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NONPARAMETRIC CONTROL CHARTS BASED ON MAHALANOBIS DEPTH

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Abstract

This study concerns the use of r and Q control charts based on data depth to control process involving multivariate quality measurements. In this paper, firstly the concept of data depth is introduced in order to construct quality control chart structures, the characteristics of data depth are given and statistics based on this concept are obtained. Following this, the structures and interpretations of mainly nonparametric r and Q control charts are explained for the Mahalanobis depth measure used in statistical quality control by means of an example.

Keywords: Control Charts, r Chart, Q Chart, Multivariate statistical process control, Depth function.

1. Introduction

In statistical process control, control charts are very important tools for monitoring and/or controlling whether a manufacturing process is statistically in control or not. Shewart's $(X, \overline{X})$ and CUSUM charts are widely used for this purpose. In addition to their efficiency, these charts are preferred because they are simple to construct and interpret. However, as these charts are based on an assumption of normality of the quality variable and are used when there is only one quality variable, they are not always appropriate. In many cases, two or more variables may need to be monitored, and following these two (or more) quality variables separately may be misleading. The Type I error α occurring when the variables are monitored separately differs from the Type I error α occurring when the variables are monitored simultaneously. Therefore, multivariate control charts are required when there is more than one quality variable. Monitoring the process of related variables is usually called a *multivariate quality control* problem.

Studies of multivariate quality control were first carried out by Hotelling in 1947; later, Hicks, Jackson, Crosier, Hawkins, Lowry, Montgomery, Pignatiello, Runger, Tracy, Young, Mason, Wadsworth, Alt and others also carried out studies on this subject. The work of these authors is given in [5], together with detailed references. The multivariate control charts considered by these authors are also based on the normality assumption

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for the quality variable. Widely used multivariate control charts are χ^2 and Hotelling's T^2 charts.

Liu defined the new, mainly nonparametric, control charts (r, Q and S) by using a depth data concept in order to monitor the process of multivariate quality measurements [4]. These charts, like $(X, \overline{X})$ and CUSUM charts, are important as they bring out the shift in location and the increase in scale, in addition to having a two dimensional graphic representation which makes them easy to interpret. Unlike χ^2 and Hotelling's T^2, the normality assumption on the quality variable is not required, which is another advantage of these charts. So, these charts serve as an important measure in quality assurance [5].

In this study, r and Q control charts based on the Mahalanobis depth for elliptical distributions are given. Firstly, depth function, its properties and statistics obtained from depth function to construct control charts based on Mahalanobis depth are explained.

2. Data Depth

Statistical depth functions are widely used in nonparametric derivations for multivariate data. A depth function suitable for a distribution F in R^p, denoted by $D(F;x)$, brings out the non-central ranking of X in R^p with respect to F. A higher value of $D(F;x)$ for X in R^p means that X based on the distribution F is deeper (more central), and vice versa. That is, a smaller value of $D(F;x)$ for X in R^p shows that the sample point is further away from the center with respect to F. Depth functions have some important characteristics, which are given as follows:

Affine Invariance: Let $\mathcal{F}$ denote the family of distributions in R^p. If X is a random vector having distribution function F in R^p, then $D(F_{Ax+b}; Ax+b) = D(F;x)$.

Here, A is a non-singular $p \times p$ dimensional matrix and b is p-dimensional vector.

Maximality at the Center: When a F$\in \mathcal{F}$ is symmetric around any point θ_0 (that is, if the distribution function of the random variable X is F, then $(X-\theta_0)$ and $(\theta_0 - X)$ have the same distribution), then

$D(F;\theta_0) = \max_{x \in R^p} D(F;x)$.

Monotonicity Relative to the Deepest Point: When $F \in \mathcal{F}$ is symmetric around a point θ_0, in other words θ_0 is the deepest point of the distribution F, then $D(F;x)$ has the monotonicity characteristic if $D(F;x) \leq D(F;\theta_0 + \alpha(x-\theta_0))$.

Here $\alpha \in [0,1]$.

Vanishing at Infinity: For all $F \in \mathcal{F}$, if $\|x\| \to \infty$ then $D(F;x) \to 0$ [6].

Many data depth concepts have been given having all of the above properties. Tukey's depth, Majority depth, Mahalanobis depth and simplicial depth are the most popular of these.

In this study, as the control charts based on Mahalanobis depth are to be used, only Mahalanobis depth is given in detail here.

Liu defined Mahalanobis depth as follows [4]:

Let X denote a random vector having distribution function F in R^p. Then the Mahalanobis measure of depth for any point x (a $p \times 1$-dimensional vector) in R^p with respect to the distribution F is defined as follows:

$$MD(F;x) = \frac{1}{[1 + (x-\mu_F)\Sigma^{-1}(x-\mu_F)]} \tag{1}$$

In (1), μ_F and Σ_F are the mean vector and covariance matrix of the distribution function F, respectively. Hence, $MD(F;x)$ is a measure showing how 'deep' or 'central' x is with respect to the distribution F. When F is unknown and a sample taken from distribution F is given, then the definition of Mahalanobis depth is:

$$(2) \qquad MD(F_m; x) = \frac{1}{\left[1 + (x - \overline{X})' S^{-1} (x - \overline{X})\right]}$$

In (2), $\overline{X}$ is a $(p \times 1)$ sample mean vector and S a $(p \times p)$ sample covariance matrix. The depth function $MD(F; \cdot)$ is a depth function satisfying all of the above properties [3, 4].

3. Some Statistics Based on Data Depth used in Constructing Control Charts

Let F and G be the distribution functions of two independent p-dimensional populations and $X = \{X_1, \ldots, X_m\}$ be a sample taken from a population having distribution F. In quality control, F can be thought of as a 'good population,' in another words considered as measurements of products produced by an in-control process.

Let $Y = \{Y_1, \ldots, Y_n\}$ be a random sample taken from a population having distribution G (that is, new measurements taken from the process). In order to test whether the process is in control, or if there is any deterioration in the quality of the product by using the observations Y_i's, the distributions F and G need to be compared. If the Y_i's do not approach to the distribution F, this means that the quality of product has deteriorated. The hypotheses to test this can be given as follows:

$$H_o : F = G \text{ vs.}$$

$$(3) \qquad H_A : \text{ There is a location shift and / or a scale increase from } F \text{ to } G$$

To test this hypothesis, the statistic $R(F; Y)$ which characterises the distance between F and G with respect to data depth when $X \sim F$ and $Y \sim G$ for a Y_i in R^p with respect to the given data depth $D(\cdot\,; \cdot)$ is defined as follows [2, 4]:

$$(4) \qquad R(F; Y_i) = P_F \{X : D(F; X) \le D(F; Y_i) / X \sim F\}$$

When $D(F; \cdot)$ has affine- invariance, then $R(F; Y)$ also has affine-invariance:

$$R(F; Y) = R(AY + b; F_{AY+b}).$$

Under the hypothesis $F = G$, if the distribution of $D(F; Y)$ is continuous, then the distribution of $R(F; Y)$ in (4) is uniformly distributed in $[0, 1]$:

$$(5) \qquad R(F; Y) \sim U(0, 1).$$

The mean of the ratios $R(F; Y)$ for all y generated from population G, denoted by $Q(F, G)$, is found as follows:

$$(6) \qquad Q(F, G) = P\{D(F; X) \le D(F; Y) / X \sim F, Y \sim G\}$$
$$(= E_G[R(F; Y)])$$

The parameter $Q(F, G)$ given in (6) is called the 'quality index' and takes values between 0 and 1 [2]. The quality index $Q(F, G)$ shows whether or not there is a difference in location and/or dispersion by comparing G and F.

When $D(F; \cdot)$ has affine-invariance, then $Q(F, G)$ also has affine-invariance:

$$(7) \qquad Q(F, G) = Q(F_{AX+b}; G_{AY+b}).$$

In (7), A is a $(p \times p)$ non-singular matrix and b is a $(p \times 1)$ vector.

Let us denote by $\text{ell}(h; \theta, \Sigma)$ an elliptical distribution with parameters θ and Σ, where $h(\,.\,)$ is a function from R to R and Σ is positive definite. When there is only a location shift but no change in dispersion, the function $Q(F; G)$ decreases as θ_1 moves away from θ_0 along any line when $F \sim \text{ell}(h; \theta_0, \Sigma_0)$ and $G \sim \text{ell}(h; \theta_1, \Sigma_0)$.

When the locations are the same but there is a difference in dispersion, for $F \sim \text{ell}(h;\theta_0,\Sigma_0)D(F;\,\cdot\,)$ and $G \sim \text{ell}(h;\theta_0,\Sigma_1)$, $R(F;Y) \overset{ss}{\leq} R(F;X)$ and $Q(F;G) \leq \frac{1}{2}$ (here "ss'" denotes stochastically smaller). When there are both location shift and scale change, then for $F \sim \text{ell}(h;\theta_0,\Sigma_0)$ and $G \sim \text{ell}(h;\theta_1,\Sigma_1)$, while the parameter θ_1 moves away from θ_0 along any line, the function $Q(F;G)$ decreases uniformly [2].

The statistics obtained from (4) and (6) will be used while constructing the structure of the control charts, and their limit distributions are given as follows:

a. Assuming the distribution F is known (meaning that F is either regarded as the collection of one (or several) acceptable lot(s) or an elliptical distribution with μ and Σ obtained from the measurements of a large acceptable batch).

When $Y = \{Y_1,\ldots,Y_n\}$ is a random sample taken from the distribution, $Q(F;G)$ is the mean of the random variables $R(F;Y)$:

$$(8) \qquad Q(F,G_n) = \frac{1}{n}\sum_{i=1}^{n} R(F;Y_i).$$

If D(F;X) has a continuous distribution, under hypothesis H_0 the distribution of $Q(F,G_n)$ is the same as the distribution of $\sum_{i=1}^{n}\frac{U_i}{n}$, when $U_1,\ U_2,\ldots,U_n$ are independent and uniformly distributed random variables in $(0,1)$.

As a result of the Central Limit Theorem,

$$(9) \qquad For\ n \to \infty\ ,\ \left[Q(F,G_n) - \frac{1}{2}\right] \to^{k} N(0,\frac{1}{12n}).$$

In (9), "k" means convergence in law.

b. If $X = \{X_1,\ldots,X_m\}$ is a random sample taken from the unknown distribution F and $Y = \{Y_1,\ldots,Y_n\}$ a random sample taken from the distribution G, then the estimation of $Q(F;G)$ is:

$$(10) \qquad Q(F_m,G_n) = \frac{1}{n}\sum_{i=1}^{n} R(F_m;Y_i).$$

In (10), $R(F_m;Y_i)$ is the ratio of the X_j's satisfying $D(F_m;X_j) \leq D(F_m;Y_i)$ when the distribution F is unknown:

$$(11) \qquad R(F_m;Y_i) = \#\left\{D(F_m;X_j) \leq D(F_m;Y_i),\ j=1,\ldots,m\right\}/m.$$

Here the values of $D(F_m;\,\cdot\,)$ are empirical depth values calculated with respect to F_m, and if the distribution F is continuous, $D(F_m;\,\cdot\,)$ converges to $D(F;\,\cdot\,)$ uniformly as $m \to \infty$. Therefore:

$$(12) \qquad R(F_m;Y_i) \to^{k} U\,[0,1]\ \text{ as } m \to \infty,\ \text{for all } X.$$

The uniform convergence of $D(F_m;\,\cdot\,)$ obtained by using Mahalanobis depth is valid when F is an elliptical distribution and the second absolute moment of the distribution F exists $\left(E_F\left\|X\right\|^2 < \infty\right)$.

As a result of this, when F is an elliptical distribution and $\left(E_F\left\|X\right\|^2 < \infty\right)$, then $MD(F_m;x)$ converges to $MD(F;x)$ uniformly as $m \to \infty$ [2, 4].

In the same way, when $D(F;Y)$ has continuous distribution, then it has also been shown that $Q(F_m,G_n)$ in (10) converges to $Q(F;G)$ as $\min(m,n) \to \infty$.

Under the condition that (11) holds, the limit distribution of $Q(F_m,G_n)$ is:

$$(13) \qquad \left[Q(F_m,G_n) - \frac{1}{2}\right] \to^{k} N\left(0,\left[\frac{1}{m}+\frac{1}{n}\right]\Big/12\right),\ \text{ as } \min(m,n) \to \infty.$$

In (13), if $Q(. , .)$ is used for the Mahalanobis depth then this equation is valid when F is continuous and the forth absolute moment exists $\left(E_F \|X\|^4 < \infty\right)$ [2, 4].

4. r and Q Control Charts based on the Mahalanobis Depth

A r control chart is alike to a univariate X chart. An X control chart reveals whether there is a deviation from a pre-determined process mean, or a trend or a non-random pattern of an observation set. However, although this chart is very simple and efficient when used to observe a univariate process, it cannot be easily generalized to a multivariate process. In studies of the bivariate normal distribution, the control limits are given as elliptical limits named as the control ellipse [1]. However, it cannot be said that they are efficient for a multivariate data set as they require a normality assumption, Type I Error α changes and the ranking of sample points with respect to time losses. An r control chart with $LCL = \alpha$ corresponds to an α-level test of the following hypothesis:

$H_0 : F = G$, vs.

H_A : there is a location shift and/or a scale increase from F to G.

In order to construct a r control chart, the values of $R(F; Y_i)$ are obtained using (4) and (11) when the distribution F is known, or the values of $R(F_m; Y_i)$, $i = 1, \dots, n$ are calculated for $X_1, \dots, X_m$ only, when the distribution F is unknown. When the distribution F is elliptical we denote by $R_{MD}(F, Y_i)$ the value of $R(F, Y_i)$ obtained by using the Mahalanobis depth given in (1) and (2). When the distribution F is known, $\mathrm{R}_{MD}(F, Y_i)$ is given by:

$$R_{MD}(F; Y_i) = P_F \left\{X : MD(F; X) \leq MD(F; Y)/X \sim F\right\}, \tag{14}$$

and when the distribution F is unknown, $\mathrm{R}_{MD}(F_m; Y_i)$ is given by:

$$R_{MD}(F_m; Y_i) = \# \left\{MD(F_m; X_j) \leq MD(F_m; Y_i),\ j = 1, \dots, m\right\}/m. \tag{15}$$

It is known from [4] that $R_{MD}(F; Y_i)$ has all the properties of $R(F, Y_i)$.

A r control chart is constructed by plotting the $R(F, Y_i)$'s or the $R(F_m, Y_i)$'s for sample points $i = 1, \dots, n$.

The center line (CL) and the lower control limit (LCL) of the chart are:

$$\begin{aligned} &CL = 0.5, \\ &LCL = \alpha. \end{aligned} \tag{16}$$

Equation (16) can be obtained from (4) and (12). As seen from these equations, the expected values of $R(F, Y_i)$ and $R(F_m, Y_i)$ are 0.5. Therefore, in a r control chart it is suitable to take CL as 0.5. Also, the values of the $R(F, Y_i)$ or the $R(F_m, Y_i)$ being higher than 0.5 indicates a decrease in scale or an omittable shift in location. This is thought of as a gain not a loss in the quality concept of statistical process control. The process is not said to be out-of-control. Therefore, in a r control chart there exists only a lower control limit LCL. However, although a UCL does not exist, CL plays the role of a reference point enabling the observation of a possible trend or non-random pattern. In a r control chart, when the values of $R(F, Y_i)$ or $R(F_m, Y_i)$ are in the region of α, this means that process is statistically out-of-control. That is, there is signal of possible quality deterioration or an out-of-control process. $R(F_m, Y_i)$ given in (11) shows how far away from the center Y is with respect to the data set X_j. If the values of the $R(F_m, Y_i)$ are small the ratio of the X_j's further from the center than Y is also small. So, Y is not fitted centrally to a good data set. Therefore, under the assumption of $Y \sim G$, a small value of the values $R(F_m, Y_i)$ shows a possible deviation of G from F. As $R(F_m, Y_i)$ is defined with respect to data depth, a possible deviation means a shift in location and/or an increase in scale [4].

The aim of a Q control chart is similar to that of a univariate $\overline{X}$ control chart. The hypotheses for a Q control chart are:

$H_0 : F = G, Q(F, G) = \frac{1}{2}$

$HA : Q(F, G) < \frac{1}{2}$

The acceptance of the hypothesis $H_A : Q(F, G) < \frac{1}{2}$ means that on average more than 50% of the population F are deeper (more central) than any of the observations Y generated from the distribution G. This shows a possible shift in location and/or an increase in scale from F to G. If $Q(F, G) > \frac{1}{2}$, then G has a very small dispersion.

As a multivariate Q chart is similar to a univariate $\overline{X}$ control chart, in order to construct Q control chart the mean of $R(F, Y_i)$ or of $R(F_m,\mathrm{Y}_i)$ for k subgroups, each of which have equal size, needs to be calculated. Assuming that the size of each subset is t, the Q control chart for $k \times t = n$, is constructed using (8) and (10) by plotting $\left\{Q(F, G_t^1), Q(F, G_t^2), \ldots\right\}$ when F is known and $\left\{Q(F_m, G_t^1), Q(F_m, G_t^2) \ldots\right\}$ when F is unknown. Here G_t^k denotes the k^{th} $(k = 1, 2, \ldots)$ subgroup of the Y_i's with size t.

The values of the CL and the LCL of a Q chart depend on the choice of the size of the subgroup. When t is large, the CL and LCL of the Q control chart for the $\left\{Q(F, G_t^k)\right\}$'s are obtained from (9):

$$(17) \qquad CL = 0.5 \text{ and } LCL = \left\{ .5 - z_\alpha \sqrt{\frac{1}{12t}} \right\},$$

and for the $\left\{Q(F_m, G_t^k)\right\}$'s they are obtained from (13):

$$(18) \qquad CL = 0.5 \text{ and } LCL = \left\{ .5 - z_\alpha \sqrt{\frac{1}{12t} \left[\left(\frac{1}{k} + \frac{1}{t} \right) \right]} \right\}.$$

This approach is valid until t=5. In applications, t can be taken as 3 or 4. In this case, the parameters for the Q chart are given as follows [4]:

$$(19) \qquad CL = 0.5 \text{ and } LCL = \frac{(t!\alpha)^{\frac{1}{t}}}{t}.$$

The means of the ratios of Y's more out-of-centre than X with respect to the Mahalanobis depth given by (14) and (15) are respectively [4]:

$$(20) \qquad Q_{MD}(F, G_t^k) = \frac{1}{t} \sum_{i=1}^{t} R_{MD}(F; Y_i), \; k = 1, 2, \ldots$$

and

$$(21) \qquad Q_{MD}(F_m, G_t^k) = \frac{1}{t} \sum_{i=1}^{t} R_{MD}(F_m; Y_i), \; k = 1, 2, \ldots$$

Hence, r control charts are constructed with respect to Mahalanobis depth by using (14) and (15) and Q control chart by using (20) and (21). It is obvious that there is no change in the central line and the control limits for these control charts obtained using Mahalanobis depht.

5. An Application

In this study, r and Q control charts based on Mahalanobis depth are obtained and interpreted for a bivariate data set by taking into account the application in the study of Liu [4]. Firstly, a random sample of size 540 is generated from the distribution $F \sim N\left[\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right]$ using the MINITAB software. And then, a sample of size 40 is generated from the distribution $G \sim N\left[\left(\begin{smallmatrix}2\\2\end{smallmatrix}\right), \left(\begin{smallmatrix}4&0\\0&4\end{smallmatrix}\right)\right]$. Although the normality assumption is not required for the construction of these charts, a normal distribution is chosen to make the evaluation of the outcome easier.

The last 40 of the 540 sample points generated from distribution F are considered as if they were generated from distribution G. So, in the charts constructed 40 sample points generated from distribution F are expected to be in-control and 40 sample points generated from distribution G are expected to reveal a shift in location and/or a change in dispersion.

For every X_i, $(i = 1, 2, \ldots, 500)$ and Y_i, $(i = 1, 2, \ldots, 80)$, the Mahalanobis measure of depth is calculated using the EXCELL program. For Y_i, $(i = 1, 2, ..., 80)$, the values of $MD(F_m; X_i)$ and $R_{MD}(F_m; X_i)$ are given in Appendix 1. Using (16), a r control chart is constructed using the 80 sample points, and are given in Figure 1. The value of LCL is equal to 0.05. From Figure 1, it can be seen that the r control chart revails a shift in the distribution mean and an increase in the scale as the last 40 sample points are out of LCL.

A Q control chart is constructed for $t = 4$ and $t = 10$. The values of $Q_{MD}(F_m, G_t)$ obtained using (21) for $t = 4$ and for $t = 10$ are given in Appendix 2a and Appendix 2b, respectively. Figure 2 and Figure 3 show the Q control charts constructed for these sample points. From Figure 2, it is seen that 10 of the last 40 sample points are beyond the lower control limit for $t = 4$, and similarly it is seen that the last 4 sample points are out-of-control for $t = 10$.

Figure 1. r control chart ($n = 80$)

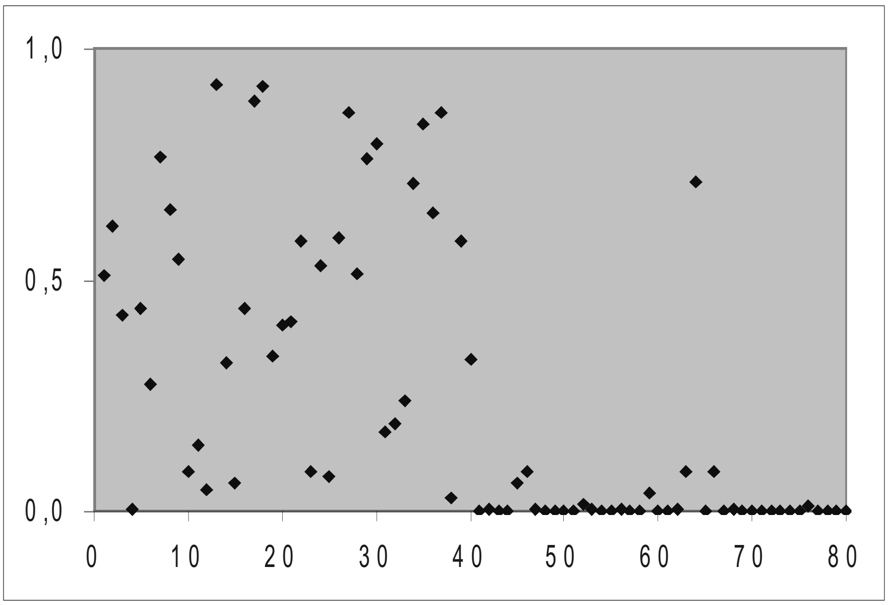


Figure 2. Q control chart ($t = 4, k = 20$)

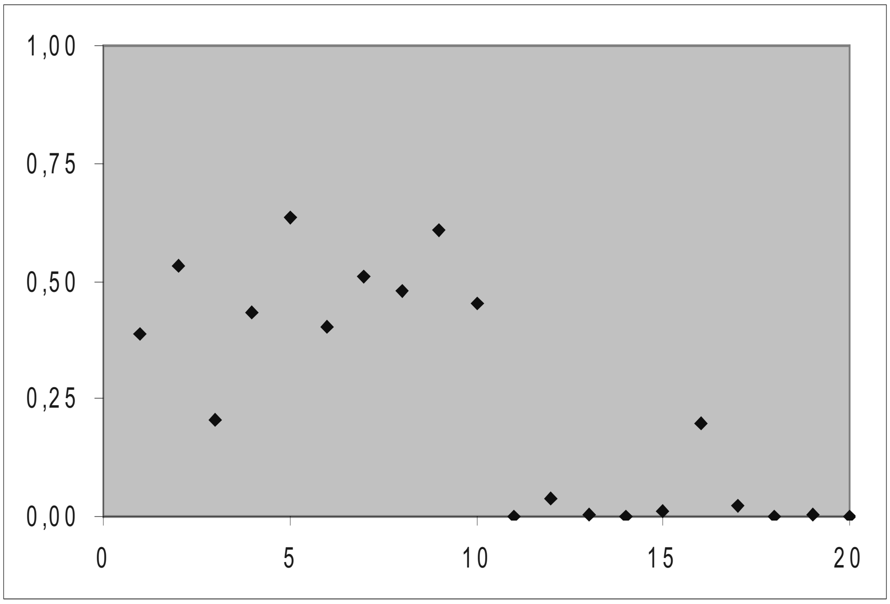
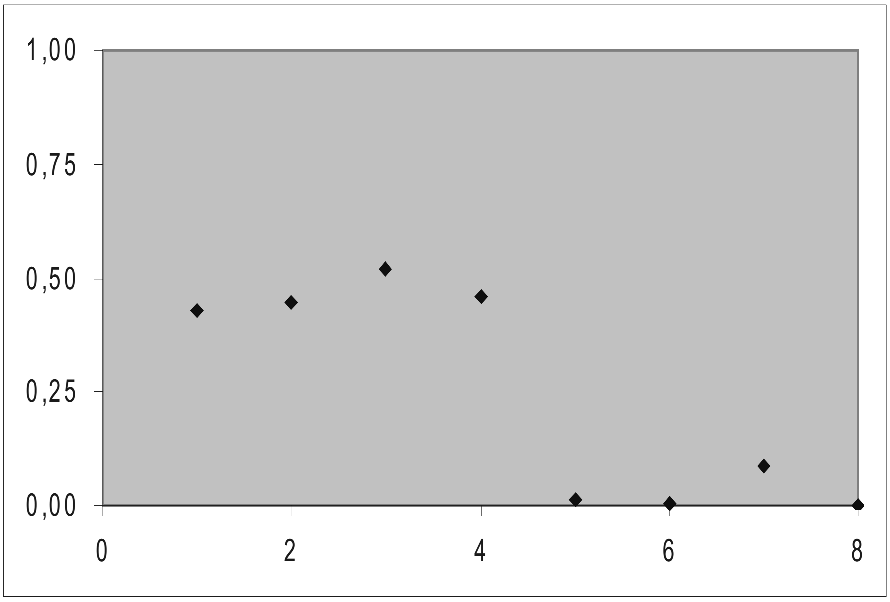


Figure 3. Q control chart ($t = 10, k = 8$)



6. Concluding Remarks

r and Q control charts, mainly nonparametric, are constructed using the data depth concept and are used for monitoring the process of multivariate quality measurements, and can be interpreted just as easily as the well-known univariate X, $\overline{X}$ and CUSUM charts. In addition they detect simultaneously the location shift and scale increase of the process [4]. Unlike χ^2 and Hotelling's T^2 charts, one of the advantages of these charts is that a normality assumption is not required.

It might be thought that r and Q control charts constructed using the Mahalanobis depth are similar to a Hotelling T^2 chart, because both of them represent quadratic distance of a point from its mean. However, while constructing r and Q control charts, the Mahalanobis depth serves only as a tool to obtain the ranks of observations. The charts are constructed with respect to the ranks of Mahalanobis depths, not with respect to Mahalanobis depth itself. Also, to decide control limits in Hotelling's T^2, the sample distribution of Hotelling T^2 statistics is needed. For r and Q control charts based on Mahalanobis depth, this is not required as the statistics are converted to ranks. Therefore, the plotting of charts based on Mahalanobis depth is different from that based on Hotelling's T^2 and for an elliptical distributions r and Q control charts based on Mahalanobis depth can be said to be more efficient

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7. Appendix 1

The values of $MD(F_m;Y_i)$ and $R_{MD}(F_m;Y_i)$ for the last 40 sample points generated from distribution F and for the 40 sample points generated from distribution G

Y_i	$MD(F_m;Y_i)$	$R_{MD}(F_m;Y_i)$	Y_i	$MD(F_m;Y_i)$	$R_{MD}(F_m;Y_i)$
1	0,6265	0,5080	41	0,0822	0,0000
2	0,6945	0,6140	42	0,1292	0,0020
3	0,5649	0,4220	43	0,1025	0,0000
4	0,1556	0,0020	44	0,0660	0,0000
5	0,5801	0,4360	45	0,2684	0,0620
6	0,4383	0,2740	46	0,2885	0,0840
7	0,8158	0,7660	47	0,1213	0,0020
8	0,7316	0,6500	48	0,0506	0,0000
9	0,6500	0,5460	49	0,0671	0,0000
10	0,2844	0,0840	50	0,0172	0,0000
11	0,3439	0,1440	51	0,0828	0,0000
12	0,2527	0,0480	52	0,2007	0,0160
13	0,9355	0,9220	53	0,1406	0,0020
14	0,4718	0,3200	54	0,0106	0,0000
15	0,2701	0,0620	55	0,0396	0,0000
16	0,5790	0,4360	56	0,1729	0,0040
17	0,9088	0,8860	57	0,0557	0,0000
18	0,9331	0,9180	58	0,0404	0,0000
19	0,4874	0,3360	59	0,2460	0,0400
20	0,5430	0,4020	60	0,0598	0,0000
21	0,5530	0,4080	61	0,0970	0,0000
22	0,6695	0,5840	62	0,1549	0,0020
23	0,2982	0,0860	63	0,2841	0,0840
24	0,6426	0,5320	64	0,7861	0,7120
25	0,2766	0,0760	65	0,0255	0,0000
26	0,6797	0,5920	66	0,2930	0,0840
27	0,8883	0,8620	67	0,0250	0,0000
28	0,6291	0,5120	68	0,1058	0,0020
29	0,8109	0,7600	69	0,0791	0,0000
30	0,8326	0,7920	70	0,0479	0,0000
31	0,3610	0,1720	71	0,0860	0,0000
32	0,3766	0,1880	72	0,0996	0,0000
33	0,4111	0,2400	73	0,0762	0,0000
34	0,7803	0,7080	74	0,0559	0,0000
35	0,8633	0,8360	75	0,0845	0,0000
36	0,7307	0,6440	76	0,1920	0,0120
37	0,8883	0,8620	77	0,0325	0,0000
38	0,2242	0,0300	78	0,0541	0,0000
39	0,6697	0,5840	79	0,0526	0,0000
40	0,4763	0,3280	80	0,0386	0,0000

8. Appendix 2

a. The values of $Q_{MD}(F_m, G_t)$ for $t = 4$

Subgroups (k)	$Q_{MD}(F_m, G_t)$ for $t = 4$
1	0,3865
2	0,5315
3	0,2055
4	0,4350
5	0,6355
6	0,4025
7	0,5105
8	0,4780
9	0,6070
10	0,4510
11	0,0005
12	0,0370
13	0,0040
14	0,0015
15	0,0100
16	0,1995
17	0,0215
18	0,0000
19	0,0030
20	0,0000

b. The values of $Q_{MD}(F_m, G_t)$ for $t = 10$

Subgroups (k)	$Q_{MD}(F_m, G_t)$ for $t = 10$
1	0,4302
2	0,4474
3	0,5204
4	0,4592
5	0,0150
6	0,0062
7	0,0884
8	0,0012