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ON JORDAN GENERALIZED DERIVATIONS IN GAMMA RINGS

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Abstract

In this study, We define a generalized derivation and a Jordan generalized derivation on Γ-rings and show that a Jordan generalized derivation on some Γ-rings is a generalized derivation.

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1. Introduction

The notion of generalized derivation was introduced by Hvala [2]. Let R be an associative ring. An additive mapping $f : R \longrightarrow R$ is called a *generalized derivation* if there exists a derivation $d : R \longrightarrow R$ such that $f(xy) = f(x)y + xd(y)$ holds for all $x, y \in R$. We call an additive mapping $f : R \longrightarrow R$ a *Jordan generalized derivation* if there exists a derivation $d : R \longrightarrow R$ such that $f(x^2) = f(x)x + xd(x)$ holds for all $x \in R$.

Let M and Γ be additive Abelian groups. Then M is called a Γ- *ring* if for any $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ the following conditions are satisfied.

(1) $x\alpha y \in M$,
(2) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)z = x\alpha z + x\beta z$, $x\alpha(y + z) = x\alpha y + x\alpha z$, and
(3) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

The notion of a Γ-ring was introduced by Nobusawa [5] and generalized by Barnes [1] as defined above. Many properties of Γ-rings were obtained by Barnes [1], Kyuno [3], Luh [4], Nobusawa [5] and others.

Let M be a $\Gamma-$ring and $D : M \longrightarrow M$ an additive map. Then D is called a derivation if

$$D(x\alpha y) = D(x)\alpha y + x\alpha D(y),\ (x, y \in M, \alpha \in \Gamma)$$

and D is called a Jordan derivation if

$$D(x\alpha x) = D(x)\alpha x + x\alpha D(x),\ (x \in M, \alpha \in \Gamma).$$

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Let M be a Γ-ring and $F : M \longrightarrow M$ an additive map. Then F is called a generalized derivation if there exists a derivation $D : M \longrightarrow M$ such that

$$F(x\alpha y) = F(x)\alpha y + x\alpha D(y)$$

for all $x, y \in M$ and $\alpha \in \Gamma$. Finally, F is called a Jordan generalized derivation if there exists a derivation $D : M \longrightarrow M$ such that

$$F(x\alpha x) = F(x)\alpha x + x\alpha D(x)$$

for all $x \in M$ and $\alpha \in \Gamma$.

The notions of derivation and Jordan derivation on a Γ-rings were defined by Sapanci and Nakajima in [6], where they showed that a Jordan derivation on a certain type of completely prime Γ-ring is a derivation. In this note we show that for our notions of generalized derivation and Jordan generalized derivation on a Γ-ring given above, a Jordan generalized derivation on certain Γ-rings is also a generalized derivation.

1.1. Example. Let $f : R \longrightarrow R$ be a generalized derivation on a ring R. Then there exists a derivation $d : R \longrightarrow R$ such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$. We take $M = M_{1,2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n \cdot 1 \\ 0 \end{pmatrix} : n \text{ is an integer} \right\}$. Then M is a Γ-ring. If we define the map $D : M \longrightarrow M$ by $D\left(\,(x, y)\,\right) = \left(\,d(x), d(y)\,\right)$ then D is a derivation on M. Let $F : M \longrightarrow M$ be the additive map defined by $F\left(\,(x, y)\,\right) = \left(\,f(x), f(y)\,\right)$. Then F is a generalized derivation on M. Let N be the subset $\{(x, x) : x \in R\}$ of M. Then N is a Γ-ring, and the map $F : N \longrightarrow N$ defined in terms of the generalized Jordan derivation $f : R \longrightarrow R$ on R by $F\left(\,(x, x)\,\right) = \left(\,f(x), f(x)\,\right)$ is a generalized Jordan derivation on N.

2. The Main Results

Throughout the following, we assume that M is an arbitrary Γ-ring and F a generalized Jordan derivation on M. Clearly, every generalized derivation on M is a Jordan generalized derivation. The converse in general is not true. In the present paper, it is shown that every Jordan generalized derivation on certain Γ-rings is a generalized derivation.

2.1. Lemma. *Let $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. Then*

(i) $F(a\alpha b + b\alpha a) = F(a)\alpha b + F(b)\alpha a + a\alpha D(b) + b\alpha D(a)$.

(ii) $F(a\alpha b\beta a + a\beta b\alpha a) = F(a)\alpha b\beta a + F(a)\beta b\alpha a + a\beta D(b)\alpha a + a\alpha D(b)\beta a + a\alpha b\beta D(a) + a\beta b\alpha D(a)$.

(iii) *In particular, if M is 2-torsion free then*

$$F(a\alpha b\alpha a) = F(a)\alpha b\alpha a + a\alpha D(b)\alpha a + a\alpha b\alpha D(a).$$

(iv) $F(a\alpha b\alpha c + c\alpha b\alpha a) = F(a)\alpha b\alpha c + F(c)\alpha b\alpha a + a\alpha D(b)\alpha c + c\alpha D(b)\alpha a + a\alpha b\alpha D(c) + c\alpha b\alpha D(a)$.

Proof. (i) is obtained by computing $F(\,(a + b)\alpha(a + b))$ and (ii) is also obtained by replacing b with $a\beta b + b\beta a$ in (i). Moreover, (iii) can be obtained by replacing β with α in (ii). If we replace a with $a + c$, we can get (iv). □

2.2. Lemma. *Let $\delta_\alpha(a, b) = F(a\alpha b) - F(a)\alpha b - a\alpha D(b)$ for $a, b \in M$ and $\alpha \in \Gamma$. Then*

(i) $\delta_\alpha(a, b) + \delta_\alpha(b, a) = 0$.

(ii) $\delta_\alpha(a, b + c) = \delta_\alpha(a, b) + \delta_\alpha(a, c)$

(iii) $\delta_\alpha(a + b, c) = \delta_\alpha(a, c) + \delta_\alpha(b, c)$.

(iv) $\delta_{\alpha+\beta}(a, b) = \delta_\alpha(a, b) + \delta_\beta(a, b)$ *for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$.*

Proof. These results follow easily by Lemma 1 (i) and the definition of $\delta_\alpha(a, b)$. □

Note that F is a generalized derivation iff $\delta_\alpha(a,b)=0$ for all $a,b\in M$ and $\alpha\in\Gamma$.

2.3. Lemma. *Let $a,b,c\in M$ and $\alpha,\beta\in\Gamma$. If M is 2-torsion free, then*

(i) $\delta_\alpha(a,b)\alpha[a,b]_\alpha=0$.
(ii) $\delta_\alpha(a,b)\beta x\beta[a,b]_\alpha=0$ *for any $x\in M$.*

Proof. (i) Replacing c by $a\alpha b$ in Lemma 1(iv), we obtain the proof.

(ii) We set $A=a\alpha b\beta x\beta b\alpha a+b\alpha a\beta x\beta a\alpha b$. Then, since $F(A)=F(a\alpha(b\beta x\beta b)\alpha a+b\alpha(a\beta x\beta a)\alpha b)$ and $F(A)=F(\,(a\alpha b)\beta x\beta(b\alpha a)+(b\alpha a)\beta x\beta(a\alpha b)\,)$, by Lemma 1(iii) and Lemma 1(iv), we have the proof. □

2.4. Theorem. *Let M be a 2-torsion free Γ-ring. If M has two elements a and b so that for any $\alpha\in\Gamma$ we have $c\gamma x\gamma[a,b]_\alpha=0$ or $[a,b]_\alpha\gamma x\gamma c=0$ implies $c=0$ for all $x\in M$, $\gamma\in\Gamma$, then every Jordan generalized derivation on M is a generalized derivation.*

Proof. Let u,v be fixed elements of M such that $c\gamma x\gamma[u,v]_\alpha=0$ or $[u,v]_\alpha\gamma x\gamma c=0$ implies $c=0$. Then by Lemma 3(ii), we get

$$\delta_\alpha(u,v)=0 \tag{2.1}$$

for all $\alpha\in\Gamma$. Our aim is to prove that $\delta_\alpha(a,b)=0$ for all $a,b\in M$ and $\alpha\in\Gamma$. In Lemma 3(ii), replacing a by $a+u$, we have $\delta_\alpha(a+u,b)\beta x\beta((a+u)\alpha b-b\alpha(a+u))=0$. Using Lemma 2(iii), we get $\delta_\alpha(a,b)\beta x\beta[a,b]_\alpha+\delta_\alpha(a,b)\beta x\beta[u,b]_\alpha+\delta_\alpha(u,b)\beta x\beta[a,b]_\alpha+\delta_\alpha(u,b)\beta x\beta[u,b]_\alpha=0$.

From Lemma 3(ii), since $\delta_\alpha(a,b)\beta x\beta[a,b]_\alpha=0$ and $\delta_\alpha(u,b)\beta x\beta[u,b]_\alpha=0$ then we have

$$\delta_\alpha(a,b)\beta x\beta[u,b]_\alpha+\delta_\alpha(u,b)\beta x\beta[a,b]_\alpha=0 \tag{2.2}$$

for all $x,a,b\in M$ and $\alpha,\beta\in\Gamma$. Now replacing b by $b+v$ in (2.2) and using Lemma 2(ii), we get $\delta_\alpha(a,b)\beta x\beta[u,b]_\alpha+\delta_\alpha(a,b)\beta x\beta[u,v]_\alpha+\delta_\alpha(a,v)\beta x\beta[u,b]_\alpha+\delta_\alpha(a,v)\beta x\beta[u,v]_\alpha+\delta_\alpha(u,b)\beta x\beta[a,b]_\alpha+\delta_\alpha(u,b)\beta x\beta[a,v]_\alpha+\delta_\alpha(u,v)\beta x\beta[a,b]_\alpha+\delta_\alpha(u,v)\beta x\beta[a,v]_\alpha=0$ or using equations (2.1) and 2.2 we have

$$\delta_\alpha(a,b)\beta x\beta[u,v]_\alpha+\delta_\alpha(a,v)\beta x\beta[u,b]_\alpha+\delta_\alpha(a,v)\beta x\beta[u,v]_\alpha+\delta_\alpha(u,b)\beta x\beta[a,v]_\alpha=0. \tag{2.3}$$

Replacing a by u in equation(2.3) and using equation (2.1) together with the fact that M is 2-torsion free, we have

$$\delta_\alpha(u,b)\beta x\beta[u,v]_\alpha=0 \tag{2.4}$$

for all $b,x\in M$ and $\alpha,\beta\in\Gamma$. Hence by hypothesis, we get

$$\delta_\alpha(u,b)=0 \tag{2.5}$$

for all $b\in M$, $\alpha\in\Gamma$. Again replacing b by v in equation (2.2), using equation (2.1) and the hypothesis, we obtain

$$\delta_\alpha(a,v)=0 \tag{2.6}$$

for all $a\in M$, $\alpha\in\Gamma$. Substituting equations (2.5) and (2.6) in equation (2.3) we have

$$\delta_\alpha(a,b)\beta x\beta[u,v]_\alpha=0,$$

and then from the hypothesis we obtain

$$\delta_\alpha(a,b)=0$$

for all $a,b\in M$, $\alpha\in\Gamma$, i.e. F is a generalized derivation on M. □

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