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Common fixed point theorems for two pairs of non-self mappings in cone metric spaces

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Abstract

Some common fixed point theorems for two pairs of non-self mappings defined on a closed subset of a metrically convex cone metric space (over the cone which is not necessarily normal) are obtained which generalize earlier results due to Imdad et al. and Jankovic et al.

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1. Introduction and preliminaries

Recently, Huang and Zhang ([14]) generalized the concept of a metric space, replacing the set of real numbers by ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions. Subsequently, the study of fixed point theorems in such spaces is followed by some other mathematicians, see [1]-[5], [7]-[12], [15]-[18], [20]-[23]. The aim of this paper is to prove some common fixed point theorems for two pairs of non-self mappings on cone metric spaces in which the cone need not be normal. This result generalizes the result of Jankovic et al.([18]).

Consistent with Huang and Zhang ([14]), the following definitions and results will be needed in the sequel.

Let E be a real Banach space. A subset P of E is called a cone if and only if:

(a) P is closed, nonempty and $P \neq \{\theta\}$;

(b) $a, b \in R, a, b \geq 0, x, y \in P$ implies $ax + by \in P$;

(c) $P \cap (-P) = \{\theta\}$.

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Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \text{ implies } \| x \| \leq K \| y \|.$$

The least positive number K satisfying the above inequality is called the normal constant of P, while $x \ll y$ stands for $y - x \in intP$ (interior of P).

1.1. Definition ([14])**.** Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d1) $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then d is called a cone metric on X and (X, d) is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space.

1.2. Definition ([14])**.** Let (X, d) be a cone metric space. We say that $\{x_n\}$ is:

(e) a Cauchy sequence if for every $c \in E$ with $\theta \ll c$, there is an N such that for all $n, m > N, d(x_n, x_m) \ll c$;

(f) a convergent sequence if for every $c \in E$ with $\theta \ll c$, there is an N such that for all $n > N, d(x_n, x) \ll c$ for some fixed $x \in X$.

A cone metric space X is said to be complete if for every Cauchy sequence in X is convergent in X. It is known that if P is normal, then $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to \theta$ as $n \to \infty$. It is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta (n, m \to \infty)$.

1.3. Remark ([24])**.** Let E be an ordered Banach (normed) space. Then c is an interior point of P, if and only if $[-c, c]$ is a neighborhood of θ.

1.4. Corollary ([19])**.** *(1) If $a \leq b$ and $b \ll c$, then $a \ll c$.*
Indeed, $c - a = (c - b) + (b - a) \geq c - b$ implies $[-(c - a), c - a] \supseteq [-(c - b), c - b]$.
(2) If $a \ll b$ and $b \ll c$, then $a \ll c$.
Indeed, $c - a = (c - b) + (b - a) \geq c - b$ implies $[-(c - a), c - a] \supseteq [-(c - b), c - b]$.
(3) If $\theta \leq u \ll c$ for each $c \in intP$ then $u = \theta$.

1.5. Remark ([18])**.** If $c \in intP, \theta \leq a_n$ and $a_n \to \theta$, then there exists an n_0 such that for all $n > n_0$ we have $a_n \ll c$.

1.6. Remark ([18])**.** If E is a real Banach space with cone P and if $a \leq ka$ where $a \in P$ and $0 < k < 1$, then $a = \theta$.

We find it convenient to introduce the following definition.

1.7. Definition ([18])**.** Let (X, d) be a complete cone metric space and C be a nonempty closed subset of X, and $f, g : C \to X$. Denote, for$x, y \in C$,

$$M_1^{f,g} = \{d(gx, gy), d(fx, gx), d(fy, gy), \frac{d(fx, gy) + d(fy, gx)}{2}\}. \tag{1.1}$$

Then f is called a generalized $g_{M_1}-$contractive mapping of C into X if for some $\lambda \in (0, \sqrt{2} - 1)$ there exists $u(x, y) \in M_1^{f,g}$ such that for all $x, y \in C$

$$d(fx, fy) \leq \lambda u(x, y). \tag{1.2}$$

1.8. Definition ([2])**.** Let f and g be self maps of a set X (i.e., $f, g : X \to X$). If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. Self maps f and g are said to be weakly compatible if they commute at their coincidence point; i.e., if $fx = gx$ for some $x \in X$, then $fgx = gfx$.

2. Main results

Recently, Jankovic et al.([18]) proved some fixed point theorems for a pair of non-self mappings defined on a nonempty closed subset of complete metrically convex cone metric spaces with new contractive conditions.

2.1. Theorem ([18])**.** *Let (X, d) be a complete cone metric space, C a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ (the boundary of C) such that*

$$d(x, z) + d(z, y) = d(x, y).$$

Suppose that $f, g : C \to X$ are such that f is a generalized $g_{M_1}-$contractive mapping of C into X, and

(i) $\partial C \subseteq gC, fC \cap C \subseteq gC$,

(ii) $gx \in \partial C$ implies that $fx \in C$,

(iii) gC is closed in X.

Then the pair (f, g) has a coincidence point. Moreover, if pair (f, g) is weakly compatible, then f and g have a unique common fixed point.

The purpose of this paper is to extend above theorem for two pairs of non-self mappings in cone metric spaces. We begin with the following definition.

2.2. Definition. Let (X, d) be a complete cone metric space and C be a nonempty closed subset of X, and $F, G, S, T : C \to X$. Denote, for $x, y \in C$,

$$M_1^{F,G,S,T} = \{d(Tx, Sy), d(Tx, Fx), d(Sy, Gy), \frac{d(Tx, Gy) + d(Fx, Sy)}{2}\}. \tag{2.1}$$

Then (F, G) is called a generalized $(T, S)_{M_1}-$contractive mappings pair of C into X if for some $\lambda \in (0, 1)$ there exists $u(x, y) \in M_1^{F,G,S,T}$ such that for all $x, y \in C$

$$d(Fx, Gy) \leq \lambda u(x, y). \tag{2.2}$$

Notice that by setting $G = F = f$ and $T = S = g$ in (2.1), one deduces a slightly generalized form of (1.1).

We state and prove our main result as follows.

2.3. Theorem. *Let (X, d) be a complete cone metric space, C a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x, z) + d(z, y) = d(x, y).$$

Suppose that $F, G, S, T : C \to X$ are such that (F, G) is a generalized $(T, S)_{M_1}-$ contractive mappings pair of C into X, and

(I) $\partial C \subseteq SC \cap TC, FC \cap C \subseteq SC, GC \cap C \subseteq TC$,

(I) $Tx \in \partial C$ implies that $Fx \in C, Sx \in \partial C$ implies that $Gx \in C$,

(III) SC and TC (or FC and GC) are closed in X.

Then

(IV) (F, T) has a point of coincidence,

(V) (G, S) has a point of coincidence.

Moreover, if (F, T) and (G, S) are weakly compatible pairs, then F, G, S and T have a unique common fixed point.

Proof. Firstly, we proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way.

Let $x \in \partial C$ be arbitrary. Then (due to $\partial C \subseteq TC$) there exists a point $x_0 \in C$ such that $x = Tx_0$. Since $Tx \in \partial C \Rightarrow Fx \in C$, one concludes that $Fx_0 \in FC \cap C \subseteq SC$.

Thus, there exist $x_1 \in C$ such that $y_1 = Sx_1 = Fx_0 \in C$. Since $y_1 = Fx_0$ there exists a point $y_2 = Gx_1$ such that

$$d(y_1, y_2) = d(Fx_0, Gx_1).$$

Suppose $y_2 \in C$. Then $y_2 \in GC \cap C \subseteq TC$ which implies that there exists a point $x_2 \in C$ such that $y_2 = Tx_2$. otherwise, if $y_2 \notin C$, then there exists a point $p \in \partial C$ such that

$$d(Sx_1, p) + d(p, y_2) = d(Sx_1, y_2).$$

Since $p \in \partial C \subseteq TC$ there exists a point $x_2 \in C$ with $p = Tx_2$ so that

$$d(Sx_1, Tx_2) + d(Tx_2, y_2) = d(Sx_1, y_2).$$

Let $y_3 = Fx_2$ be such that $d(y_2, y_3) = d(Gx_1, Fx_2)$. Thus, repeating the foregoing arguments, one obtains two sequences $\{x_n\}$ and $\{y_n\}$ such that

(a) $y_{2n} = Gx_{2n-1}, y_{2n+1} = Fx_{2n}$,

(b) $y_{2n} \in C \Rightarrow y_{2n} = Tx_{2n}$ or $y_{2n} \notin C \Rightarrow Tx_{2n} \in \partial C$ and

$$d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, y_{2n}) = d(Sx_{2n-1}, y_{2n}).$$

(c) $y_{2n+1} \in C \Rightarrow y_{2n+1} = Sx_{2n+1}$ or $y_{2n+1} \notin C \Rightarrow Sx_{2n+1} \in \partial C$ and

$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1}).$$

We denote

$$\begin{aligned} P_0 &= \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} = y_{2i}\}, \\ P_1 &= \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} \neq y_{2i}\}, \\ Q_0 &= \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} = y_{2i+1}\}, \\ Q_1 &= \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} \neq y_{2i+1}\}. \end{aligned}$$

Note that $(Tx_{2n}, Sx_{2n+1}) \notin P_1 \times Q_1$, as if $Tx_{2n} \in P_1$, then $y_{2n} \neq Tx_{2n}$ and one infers that $Tx_{2n} \in \partial C$ which implies that $y_{2n+1} = Fx_{2n} \in C$. Hence $y_{2n+1} = Sx_{2n+1} \in Q_0$. Similarly, one can argue that $(Sx_{2n-1}, Tx_{2n}) \notin Q_1 \times P_1$.

Now, we distinguish the following three cases.

Case 1. If $(Tx_{2n}, Sx_{2n+1}) \in P_0 \times Q_0$, then from (2.2)

$$d(Tx_{2n}, Sx_{2n+1}) = d(Fx_{2n}, Gx_{2n-1}) \leq \lambda u_{2n-1},$$

where

$$\begin{aligned} u_{2n-1} \in \{&d(Sx_{2n-1}, Tx_{2n}), d(Sx_{2n-1}, Gx_{2n-1}), d(Tx_{2n}, Fx_{2n}), \\ &\frac{d(Tx_{2n}, Gx_{2n-1}) + d(Sx_{2n-1}, Fx_{2n})}{2}\} \\ = \{&d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \frac{d(y_{2n-1}, y_{2n+1})}{2}\}. \end{aligned}$$

Clearly, there are infinitely many n such that at least one of the following three cases holds:

(1) $d(Tx_{2n}, Sx_{2n+1}) \leq \lambda d(y_{2n-1}, y_{2n}) = \lambda d(Sx_{2n-1}, Tx_{2n})$;

(2) $d(Tx_{2n}, Sx_{2n+1}) \leq \lambda d(y_{2n}, y_{2n+1}) \Rightarrow d(Tx_{2n}, Sx_{2n+1}) = \theta \leq \lambda d(Sx_{2n-1}, Tx_{2n})$;

(3) $d(Tx_{2n}, Sx_{2n+1}) \leq \lambda \frac{d(y_{2n-1}, y_{2n+1})}{2} \leq \frac{\lambda}{2} d(y_{2n-1}, y_{2n}) + \frac{1}{2} d(y_{2n}, y_{2n+1}) \Rightarrow d(Tx_{2n}, Sx_{2n+1}) \leq \lambda d(Sx_{2n-1}, Tx_{2n})$.

From (1), (2), (3) it follows that

$$d(Tx_{2n}, Sx_{2n+1}) \leq \lambda d(Sx_{2n-1}, Tx_{2n}). \tag{2.3}$$

Similarly, if $(Sx_{2n+1}, Tx_{2n+2}) \in Q_0 \times P_0$, we have

$$d(Sx_{2n+1}, Tx_{2n+2}) = d(Fx_{2n}, Gx_{2n+1}) \leq \lambda d(Tx_{2n}, Sx_{2n+1}). \tag{2.4}$$

If $(Sx_{2n-1}, Tx_{2n}) \in Q_0 \times P_0$, we have

$$d(Sx_{2n-1}, Tx_{2n}) = d(Fx_{2n-2}, Gx_{2n-1}) \leq \lambda d(Tx_{2n-2}, Sx_{2n-1}). \tag{2.5}$$

Case 2. If $(Tx_{2n}, Sx_{2n+1}) \in P_0 \times Q_1$, then $Sx_{2n+1} \in Q_1$ and

$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1}) \tag{2.6}$$

which in turn yields

$$d(Tx_{2n}, Sx_{2n+1}) \leq d(Tx_{2n}, y_{2n+1}) = d(y_{2n}, y_{2n+1}) \tag{2.7}$$

and hence

$$d(Tx_{2n}, Sx_{2n+1}) \leq d(y_{2n}, y_{2n+1}) = d(Fx_{2n}, Gx_{2n-1}). \tag{2.8}$$

Now, proceeding as in Case 1, we have that (2.3) holds.

If $(Sx_{2n+1}, Tx_{2n+2}) \in Q_1 \times P_0$, then $Tx_{2n} \in P_0$. We show that

$$d(Sx_{2n+1}, Tx_{2n+2}) \leq \lambda d(Tx_{2n}, Sx_{2n-1}). \tag{2.9}$$

Using (2.6), we get

$$\begin{aligned} d(Sx_{2n+1}, Tx_{2n+2}) &\leq d(Sx_{2n+1}, y_{2n+1}) + d(y_{2n+1}, Tx_{2n+2}) \\ &= d(Tx_{2n}, y_{2n+1}) - d(Tx_{2n}, Sx_{2n+1}) + d(y_{2n+1}, Tx_{2n+2}). \end{aligned} \tag{2.10}$$

By noting that $Tx_{2n+2}, Tx_{2n} \in P_0$, one can conclude that

$$d(y_{2n+1}, Tx_{2n+2}) = d(y_{2n+1}, y_{2n+2}) = d(Fx_{2n}, Gx_{2n+1}) \leq \lambda d(Tx_{2n}, Sx_{2n+1}), \tag{2.11}$$

and

$$d(Tx_{2n}, y_{2n+1}) = d(y_{2n}, y_{2n+1}) = d(Fx_{2n}, Gx_{2n-1}) \leq \lambda d(Sx_{2n-1}, Tx_{2n}), \tag{2.12}$$

in view of Case 1.

Thus,

$$d(Sx_{2n+1}, Tx_{2n+2}) \leq \lambda d(Sx_{2n-1}, Tx_{2n}) - (1-\lambda) d(Tx_{2n}, Sx_{2n+1}) \leq \lambda d(Sx_{2n-1}, Tx_{2n}),$$

and we proved (2.9).

Case 3. If $(Tx_{2n}, Sx_{2n+1}) \in P_1 \times Q_0$, then $Sx_{2n-1} \in Q_0$. We show that

$$d(Tx_{2n}, Sx_{2n+1}) \leq \lambda d(Sx_{2n-1}, Tx_{2n-2}). \tag{2.13}$$

Since $Tx_{2n} \in P_1$, then

$$d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, y_{2n}) = d(Sx_{2n-1}, y_{2n}). \tag{2.14}$$

From this, we get

$$\begin{aligned} d(Tx_{2n}, Sx_{2n+1}) &\leq d(Tx_{2n}, y_{2n}) + d(y_{2n}, Sx_{2n+1}) \\ &= d(Sx_{2n-1}, y_{2n}) - d(Sx_{2n-1}, Tx_{2n}) + d(y_{2n}, Sx_{2n+1}). \end{aligned} \tag{2.15}$$

By noting that $Sx_{2n+1}, Sx_{2n-1} \in Q_0$, one can conclude that

$$d(y_{2n}, Sx_{2n+1}) = d(y_{2n}, y_{2n+1}) = d(Fx_{2n}, Gx_{2n-1}) \leq \lambda d(Sx_{2n-1}, Tx_{2n}), \tag{2.16}$$

and

$$d(Sx_{2n-1}, y_{2n}) = d(y_{2n-1}, y_{2n}) = d(Fx_{2n-2}, Gx_{2n-1}) \leq \lambda d(Sx_{2n-1}, Tx_{2n-2}), \tag{2.17}$$

in view of Case 1.

Thus,

$$d(Tx_{2n}, Sx_{2n+1}) \leq \lambda d(Sx_{2n-1}, Tx_{2n-2}) - (1-\lambda) d(Sx_{2n-1}, Tx_{2n}) \leq \lambda d(Sx_{2n-1}, Tx_{2n-2}),$$

and we proved (2.13).

Similarly, If $(Sx_{2n+1}, Tx_{2n+2}) \in Q_0 \times P_1$, then $Tx_{2n+2} \in P_1$, and

$$d(Sx_{2n+1}, Tx_{2n+2}) + d(Tx_{2n+2}, y_{2n+2}) = d(Sx_{2n+1}, y_{2n+2}).$$

From this, we have

$$
\begin{aligned}
d(Sx_{2n+1}, Tx_{2n+2}) &\le d(Sx_{2n+1}, y_{2n+2}) + d(y_{2n+2}, Tx_{2n+2}) \\
&\le d(Sx_{2n+1}, y_{2n+2}) + d(Sx_{2n+1}, y_{2n+2}) - d(Sx_{2n+1}, Tx_{2n+2}) \\
&= 2d(Sx_{2n+1}, y_{2n+2}) - d(Sx_{2n+1}, Tx_{2n+2}) \\
&\Rightarrow d(Sx_{2n+1}, Tx_{2n+2}) \le d(Sx_{2n+1}, y_{2n+2}).
\end{aligned}
$$

By noting that $Sx_{2n+1} \in Q_0$, one can conclude that

$$d(Sx_{2n+1}, Tx_{2n+2}) \le d(Sx_{2n+1}, y_{2n+2}) = d(Fx_{2n}, Gx_{2n+1}) \le \lambda d(Tx_{2n}, Sx_{2n+1}), \tag{2.18}$$

in view of Case 1.

Thus, in all the cases 1-3, there exists $w_{2n} \in \{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n-2}, Sx_{2n-1})\}$ such that

$$d(Tx_{2n}, Sx_{2n+1}) \le \lambda w_{2n}$$

and exists $w_{2n+1} \in \{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n}, Sx_{2n+1})\}$ such that

$$d(Sx_{2n+1}, Tx_{2n+2}) \le \lambda w_{2n+1}.$$

Following the procedure of Assad and Kirk ([6]), it can easily be shown by induction that, for $n \ge 1$, there exists $w_2 \in \{d(Tx_0, Sx_1), d(Sx_1, Tx_2)\}$ such that

$$d(Tx_{2n}, Sx_{2n+1}) \le \lambda^{n-\frac{1}{2}} w_2 \quad \text{and} \quad d(Sx_{2n+1}, Tx_{2n+2}) \le \lambda^n w_2. \tag{2.19}$$

From (2.19) and by the triangle inequality, for $n > m$ we have

$$
\begin{aligned}
d(Tx_{2n}, Sx_{2m+1}) &\le d(Tx_{2n}, Sx_{2n-1}) + d(Sx_{2n-1}, Tx_{2n-2}) + \cdots + d(Tx_{2m+2}, Sx_{2m+1}) \\
&\le (\lambda^m + \lambda^{m+\frac{1}{2}} + \cdots + \lambda^{n-1}) w_2 \le \frac{\lambda^m}{1-\sqrt{\lambda}} w_2 \to \theta, \quad \text{as} \quad m \to \infty.
\end{aligned}
$$

From Remark 1.3 and Corollary 1.4 (1) $d(Tx_{2n}, Sx_{2m+1}) \ll c$.

Thus, the sequence $\{Tx_0, Sx_1, Tx_2, Sx_3, \cdots, Sx_{2n-1}, Tx_{2n}, Sx_{2n-1}, \cdots\}$ is a Cauchy sequence. Then, as noted in [13], there exists at least one subsequence $\{Tx_{2n_k}\}$ or $\{Sx_{2n_k+1}\}$ which is contained in P_0 or Q_0 respectively and finds its limit $z \in C$. Furthermore, subsequences $\{Tx_{2n_k}\}$ and $\{Sx_{2n_k+1}\}$ both converge to $z \in C$ as C is a closed subset of complete cone metric space (X, d). We assume that there exists a subsequence $\{Tx_{2n_k}\} \subseteq P_0$ for each $k \in N$, then $Tx_{2n_k} = y_{2n_k} = Gx_{2n_k-1} \in C \cap GC \subseteq TC$ Since TC as well as SC are closed in X and $\{Tx_{2n_k}\}$ is Cauchy sequence in TC, it converges to a point $z \in TC$. Let $w \in T^{-1}z$, then $Tw = z$. Similarly, $\{Sx_{2n_k+1}\}$ being a subsequence of Cauchy sequence $\{Tx_0, Sx_1, Tx_2, Sx_3, \cdots, Sx_{2n-1}, Tx_{2n}, Sx_{2n-1}, \cdots\}$ also converges to z as SC is closed. Using (2.2), one can write

$$d(Fw, z) \le d(Fw, Gx_{2n_k-1}) + d(Gx_{2n_k-1}, z) \le \lambda u_{2n_k-1} + d(Gx_{2n_k-1}, z),$$

where

$$
\begin{aligned}
u_{2n_k-1} \in \{&d(Tw, Sx_{2n_k-1}), d(Tw, Fw), d(Sx_{2n_k-1}, Gx_{2n_k-1}), \\
&\frac{d(Tw, Gx_{2n_k-1}) + d(Fw, Sx_{2n_k-1})}{2}\} \\
= \{&d(z, Sx_{2n_k-1}), d(z, Fw), d(Sx_{2n_k-1}, Gx_{2n_k-1}), \\
&\frac{d(z, Gx_{2n_k-1}) + d(Fw, Sx_{2n_k-1})}{2}\}.
\end{aligned}
$$

Let $\theta \ll c$. Clearly at least one of the following four cases holds for infinitely many n.

(1) $d(Fw, z) \le \lambda d(z, Sx_{2n_k-1}) + d(Gx_{2n_k-1}, z) \ll \lambda \frac{c}{2\lambda} + \frac{c}{2} = c$;

(2) $d(Fw, z) \le \lambda d(z, Fw) + d(Gx_{2n_k-1}, z) \Rightarrow d(Fw, z) \le \frac{1}{1-\lambda} d(Gx_{2n_k-1}, z) \ll \frac{1}{1-\lambda}(1-\lambda)c = c$;

(3) $d(Fw,z) \leq \lambda d(Sx_{2n_k-1}, Gx_{2n_k-1}) + d(Gx_{2n_k-1}, z) \leq \lambda(d(Sx_{2n_k-1}, z) + d(z, Gx_{2n_k-1})) + d(Gx_{2n_k-1}, z)$
$\leq (\lambda+1)d(Gx_{2n_k-1}, z) + \lambda d(Sx_{2n_k-1}, z) \ll (\lambda+1)\frac{c}{2(\lambda+1)} + \lambda\frac{c}{2\lambda} = c$;

(4) $d(Fw,z) \leq \lambda\frac{d(z,Gx_{2n_k-1}) + d(Fw,Sx_{2n_k-1})}{2} + d(Gx_{2n_k-1}, z)$
$\leq \lambda\frac{d(z,Gx_{2n_k-1}) + d(z,Sx_{2n_k-1})}{2} + \frac{1}{2}d(Fw,z) + d(Gx_{2n_k-1}, z)$
$\Rightarrow d(Fw,z) \leq (2+\lambda)d(Gx_{2n_k-1}, z) + \lambda d(z, Sx_{2n_k-1}) \ll (2+\lambda)\frac{c}{2(2+\lambda)} + \lambda\frac{c}{2\lambda} = c$

In all the cases we obtain $d(Fw,z) \ll c$ for each $c \in intP$, using Corollary 1.4 (3) it follows that $d(Fw,z) = \theta$ or $Fw = z$. Thus, $Fw = z = Tw$, that is z is a coincidence point of F, T.

Further, since Cauchy sequence $\{Tx_0, Sx_1, Tx_2, Sx_3, \cdots, Sx_{2n-1}, Tx_{2n}, Sx_{2n-1}, \cdots\}$ converges to $z \in C$ and $z = Fw, z \in FC \cap C \subseteq SC$, there exists $v \in C$ such that $Sv = z$. Again using (2.2), we get

$$d(Sv, Gv) = d(z, Gv) = d(Fw, Gv) \leq \lambda u,$$

where

$$\begin{aligned} u &\in \{d(Tw,Sv), d(Tw,Fw), d(Sv,Gv), \frac{d(Tw,Gv) + d(Fw,Sv)}{2}\} \\ &= \{\theta, \theta, d(Sv,Gv), \frac{d(z,Gv) + \theta}{2}\} = \{\theta, d(Sv,Gv), \frac{d(Sv,Gv)}{2}\}. \end{aligned}$$

Hence, we get the following cases:

$$d(Sv,Gv) \leq \lambda\theta = \theta, d(Sv,Gv) \leq \lambda d(Sv,Gv)$$

and

$$d(Sv,Gv) \leq \frac{\lambda}{2}d(Sv,Gv) \leq \lambda d(Sv,Gv).$$

Using Remark 1.3 and Corollary 1.4 (3), it follows that $Sv = Gv$, therefore, $Sv = z = Gv$, that is z is a coincidence point of (G, S).

In case FC and GC are closed in X, then $z \in FC \cap C \subseteq SC$ or $z \in GC \cap C \subseteq TC$. The analogous arguments establish (IV) and (V). If we assume that there exists a subsequence $\{Sx_{2n_k+1}\} \subseteq Q_0$ with TC as well SC are closed in X, then noting that $\{Sx_{2n_k+1}\}$ is a Cauchy sequence in SC, foregoing arguments establish (IV) and (V).

Suppose now that (F,T) and (G,S) are weakly compatible pairs, then

$$z = Fw = Tw \Rightarrow Fz = FTw = TFw = Tz$$

and

$$z = Gv = Sv \Rightarrow Gz = GSv = SGv = Sz.$$

Then, from (2.2),

$$d(Fz, z) = d(Fz, Gv) \leq \lambda u,$$

where

$$\begin{aligned} u &\in \{d(Sv,Tz), d(Tz,Fz), d(Sv,Gv), \frac{d(Tz,Gv) + d(Sv,Fz)}{2}\} \\ &= \{d(z,Fz), d(z,z), \frac{d(Fz,z) + d(z,Fz)}{2}\} \\ &= \{d(z,Fz), \theta\}. \end{aligned}$$

Hence, we get the following cases:

$$d(Fz,z) \leq \lambda d(z,Fz) \Rightarrow d(Fz,z) = 0,$$

$$d(Fz,z) \leq \lambda\theta = \theta \Rightarrow d(Fz,z) = 0.$$

Using Remark 1.3 and Corollary 1.1 (3), it follows that $Fz = z$. Thus, $Fz = z = Tz$

Similarly, we can prove $Gz = z = Sz$. Therefore $z = Fz = Gz = Sz = Tz$, that is, z is a common fixed point of F, G, S and T.

Uniqueness of the common fixed point follows easily from (2.2). □

2.4. Remark. 1. Theorem 2.2 in [18] is a special case of Theorem 2.3 with $G = F = f, T = S = g$ and $\lambda \in (0, \sqrt{2} - 1)$.

2. Setting $G = F = f$ and $T = S = I_X$ (the identity mapping on X) in Theorem 2.3, we obtain the following result:

2.5. Corollary. *Let (X, d) be a complete cone metric space, and C a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x, z) + d(z, y) = d(x, y).$$

Suppose that $f : C \to X$ satisfying the condition

$$d(fx, fy) \leq \lambda u(x, y),$$

where

$$u(x, y) \in \{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}\}$$

for all $x, y \in C, 0 < \lambda < 1$ and f has the additional property that for each $x \in \partial C, fx \in C$. Then f has a unique fixed point.

2.6. Remark. The following definition is a special case of Definition 2.2 when (X, d) is a metric space. But when (X, d) is a cone metric space, which is not a metric space, this is not true. Indeed, there may exist $x, y \in X$ such that the vectors $d(Tx, Fx), d(Sy, Gy)$ and $\frac{d(Tx,Fx)+d(Sy,Gy)}{2}$ are incomparable. For the same reason Theorems 2.3 and 2.8 (given below) are incomparable.

2.7. Definition. Let (X, d) be a complete cone metric space and C be a nonempty closed subset of X, and $F, G, S, T : C \to X$. Denote, for$x, y \in C$,

$$M_2^{F,G,S,T} = \{d(Tx, Sy), \frac{d(Tx, Fx) + d(Sy, Gy)}{2}, \frac{d(Tx, Gy) + d(Fx, Sy)}{2}\}. \tag{2.20}$$

Then (F, G) is called a generalized $(T, S)_{M_2}-$contractive mapping of C into X if for some $\lambda \in (0, 1)$ there exists $u(x, y) \in M_2^{F,G,S,T}$ such that for all $x, y \in C$

$$d(Fx, Gy) \leq \lambda u(x, y). \tag{2.21}$$

Our next result is the following.

2.8. Theorem. *Let (X, d) be a complete cone metric space, C a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x, z) + d(z, y) = d(x, y).$$

Suppose that $F, G, S, T : C \to X$ are such that (F, G) is a generalized $(T, S)_{M_2}-$contractive mappings pair of C into X, and

(I) $\partial C \subseteq SC \cap TC, FC \cap C \subseteq SC, GC \cap C \subseteq TC$,

(II) $Tx \in \partial C$ implies that $Fx \in C, Sx \in \partial C$ implies that $Gx \in C$,

(III) SC and TC (or FC and GC) are closed in X.

Then

(IV) (F, T) has a point of coincidence,

(V) (G, S) has a point of coincidence.

Moreover, if (F, T) and (G, S) are weakly compatible pairs, then F, G, S and T have a unique common fixed point.

The proof of this theorem is very similar to the proof of Theorem 2.3 and it is omitted.

We now list some corollaries of Theorems 2.3 and 2.8.

2.9. Corollary. *Let (X,d) be a complete cone metric space, C a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x,z) + d(z,y) = d(x,y).$$

Let $F,G,S,T : C \to X$ be such that

$$d(Fx,Gy) \leq \lambda d(Tx,Sy), \tag{2.22}$$

for some $\lambda \in (0,1)$ and for all $x,y \in C$.

Suppose, further, that F,G,S,T and C satisfy the following conditions:

(I) $\partial C \subseteq SC \cap TC, FC \cap C \subseteq SC, GC \cap C \subseteq TC$,

(II) $Tx \in \partial C$ implies that $Fx \in C, Sx \in \partial C$ implies that $Gx \in C$,

(III) SC and TC (or FC and GC) are closed in X.

Then

(IV) (F,T) has a point of coincidence,

(V) (G,S) has a point of coincidence.

Moreover, if (F,T) and (G,S) are weakly compatible pairs, then F,G,S and T have a unique common fixed point.

2.10. Corollary. *Let (X,d) be a complete cone metric space, C a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x,z) + d(z,y) = d(x,y).$$

Let $F,G,S,T : C \to X$ be such that

$$d(Fx,Gy) \leq \lambda(d(Tx,Fx) + d(Sy,Gy)), \tag{2.23}$$

for some $\lambda \in (0,1/2)$ and for all $x,y \in C$.

Suppose, further, that F,G,S,T and C satisfy the following conditions:

(I) $\partial C \subseteq SC \cap TC, FC \cap C \subseteq SC, GC \cap C \subseteq TC$,

(II) $Tx \in \partial C$ implies that $Fx \in C, Sx \in \partial C$ implies that $Gx \in C$,

(III) SC and TC (or FC and GC) are closed in X.

Then

(IV) (F,T) has a point of coincidence,

(V) (G,S) has a point of coincidence.

Moreover, if (F,T) and (G,S) are weakly compatible pairs, then F,G,S and T have a unique common fixed point.

2.11. Corollary. *Let (X,d) be a complete cone metric space, C a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x,z) + d(z,y) = d(x,y).$$

Let $F,G,S,T : C \to X$ be such that

$$d(Fx,Gy) \leq \lambda(d(Tx,Gy) + d(Fx,Sy)), \tag{2.24}$$

for some $\lambda \in (0,1/2)$ and for all $x,y \in C$.

Suppose, further, that F,G,S,T and C satisfy the following conditions:

(I) $\partial C \subseteq SC \cap TC, FC \cap C \subseteq SC, GC \cap C \subseteq TC$,

(II) $Tx \in \partial C$ implies that $Fx \in C, Sx \in \partial C$ implies that $Gx \in C$,

(III) SC and TC (or FC and GC) are closed in X.

Then

(IV) (F,T) has a point of coincidence,

(V) (G,S) has a point of coincidence.

Moreover, if (F,T) and (G,S) are weakly compatible pairs, then F,G,S and T have a unique common fixed point.

2.12. Remark. Setting $G = F = f$ and $T = S = g$ (the identity mapping on X) in Corollary 2.9-2.11, we obtain the following result:

2.13. Corollary. *Let (X, d) be a complete cone metric space, C a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x, z) + d(z, y) = d(x, y).$$

Let $f, g : C \to X$ be such that

$$d(fx, fy) \leq \lambda d(gx, gy), \tag{2.25}$$

for some $\lambda \in (0, 1)$ and for all $x, y \in C$. Suppose, further, that f, g and C satisfy the following conditions:

(I) $\partial C \subseteq gC, fC \cap C \subseteq gC$,
(II) $gx \in \partial C$ implies that $fx \in C$,
(III) gC is closed in X.

Then there exists a coincidence point z of f, g in C. Moreover, if (f, g) are weakly compatible, then z is the unique common fixed point of f and g.

2.14. Corollary. *Let (X, d) be a complete cone metric space, C a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x, z) + d(z, y) = d(x, y).$$

Let $f, g : C \to X$ be such that

$$d(fx, fy) \leq \lambda(d(fx, gx) + d(fy, gy)), \tag{2.26}$$

for some $\lambda \in (0, 1/2)$ and for all $x, y \in C$. Suppose, further, that f, g and C satisfy the following conditions:

(I) $\partial C \subseteq gC, fC \cap C \subseteq gC$,
(II) $gx \in \partial C$ implies that $fx \in C$,
(III) gC is closed in X.

Then there exists a coincidence point z of f, g in C. Moreover, if (f, g) are weakly compatible, then z is the unique common fixed point of f and g.

2.15. Corollary. *Let (X, d) be a complete cone metric space, C a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x, z) + d(z, y) = d(x, y).$$

Let $f, g : C \to X$ be such that

$$d(fx, fy) \leq \lambda(d(fx, gy) + d(fy, gx)), \tag{2.27}$$

for some $\lambda \in (0, 1/2)$ and for all $x, y \in C$. Suppose, further, that f, g and C satisfy the following conditions:

(I) $\partial C \subseteq gC, fC \cap C \subseteq gC$,
(II) $gx \in \partial C$ implies that $fx \in C$,
(III) gC is closed in X.

Then there exists a coincidence point z of f, g in C. Moreover, if (f, g) are weakly compatible, then z is the unique common fixed point of f and g.

2.16. Remark. Corollaries 2.13-2.15 are the corresponding theorems of Abbas and Jungck from [2] in the case that f, g are non-self mappings.

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