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A RELATED FIXED POINT THEOREM FOR THREE METRIC SPACES

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Abstract

A related fixed point theorem for set valued mappings on three complete metric spaces is obtained which generalizes a result of [4].

Key Words: Complete metric space, common fixed point.

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1. Introduction and Preliminaries

Let (X, d) be a complete metric space and let $B(X)$ be the set of all non-empty subsets of X. The function $\delta(A, B)$ with A and B in $B(X)$ is defined by

$$\delta(A, B) = \sup\{d(a, b) : a \in A,\ b \in\ B\}.$$

If A consists of a single point a, we write $\delta(A, B) = \delta(a, B)$ and if B also consists of a single point b, we write $\delta(A, B) = \delta(a, b) = d(a, b)$. It follows immediately from the definition that

$$\begin{aligned} \delta(A, B) &= \delta(B, A) \geq 0, \\ \delta(A, B) &\leq \delta(A, C) + \delta(C, B) \end{aligned}$$

for all A, B, C in $B(X)$. For some preliminary definitions such as the convergence of a sequence of sets in $B(X)$ and the continuity of a mapping F of X into $B(X)$, see [1] or [2]. We need the following lemma in the sequel.

1.1. Lemma. (Fisher [1]). *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of a complete metric space (X, d) which converge to the bounded subsets A and B respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

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Generalizing a result of Fisher [3] from two metric spaces to three metric spaces, the following related fixed point theorem was proved in [4].

1.2. Theorem: *Let (X, d_1), (Y, d_2) and (Z, d_3) be complete metric spaces. If T is a continuous mapping of X into Y, S is a continuous mapping of Y into Z and R is a mapping of Z into X satisfying the inequalities*

$$\begin{aligned} d_1(RSTx, RSTx') &\leq c\max\{d_1(x,x'), d_1(x,RSTx), d_1(x',RSTx'), \\ &\qquad d_2(Tx,Tx'), d_3(STx,STx')\} \\ d_2(TRSy, TRSy') &\leq c\max\{d_2(y,y'), d_2(y,TRSy), d_2(y',TRSy'), \\ &\qquad d_3(Sy,Sy'), d_1(RSy,RSy')\} \\ d_3(STRz, STRz') &\leq c\max\{d_3(z,z'), d_3(z,STRz), d_3(z',STRz'), \\ &\qquad d_1(Rz,Rz'), d_2(TRz,TRz')\} \end{aligned}$$

for all x, x' in X, y, y' in Y and z, z' in Z, where $0 \leq c < 1$, then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further, $Tu = v,$, $Sv = w$ and $Rw = u$.

We now generalize Theorem 1.2 for set valued mappings.

2. Main Result

2.1. Theorem: *Let (X, d_1), (Y, d_2) and (Z, d_3) be complete metric spaces. If F is a continuous mapping of X into $B(Y)$, G is a continuous mapping of Y into $B(Z)$ and H is a mapping of Z into $B(X)$ satisfying the inequalities*

$$\begin{aligned} \delta_1(HGFx, HGFx') &\leq c\max\{d_1(x,x'), \delta_1(x,HGFx), \delta_1(x',HGFx'), \\ &\qquad \delta_2(Fx,Fx'), \delta_3(GFx,GFx')\} \quad (1) \\ \delta_2(FHGy, FHGy') &\leq c\max\{d_2(y,y'), \delta_2(y,FHGy), \delta_2(y',FHGy'), \\ &\qquad \delta_3(Gy,Gy'), \delta_1(HGy,HGy')\} \quad (2) \\ \delta_3(GFHz, GFHz') &\leq c\max\{d_3(z,z'), \delta_3(z,GFHz), \delta_3(z',GFHz'), \\ &\qquad \delta_1(Hz,Hz'), \delta_2(FHz,FHz')\} \quad (3) \end{aligned}$$

for all x, x' in X, y, y' in Y and z, z' in Z, where $0 \leq c < 1$, then HGF has a unique fixed point u in X, FHG has a unique fixed point v in Y and GFH has a unique fixed point w in Z. Further, $Fu = \{v\}$, $Gv = \{w\}$ and $Hw = \{u\}$.

Proof. Let $x = x_1$ be an arbitrary point in X. We define the sequences $\{x_n\}$ in X, $\{y_n\}$ in Y and $\{z_n\}$ in Z inductively as follows. Choose a point y_1 in Fx_1, then a point z_1 in Gy_1 and then a point x_2 in Hz_1. In general, having chosen x_n in X, y_n in Y and z_n in Z, choose a point x_{n+1} in Hz_n, then a point y_{n+1} in Fx_{n+1} and then a point z_{n+1} in Gy_{n+1} for $n = 1, 2, \ldots$. Applying inequality (1), we get

$$\begin{aligned} d_1(x_{n+1}, x_{n+2}) &\leq \delta_1(HGFx_n, HGFx_{n+1}) \\ &\leq c\max\{d_1(x_n,x_{n+1}), \delta_1(x_n,HGFx_n), \\ &\qquad \delta_1(x_{n+1},HGFx_{n+1}), \delta_2(Fx_n,Fx_{n+1}), \delta_3(GFx_n,GFx_{n+1})\} \end{aligned}$$

$$
\begin{aligned}
&\le c\max\{\delta_1(HGFx_{n-1},HGFx_n),\delta_1(HGFx_n,HGFx_{n+1}),\\
&\qquad \delta_2(FHGy_{n-1},FHGy_n),\delta_3(GFHz_{n-1},GFHz_n)\}\\
&\le c\max\{\delta_1(HGFx_{n-1},HGFx_n),\delta_2(FHGy_{n-1},FHGy_n),\\
&\qquad \delta_3(GFHz_{n-1},GFHz_n)\}. \qquad (4)
\end{aligned}
$$

Using inequality (2), we get

$$
\begin{aligned}
d_2(y_{n+1},y_{n+2}) &\le \delta_2(FHGy_n,FHGy_{n+1})\\
&\le c\max\{d_2(y_n,y_{n+1}),\delta_2(y_n,FHGy_n),\delta_2(y_{n+1},FHGy_{n+1}),\\
&\qquad \delta_3(Gy_n,Gy_{n+1}),\delta_1(HGy_n,HGy_{n+1})\}\\
&\le c\max\{\delta_2(FHGy_{n-1},FHGy_n),\delta_2(FHGy_n,FHGy_{n+1}),\\
&\qquad \delta_3(GFHz_{n-1},GFHz_n),\delta_1(HGFx_n,HGFx_{n+1})\}\\
&\le c\max\{\delta_2(FHGy_{n-1},FHGy_n),\delta_3(GFHz_{n-1},GFHz_n),\\
&\qquad \delta_1(HGFx_n,HGFx_{n+1})\}\\
&\le c\max\{\delta_2(FHGy_{n-1},FHGy_n),\delta_3(GFHz_{n-1},GFHz_n),\\
&\qquad \delta_1(HGFx_{n-1},HGFx_n)\} \qquad (5)
\end{aligned}
$$

on using inequality (4). Further, on using inequality (3), we have

$$
\begin{aligned}
d_3(z_{n+1},z_{n+2}) &\le \delta_3(GFHz_n,GFHz_{n+1})\\
&\le c\max\{d_3(z_n,z_{n+1}),\delta_3(z_n,GFHz_n),\delta_3(z_{n+1},GFHz_{n+1}),\\
&\qquad \delta_1(Hz_n,Hz_{n+1}),\delta_2(FHz_n,FHz_{n+1})\}\\
&\le c\max\{\delta_3(GFHz_{n-1},GFHz_n),\delta_3(GFHz_n,GFHz_{n+1}),\\
&\qquad \delta_1(HGFx_n,HGFx_{n+1}),\delta_2(FHGy_n,FHGy_{n+1})\}\\
&\le c\max\{\delta_3(GFHz_{n-1},GFHz_n),\delta_1(HGFx_{n-1},HGFx_n),\\
&\qquad \delta_2(FHGy_{n-1},FHGy_n)\} \qquad (6)
\end{aligned}
$$

on using inequalities (4) and (5). It now follows easily by induction on using inequalities (4), (5) and (6) that

$$
\begin{aligned}
d_1(x_{n+1},x_{n+2}) &\le c^{n-1}\max\{\delta_1(HGFx_1,HGFx_2),\delta_2(FHGy_1,FHGy_2),\\
&\qquad \delta_3(GFHz_1,GFHz_2)\}\\
d_2(y_{n+1},y_{n+2}) &\le c^{n-1}\max\{\delta_1(HGFx_1,HGFx_2),\delta_2(FHGy_1,FHGy_2),\\
&\qquad \delta_3(GFHz_1,GFHz_2)\}\\
d_3(z_{n+1},z_{n+2}) &\le c^{n-1}\max\{\delta_1(HGFx_1,HGFx_2),\delta_2(FHGy_1,FHGy_2),\\
&\qquad \delta_3(GFHz_1,GFHz_2)\}.
\end{aligned}
$$

Then, for $r = 1, 2, \ldots$ and arbitrary $\epsilon > 0$, we have

$$
\begin{aligned}
d_1(x_{n+1},x_{n+r+1}) &\le \delta_1(HGFx_n,HGFx_{n+r})\\
&\le \delta_1(HGFx_n,HGFx_{n+1})+\cdots+\delta_1(HGFx_{n+r-1},HGFx_{n+r})\\
&\le (c^{n-1}+c^n+\ldots+c^{n+r-2})\times\\
&\qquad \max\{\delta_1(HGFx_1,HGFx_2),\delta_2(FHGy_1,FHGy_2),\delta_3(GFHz_1,GFHz_2)\}\\
&< \epsilon \qquad (7)
\end{aligned}
$$

for n greater than some N, since $c < 1$. The sequence $\{x_n\}$ is therefore a Cauchy sequence in the complete metric space X and so has a limit u in X. Similarly, the sequences $\{y_n\}$ and $\{z_n\}$ are also Cauchy sequences with limits v and w in the complete metric spaces Y and Z respectively. Further, inequality (7) gives

$$\begin{aligned}\delta_1(u, HGFx_n) &\leq d_1(u, x_{m+1}) + \delta_1(x_{m+1}, HGFx_n) \\ &\leq d_1(u, x_{m+1}) + \delta_1(HGFx_m, HGFx_n) \\ &\leq d_1(u, x_{m+1}) + \epsilon\end{aligned}$$

for $m, n \geq N$. Letting m tend to infinity, it follows that

$$\delta_1(u, HGFx_n) \leq \epsilon,$$

for $n > N$ and so

$$\lim_{n\to\infty} HGFx_n = \{u\} = \lim_{n\to\infty} HGy_n, \tag{8}$$

since ϵ is arbitrary. Similarly,

$$\lim_{n\to\infty} FHGy_n = \{v\} = \lim_{n\to\infty} FHz_n, \tag{9}$$

$$\lim_{n\to\infty} GFHz_n = \{w\} = \lim_{n\to\infty} GFx_n. \tag{10}$$

Using the continuity of F and G, we obtain

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Fx_n = Fu = \{v\}, \tag{11}$$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} Gy_n = Gv = \{w\} \tag{12}$$

and then we see that

$$GFu = \{w\}. \tag{13}$$

We now show that u is a fixed point of HGF. Applying inequality (1), we have

$$\begin{aligned}\delta_1(HGFu, x_n) &\leq \delta_1(HGFu, HGFx_{n-1}) \\ &\leq c\max\{d_1(u, x_{n-1}), \delta_1(u, HGFu), \delta_1(x_{n-1}, HGFx_{n-1}), \\ &\qquad \delta_2(Fu, Fx_{n-1}), \delta_3(GFu, GFx_{n-1})\}.\end{aligned}$$

Since F and G are continuous, it follows on letting n tend to infinity and using equation (8) that

$$\delta_1(HGFu, u) \leq c\delta_1(u, HGFu).$$

Since $c < 1$, we have $HGFu = \{u\}$ and so u is a fixed point of HGF. We now have

$$HGFu = Hw = \{u\}$$

on using equation (13) and then

$$\begin{aligned}FHGv &= FHGFu = Fu = \{v\}, \\ GFHw &= GFHGv = Gv = \{w\}\end{aligned}$$

on using equations (11) and (12). Hence, v and w are fixed points of FHG and GFH respectively. Further, we see that

$$HGv = \{u\}, \quad FHw = \{v\}.$$

To prove the uniqueness of u, suppose that HGF has a second fixed point u'. Then, using inequality (1), we have

$$\begin{aligned}\delta_1(u', HGFu') &\leq \delta_1(HGFu', HGFu') \\ &\leq c\max\{d_1(u', u'), \delta_1(u', HGFu'), \delta_2(Fu', Fu'), \\ &\qquad \delta_3(GFu', GFu')\} \\ &\leq c\max\{\delta_2(Fu', Fu'), \delta_3(GFu', GFu')\}. \end{aligned} \tag{14}$$

Next, using inequality (2), we have

$$\begin{aligned}\delta_2(Fu', Fu') &\leq \delta_2(FHGFu', FHGFu') \\ &\leq c\max\{d_2(Fu', Fu'), \delta_2(Fu', FHGFu'), \\ &\qquad \delta_3(GFu', GFu'), \delta_1(HGFu', HGFu')\} \\ &\leq c\max\{\delta_3(GFu', GFu'), \delta_1(HGFu', HGFu')\} \end{aligned} \tag{15}$$

and using inequality (3), we have

$$\begin{aligned}\delta_3(GFu', GFu') &\leq \delta_3(GFHGFu', GFHGFu') \\ &\leq c\max\{d_3(GFu', GFu'), \delta_3(GFu', GFHGFu'), \\ &\qquad \delta_1(HGFu', HGFu'), \delta_2(FHGFu', FHGFu')\} \\ &\leq c\max\{\delta_1(HGFu', HGFu'), \delta_2(FHGFu', FHGFu')\}. \end{aligned} \tag{16}$$

It now follows easily from inequalities (15) and (16) that

$$\delta_2(FHGFu', FHGFu') \leq c\delta_1(HGFu', HGFu') \tag{17}$$

and then

$$\delta_3(GFu', GFu') \leq c\delta_1(HGFu', HGFu'). \tag{18}$$

Using inequalities (14), (17) and (18), we now have

$$\delta_1(u', HGFu') \leq \delta_1(HGFu', HGFu') \leq c^2\delta_1(HGFu', HFGu')$$

and so $HGFu'$ is a singleton and $HGFu' = \{u'\}$, since $c < 1$. It then follows from inequality (18) that GFu' is a singleton and from inequality (17) that Fu' is a singleton. Using inequality (1) again, we have

$$\begin{aligned}d_1(u, u') &= \delta_1(HGFu, HGFu') \\ &\leq c\max\{d_1(u, u'), d_1(u, u), d_1(u', u'), d_2(Fu, Fu'), \\ &\qquad d_3(GFu, GFu')\} \\ &= c\max\{d_2(Fu, Fu'), d_3(GFu, GFu')\}. \end{aligned} \tag{19}$$

Next, using inequality (2), we have

$$\begin{aligned} d_2(Fu, Fu') &= \delta_2(FHGFu, FHGFu') \\ &\leq c \max\{d_2(Fu, Fu'), d_2(Fu, FHGFu), d_2(Fu', FHGFu'), \\ &\qquad d_3(GFu, GFu'), d_1(HGFu, HGFu')\} \\ &= c \max\{d_3(GFu, GFu'), d_1(u, u')\} \end{aligned} \tag{20}$$

and using inequality (3), we have

$$\begin{aligned} d_3(GFu, GFu') &= d_3(GFHGFu, GFHGFu') \\ &\leq c \max\{d_3(GFu, GFu'), d_3(GFu, GFHGFu), \\ &\qquad d_3(GFu', GFHGFu'), d_1(HGFu, HGFu'), \\ &\qquad d_2(FHGFu, FHGFu')\} \\ &= c \max\{d_1(u, u'), d_2(Fu, Fu')\}. \end{aligned} \tag{21}$$

It now follows from inequalities (19), (20) and (21) that $d_1(u, u') = 0$, proving the uniqueness of u. The uniqueness of v and w follow similarly.

2.2. Remark. If we let F be a single valued mapping T of X into Y, G be a single valued mapping S of Y into Z and H be a single valued mapping R of Z into X, we obtain Theorem 1.2.

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