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CONTINUOUS DEPENDENCE ON THE PARAMETERS OF PHASE FIELD EQUATIONS

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Abstract

Phase field equations are analyzed. It is shown that the solution of the problem considered depends continuously on changes in the parameters.

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1. Introduction

We consider the problem

(1) $$\tau\phi_t - \xi^2\Delta\phi + f(x,\phi) = 2u + h_1(x,t),\ (x,t) \in Q_T$$

(2) $$u_t + \frac{l}{2}\phi_t = K\Delta u + h_2(x,t),\ (x,t) \in Q_T$$

(3) $$\phi\,|_\Gamma = \phi_\partial(x,t),\ u\,|_\Gamma = u_\partial(x,t),\ (x,t) \in \partial\Omega \times (0,T]$$

(4) $$\phi(x,0) = \phi_0(x),\ u(x,0) = u_0(x),\ x \in \Omega,$$

where $Q_T = \Omega \times (0,T]$, $T > 0$, $\Omega \subset \mathbb{R}^n$, $(n \geq 1)$ is a bounded domain with a sufficiently smooth boundary, $\partial\Omega$; ξ, τ, l and K are positive constants characterizing the length scale, the relaxation time, the latent heat and the thermal diffusivity, respectively. Also, $\phi_0(x)$, $u_0(x)$, $\phi_\partial(x,t)$, $u_\partial(x,t)$, $h_1(x,t)$, $h_2(x,t)$ and $f(x,s)$ are given functions.

In [1], G. Caginalp considered the following system of equations as a model describing the phase transitions with a separation surface of finite thickness:

(5) $$\tau\phi_t = \xi^2\Delta\phi + \frac{1}{2}(\phi - \phi^3) + 2u,\ x \in \Omega,\ t \in \mathbb{R}^+$$

(6) $$u_t + \frac{l}{2}\phi_t = K\Delta u,\ x \in \Omega,\ t \in \mathbb{R}^+.$$

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Under the assumption $\frac{\xi^2}{\tau} < K$, a global existence theorem was proved for the classical solution of the initial boundary value problem for the system (5)-(6) with non-homogeneous Dirichlet boundary conditions of the form

$$\phi(t,x)\,|_{\partial\Omega} = \phi_\partial(x),\ u(t,x)\,|_{\partial\Omega} = u_\partial(x),\ t \in \mathbb{R}^+. \tag{7}$$

Several scientists have investigated problems based on Caginalp's model, and made a few modifications. In [2], Caginalp and Hastings investigated the existence of stationary solutions of problem (5)-(7) in $\Omega \subset \mathbb{R}^1$. In [3], C. M. Eliott and Song Mu Zheng proved the global unique solvability of initial boundary value problems for the system (5)-(6) in the class $H^2(\Omega) \times H^2(\Omega),\ \Omega \subset \mathbb{R}^n, n \leq 3$, without the assumption $\frac{\xi^2}{\tau} < K$, for boundary conditions of the form (7), as well as for conditions of the form

$$\frac{\partial \phi}{\partial n}\,|_{\partial\Omega} = 0,\ \frac{\partial u}{\partial n}\,|_{\partial\Omega} = 0,\ t \in \mathbb{R}^+,$$
$$\phi\,|_{\partial\Omega} = \phi_\partial(x),\quad \frac{\partial u}{\partial n}\,|_{\partial\Omega} = 0,\ t \in \mathbb{R}^+.$$

They also studied the behaviour of the solutions of the system (5),(6) when $t \to \infty$. In [4], Kalantarov proved that the initial boundary value problem for system (5)-(6), under homogeneous boundary conditions of the form (7), is globally uniquely solvable in $C(\mathbb{R}^+, X),\ X = \overset{0}{H}{}^1(\Omega) \times \overset{0}{H}{}^1(\Omega)$, and established the existence of a global attractor. In [5], Brochet, Hilhorst and Chen investigated problem (1)-(4), considering $v = u + \frac{l}{2}\phi$, $f(s) = \sum_{j=0}^{2p-1} b_j s^j,\ b_{2p-1} > 0,\ p \geq 2,\ h_i(x,t) = 0,\ (i = 1,2)$ and homogeneous Neumann boundary conditions, proving this problem to be well posed if $(\phi_0, u_0) \in (L_2(\Omega))^2$.

2. Continuous Dependence of Solutions

We investigate the continuous dependence on the parameters $\xi,\ \tau,\ l$ and K of solutions of problem (1)–(4) in the class $V(Q_T) \times V(Q_T)$, where

$$V(Q_T) = W_2^1(Q_T) \cap \{v(x,t) : \Delta v \in L_2(Q_T)\}. \tag{8}$$

The existence of a solution to this problem can be seen from the general results of [7] and [8], but to the best of our knowledge an investigation of continuous dependence does not occur in the literature. Investigations of this type are of interest in physical problems, and can lead to useful applications.

We assume that $f(x,\phi)$ is the Caratheodory function which satisfies the local Lipschitz condition:

$$|f(x,s_1) - f(x,s_2)| \leq c(1 + |s_1|^{p-1} + |s_2|^{p-1})\,|s_1 - s_2|\,,\ \forall s_1,\ s_2 \in \mathbb{R}^1, \tag{9}$$

where $p \in [1,\infty)$ if $n = 1,2$, and $p \in \left[1, \frac{n}{n-2}\right]$ if $n \geq 3$. We have used standard techniques for the calculations (cf. [6], which considers this type of question for a different problem). Let $\{\phi_1, u_1\}$ and $\{\phi_2, u_2\}$ be the solutions from $V(Q_T) \times V(Q_T)$ of the following initial-boundary value problems for different coefficients ξ_1,τ_1,l_1,K_1 and ξ_2,τ_2,l_2,K_2

respectively.

$$
\begin{aligned}
&\tau_1(\phi_1)_t - \xi_1^2\Delta\phi_1 + f(x,\phi_1) = 2u_1 + h_1(x,t),\ \ (x,t)\in Q_T\\
&(u_1)_t + \frac{l_1}{2}(\phi_1)_t = K_1\Delta u_1 + h_2(x,t),\ \ (x,t)\in Q_T\\
&\phi_1\,|_\Gamma\ = \phi_\partial(x,t), u_1\,|_\Gamma\ = u_\partial(x,t),\ \ (x,t)\in\partial\Omega\times(0,T]\\
&\phi_1(x,0) = \phi_0(x),\ \ u_1(x,0) = u_0(x),\ \ x\in\Omega\\
&\tau_2(\phi_2)_t - \xi_2^2\Delta\phi_2 + f(x,\phi_2) = 2u_2 + h_1(x,t),\ \ (x,t)\in Q_T\\
&(u_2)_t + \frac{l_2}{2}(\phi_2)_t = K_2\Delta u_2 + h_2(x,t),\ \ (x,t)\in Q_T\\
&\phi_2\,|_\Gamma\ = \phi_\partial(x,t),\ \ u_2\,|_\Gamma\ = u_\partial(x,t),\ \ (x,t)\in\partial\Omega\times(0,T]\\
&\phi_2(x,0) = \phi_0(x),\ \ u_2(x,0) = u_0(x),\ \ x\in\Omega.
\end{aligned}
$$

We define the difference variables ϕ, u, ξ, τ, l and K by

$$
\begin{aligned}
\phi &= \phi_1-\phi_2, &\qquad \tau &= \tau_1-\tau_2,\\
u &= u_1-u_2, & l &= l_1-l_2,\\
\xi^2 &= \xi_1^2-\xi_2^2, & K &= K_1-K_2
\end{aligned}
$$

Then $\{\phi, u\}$ satisfies the initial-boundary value problem:

$$\tau_1\phi_t + \tau(\phi_2)_t - \xi_1^2\Delta\phi - \xi^2\Delta\phi_2 + f(x,\phi_1) - f(x,\phi_2) = 2u, \tag{10}$$

$$u_t + \frac{l_1}{2}\phi_t + \frac{l}{2}(\phi_2)_t = K_1\Delta u + K\Delta u_2 \tag{11}$$

$$\phi\,|_\Gamma\ = u\,|_\Gamma\ = 0 \tag{12}$$

$$\phi(x,0) = u(x,0) = 0 \tag{13}$$

If we take the inner product in $L_2(\Omega)$ of (10) by $\phi_t + \phi$ and of (11) by $\frac{2\tau_1}{l_1^2}u_t + \frac{4}{l_1}u$ and then add the equations obtained, we obtain

$$
\begin{aligned}
&\tau_1\left\|\phi_t\right\|^2 + \xi_1^2\left\|\nabla\phi\right\|^2 + \frac{4K_1}{l_1}\left\|\nabla u\right\|^2 + \frac{2\tau_1}{l_1^2}\left\|u_t\right\|^2 +\\
&+ \frac{d}{dt}\left[\frac{\xi_1^2}{2}\left\|\nabla\phi\right\|^2 + \frac{2}{l_1}\left\|u\right\|^2 + \frac{\tau_1}{2}\left\|\phi\right\|^2 + \frac{\tau_1K_1}{l_1^2}\left\|\nabla u\right\|^2\right] \le\\
\le\ &\left|\int_\Omega\left(f(x,\phi_1)-f(x,\phi_2)\right)\phi_t dx\right| + \left|\int_\Omega\left(f(x,\phi_1)-f(x,\phi_2)\right)\phi dx\right| +\\
&+ 2\left|(u,\phi)\right| + \frac{4\left|K\right|}{l_1}\left|(\Delta u_2,u)\right| + \frac{2\tau_1\left|K\right|}{l_1^2}\left|(\Delta u_2,u_t)\right| + \frac{\tau_1}{l_1}\left|(\phi_t,u_t)\right| +\\
&+ \left|\tau\right|\left|((\phi_2)_t,\phi_t)\right| + \left|\tau\right|\left|((\phi_2)_t,\phi)\right| + \left|\xi^2\right|\left|(\Delta\phi_2,\phi_t)\right| + \left|\xi^2\right|\left|(\Delta\phi_2,\phi)\right| +\\
&\frac{2\left|l\right|}{l_1}\left|((\phi_2)_t,u)\right| + \frac{\tau_1\left|l\right|}{l_1^2}\left|((\phi_2)_t,u_t)\right|,
\end{aligned}
\tag{14}
$$

where $\|.\|$ denotes the norm on $L_2(\Omega)$. Using (9) and Hölder's inequality, the first term on the right hand side of (14) can be estimated in the following manner:

$$\left|\int\limits_\Omega (f(x,\phi_1)-f(x,\phi_2))\,\phi_t dx\right| \le \int\limits_\Omega |(f(x,\phi_1)-f(x,\phi_2))|\,|\phi_t|\,dx \le$$
$$\le \int\limits_\Omega c(1+|\phi_1|^{p-1}+|\phi_2|^{p-1})\,|\phi|\,|\phi_t|\,dx \le$$
$$\le c\,\|\phi\|\,\|\phi_t\| + c\int\limits_\Omega |\phi_1|^{p-1}\,|\phi|\,|\phi_t|\,dx + c\int\limits_\Omega |\phi_2|^{p-1}\,|\phi|\,|\phi_t|\,dx \le$$
$$\le c\,\|\phi\|\,\|\phi_t\| + c\,\|\phi\|_{L_{\frac{2n}{n-2}}}\,\|\phi_t\|\left(\|\phi_1\|^{p-1}_{L_{(p-1)n}} + \|\phi_2\|^{p-1}_{L_{(p-1)n}}\right).$$

By the Sobolev imbedding theorem the following inequality holds:

$$\|\phi\|_{L_{\frac{2n}{n-2}}(\Omega)} \le c_2\,\|\nabla\phi\|\,.$$

Therefore,

$$\left|\int\limits_\Omega (f(x,\phi_1)-f(x,\phi_2))\,\phi_t dx\right| \le c\,\|\phi\|\,\|\phi_t\| +$$
$$+\,cc_2\,\|\phi_t\|\,\|\nabla\phi\|\left(\|\phi_1\|^{p-1}_{L_{(p-1)n}} + \|\phi_2\|^{p-1}_{L_{(p-1)n}}\right).$$

Since $\{\phi_i, u_i\} \in V(Q_T)\times V(Q_T)$ are fixed,

$$\|\phi_1\|^{p-1}_{L_{(p-1)n}} + \|\phi_2\|^{p-1}_{L_{(p-1)n}} \le c_1(t).$$

Hence

$$\left|\int\limits_\Omega (f(x,\phi_1)-f(x,\phi_2))\,\phi_t dx\right| \le c\,\|\phi\|\,\|\phi_t\| + cc_1(t)c_2\,\|\phi_t\|\,\|\nabla\phi\|\,, \tag{15}$$

and similarly,

$$\left|\int\limits_\Omega (f(x,\phi_1)-f(x,\phi_2))\,\phi dx\right| \le c\,\|\phi\|^2 + cc_1(t)c_2\,\|\phi\|\,\|\nabla\phi\|\,. \tag{16}$$

Taking into account (15) and (16), we obtain from (14)

$$\begin{aligned}
&\tau_1\,\|\phi_t\|^2 + \xi_1^2\,\|\nabla\phi\|^2 + \frac{4K_1}{l_1}\,\|\nabla u\|^2 + \frac{2\tau_1}{l_1^2}\,\|u_t\|^2 + \\
&+\frac{d}{dt}\left[\frac{\xi_1^2}{2}\,\|\nabla\phi\|^2 + \frac{2}{l_1}\,\|u\|^2 + \frac{\tau_1}{2}\,\|\phi\|^2 + \frac{\tau_1 K_1}{l_1^2}\,\|\nabla u\|^2\right] \le \\
\le\ &2\,|(u,\phi)| + \frac{4\,|K|}{l_1}\,|(\Delta u_2,u)| + \frac{2\tau_1\,|K|}{l_1^2}\,|(\Delta u_2,u_t)| + \frac{\tau_1}{l_1}\,|(\phi_t,u_t)| + \\
&+|\tau|\,|((\phi_2)_t,\phi_t)| + |\tau|\,|((\phi_2)_t,\phi)| + \left|\xi^2\right|\,|(\Delta\phi_2,\phi_t)| + \left|\xi^2\right|\,|(\Delta\phi_2,\phi)| + \\
&+\frac{2\,|l|}{l_1}\,|((\phi_2)_t,u)| + \frac{\tau_1\,|l|}{l_1^2}\,|((\phi_2)_t,u_t)| + c\,\|\phi\|\,\|\phi_t\| + \\
&+cc_1(t)c_2\,\|\phi_t\|\,\|\nabla\phi\| + c\,\|\phi\|^2 + cc_1(t)c_2\,\|\phi\|\,\|\nabla\phi\|\,.
\end{aligned} \tag{17}$$

Making use of Cauchy's inequality with ε, the right hand side of (17) can be estimated. If we select the number $\varepsilon > 0$ sufficiently small then we obtain

$$\begin{aligned}
&\frac{\tau_1}{4}\|\phi_t\|^2 + \frac{4K_1}{l_1}\|\nabla u\|^2 + \frac{\tau_1}{2l_1^2}\|u_t\|^2 + \\
&\frac{d}{dt}\left[\frac{\xi_1^2}{2}\|\nabla\phi\|^2 + \frac{2}{l_1}\|u\|^2 + \frac{\tau_1}{2}\|\phi\|^2 + \frac{\tau_1 K_1}{l_1^2}\|\nabla u\|^2\right] \leq \\
\leq\ & a_1(t)\|\nabla\phi\|^2 + (\frac{3+l_1}{l_1})\|u\|^2 + a_2(t)\|\phi\|^2 + \\
&+\left[\left(\frac{\tau_1+2l_1}{2l_1^2}\right)l^2 + \left(\frac{8+\tau_1}{2\tau_1}\right)\tau^2\right]\|(\phi_2)_t\|^2 + \\
&+\left(\frac{2(\tau_1+l_1)}{l_1^2}\right)K^2\|\Delta u_2\|^2 + \left(\frac{8+\tau_1}{2\tau_1}\right)\xi^4\|\Delta\phi_2\|^2,
\end{aligned} \tag{18}$$

where,

$$a_1(t) = \frac{4c^2c_1^2(t)c_2^2}{\tau_1} \text{ and } a_2(t) = \frac{4c^2}{\tau_1} + \frac{c^2c_1^2(t)c_2^2}{4\xi_1^2} + c + 2.$$

If we set

$$c_2(t) = \max\left\{\frac{2a_1(t)}{\xi_1^2}, \frac{3+l_1}{2}, \frac{2a_2(t)}{\tau_1}, 1\right\},$$

and

$$Y(t) = \frac{\xi_1^2}{2}\|\nabla\phi\|^2 + \frac{2}{l_1}\|u\|^2 + \frac{\tau_1}{2}\|\phi\|^2 + \frac{\tau_1K_1}{l_1^2}\|\nabla u\|^2,$$

then from (18) we obtain

$$\begin{cases} \dfrac{dY(t)}{dt} \leq c_2(t)Y(t) + \left(c_3l^2 + c_4\tau^2\right)\|(\phi_2)_t\|^2 + c_4\xi^4\|\Delta\phi_2\|^2 + c_5K^2\|\Delta u_2\|^2, \\ Y(0) = 0, \end{cases}$$

where, $c_3 = \frac{\tau_1+2l_1}{2l_1^2}$, $c_4 = \frac{8+\tau_1}{2\tau_1}$ and $c_5 = \frac{2(\tau_1+l_1)}{l_1^2}$. According to Gronwall's lemma, we have

$$\begin{aligned}
Y(t) \leq \exp\left\{\int_0^T c_2(s)ds\right\}&\Big\{(c_3l^2 + c_4\tau^2)\|(\phi_2)_t\|^2_{L_2(Q_T)} + \\
&+c_4\xi^4\|\Delta\phi_2\|^2_{L_2(Q_T)} + c_5K^2\|\Delta u_2\|^2_{L_2(Q_T)}\Big\}.
\end{aligned} \tag{19}$$

Since $\{\phi_i, u_i\} \in V(Q_T) \times V(Q_T)$, we have

$$\begin{aligned}
&\|(\phi_2)_t\|^2_{L_2(Q_T)} \leq C, \\
&\|\Delta\phi_2\|^2_{L_2(Q_T)} \leq C, \text{ and} \\
&\|\Delta u_2\|^2_{L_2(Q_T)} \leq C.
\end{aligned}$$

If we set

$$\max\{c_3C, c_4C, c_4C, c_5C\} = C_6$$

and

$$C_6 \exp\left\{\int_0^T c_2(s)ds\right\} = C_7$$

then from (19) we have

$$Y(t) \leq C_7 \left[K^2 + \xi^4 + \tau^2 + l^2\right]$$

Hence we have proved the following theorem.

2.1. Theorem. *Assume that (9) is satisfied. Then the solution of problem (1)-(4) from* $V(Q_T) \times V(Q_T)$ *depends continuously on the parameters* ξ, τ, l *and* K. *Moreover,*

$$\|\phi_1 - \phi_2\|^2_{C\left(0,T;W_2^1(\Omega)\right)} \leq C_7 \left[(K_1 - K_2)^2 + (\xi_1 - \xi_2)^4 + (\tau_1 - \tau_2)^2 + (l_1 - l_2)^2\right],$$

and

$$\|u_1 - u_2\|^2_{C\left(0,T;W_2^1(\Omega)\right)} \leq C_7 \left[(K_1 - K_2)^2 + (\xi_1 - \xi_2)^4 + (\tau_1 - \tau_2)^2 + (l_1 - l_2)^2\right].$$

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