nH_{n-1}\left(2x, -\frac{1}{t}\right). \end{aligned} \tag{2.9}$$

From the integral representations stated in the relations (50) and (53), relevant to the Chebyshev polynomials of the first and second kind, and by using the second of the identities written above, we obtain

$$\frac{d}{dx}U_n(x) = \frac{2n}{n!}\int_0^{+\infty} e^{-t}t^nH_{n-1}\left(2x, -\frac{1}{t}\right)dt \tag{2.10}$$

and

$$\frac{d}{dx}T_n(x) = \frac{n}{(n-1)!}\int_0^{+\infty} e^{-t}t^{n-1}H_{n-1}\left(2x, -\frac{1}{t}\right)dt. \tag{2.11}$$

It is easy to note that the above relation gives a link between the polynomials $T_n(x)$ and $U_n(x)$; in fact, since:

$$U_{n-1}(x) = \frac{1}{(n-1)!}\int_0^{+\infty} e^{-t}t^{n-1}H_{n-1}\left(2x, -\frac{1}{t}\right)dt$$

we immediately get:

$$\frac{d}{dx}T_n(x) = nU_{n-1}(x). \tag{2.12}$$

By applying the multiplication operator to the second kind Chebyshev polynomials, stated in the first of the identities (56), we can write

$$U_{n+1}(x) = \frac{1}{(n+1)!}\int_0^{+\infty} e^{-t}t^{n+1}\left[(2x) + \left(-\frac{1}{t}\right)\frac{\partial}{\partial x}\right]H_n\left(2x, -\frac{1}{t}\right)dt$$

that is

(2.13) $\quad U_{n+1}(x) =$

$$= x\frac{2}{(n+1)!}\int_0^{+\infty} e^{-t}t^{n+1}H_n\left(2x,-\frac{1}{t}\right)dt - \frac{n}{n+1}\frac{2}{n!}\int_0^{+\infty} e^{-t}t^{n}H_{n-1}\left(2x,-\frac{1}{t}\right)dt.$$

The second member of the r.h.s. of the above relation suggests us to introduce the following polynomials:

$$W_n(x) = \frac{2}{(n+1)!}\int_0^{+\infty} e^{-t}t^{n+1}H_n\left(2x,-\frac{1}{t}\right)dt \tag{2.14}$$

recognized as belonging to the families of the Chebyshev polynomials. Thus, from the relation (57), we have:

$$\frac{d}{dx}U_n(x) = nW_{n-1}(x) \tag{2.15}$$

and, from the identity (60), we get

$$U_{n+1}(x) = xW_n(x) - \frac{n}{n+1}W_{n-1}(x). \tag{2.16}$$

Finally, by using the multiplication operator for the first kind Chebyshev polynomials, we can write

$$T_{n+1}(x) = \frac{1}{2n!}\int_0^{+\infty} e^{-t}t^{n}\left[(2x)+\left(-\frac{1}{t}\right)\frac{\partial}{\partial x}\right]H_n\left(2x,-\frac{1}{t}\right)dt \tag{2.17}$$

and then, after exploiting the r.h.s. of the above relation, we can find

$$T_{n+1}(x) = xU_n(x) - U_{n-1}(x) \tag{2.18}$$

which completely prove the theorem.

□

3. Generating functions

By using the integral representations and the related recurrence relations, stated in the previous Section, for the Chebyshev polynomials of the first and second kind, it is possible to derive a slight different relations linking these polynomials and their generating functions [1, 2, 3, 4, 5, 7, 8, 9, 15].

We note indeed, for the Chebyshev polynomials $U_n(x)$, that by multiplying both sides of equation (50) by ξ^n, $|\xi| < 1$ and by summing up over n, it follows that

$$\sum_{n=0}^{+\infty}\xi^n U_n(x) = \int_0^{+\infty} e^{-t}\sum_{n=0}^{+\infty}\frac{(t\xi)^n}{n!}H_n\left(2x,-\frac{1}{t}\right)dt. \tag{3.1}$$

By recalling the generating function of the polynomials $H_n(x,y)$ stated in the relation (12) and by integrating over t, we end up with

$$\sum_{n=0}^{+\infty}\xi^n U_n(x) = \frac{1}{1-2\xi x+\xi^2}. \tag{3.2}$$

We can now state the related generating function for the first kind Chebyshev polynomials $T_n(x)$ and for the polynomials $W_n(x)$, by using the results proved in the previous theorem.

3.1. Corollary. *Let $x, \xi \in \mathbf{R}$, such that $|x| < 1, |\xi| < 1$; the generating functions of the polynomials $T_n(x)$ and $W_n(x)$ read*

$$\sum_{n=0}^{+\infty} \xi^n T_{n+1}(x) = \frac{x-\xi}{1-2\xi x+\xi^2} \tag{3.3}$$

and

$$\sum_{n=0}^{+\infty} (n+1)(n+2\xi^n W_{n+1}(x) = \frac{8(x-\xi)}{(1-2\xi x+\xi^2)^3}. \tag{3.4}$$

Proof. By multiplying both sides of the relation (2.8) by ξ^n and by summing up over n, we obtain

$$\sum_{n=0}^{+\infty} \xi^n T_{n+1}(x) = x\sum_{n=0}^{+\infty} \xi^n U_n(x) - \sum_{n=0}^{+\infty} \xi^n U_{n-1}(x)$$

that is

$$\sum_{n=0}^{+\infty} \xi^n T_{n+1}(x) = \frac{x}{1-2\xi x+\xi^2} - \frac{\xi}{1-2\xi x+\xi^2}$$

which gives the (68).

In the same way, by multiplying both sides of the second relation stated in the (54) by ξ^n and by summing up over n, we get

$$\sum_{n=0}^{+\infty} \xi^n U_{n+1}(x) = x\sum_{n=0}^{+\infty} \xi^n W_n(x) - \sum_{n=0}^{+\infty} \frac{n}{n+1}\xi^n W_{n-1}(x)$$

and then the thesis.

These results allows us to note that the use of integral representations relating Chebyshev and Hermite polynomials are a fairly important tool of analysis allowing the derivation of a wealth of relations between first and second kind Chebyshev polynomials and the Chebyshev-like polynomials $W_n(x)$. In a forthcoming paper, we will deeper investigate other generalizations for these families of polynomials, recognized as Chebyshev polynomials, by using the instruments of integral representations.

□

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