Hacettepe Journal of Mathematics and Statistics
Volume 34 (2005), 17 – 25

ON L-FUZZY PRIME SUBMODULES

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Received 06 : 01 : 2005 : Accepted 16 : 12 : 2005

Abstract

In this paper the concept of an L-fuzzy prime submodule of M is given, and some fundamental lemmas are proved. Also a characterization of an L-fuzzy prime submodule is given. Finally, we show that an L-fuzzy prime submodule is inherited by an R-module epimorphism.

Keywords: L-Fuzzy submodule, L-Fuzzy prime submodule.

2000 AMS Classification: 08A72, 03E72.

1. Introduction

Zadeh in [6] introduced the notion of a fuzzy subset μ of a non-empty set X as a function from X to $[0, 1]$. Goguen in [1] generalized the notion of fuzzy subset of X to that of an L-fuzzy subset, namely a function from X to a lattice L.

In [5], Rosenfeld considered the fuzzification of algebraic structures. Liu [2], introduced and examined the notion of a fuzzy ideal of a ring. Since then several authors have obtained interesting results on L-fuzzy ideals of R and L-fuzzy modules. See [4] for a comprehensive survey of the literature on these developments.

In [3] the notion of fuzzy prime submodule of M over $[0, 1]$ is given in terms of fuzzy singletons. In Section 3 of this paper, we generalize their definition to any complete lattice L when R is a commutative ring with identity. In Theorem 3.6 we give a characterization of L-fuzzy prime submodules which is one of the original results obtained in this paper. In Section 4, we investigate the behaviour of L-fuzzy prime submodules under R-module homomorphisms, which constitutes another original result of our work.

2. Preliminaries

Throughout this paper R is a commutative ring with identity, M a unitary R-module and L stands for a complete lattice with least element 0 and greatest element 1. 0_M denotes the zero element of M.

An element $\alpha \in L$, $1 \neq \alpha$, is called a *prime element* in L if for all $a, b \in L$ if $a \wedge b \leq \alpha$ implies $a \leq \alpha$ or $b \leq \alpha$.

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Given a nonempty set X, an L-fuzzy subset μ is a function from X to L. We denote by $F(X)$ the set of all L-fuzzy subsets of X. For $\mu, \nu \in F(X)$ we say $\mu \subseteq \nu$ if and only if $\mu(x) \leq \nu(x)$, for all $x \in X$. Also, $\mu \subset \nu$ if and only if $\mu \subseteq \nu$ and $\mu \neq \nu$.

Let $\mu \in F(X)$ and $t \in L$. Then the set $\mu_t = \{x \in X \mid \mu(x) \geq t\}$ is called the *level subset of X with respect to μ*. By an *L-fuzzy point x_r* of X, $x \in X$, $r \in L \setminus \{0\}$, we mean $x_r \in F(X)$ defined by

$$x_r(y) = \begin{cases} r & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

If x_r is an L-fuzzy point of X and $x_r \subseteq \mu \in F(X)$, we write $x_r \in \mu$. For $A \subseteq X$ the characteristic function of A, $\chi_A \in F(X)$, is defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

The following are two very basic definitions given [4].

2.1. Definition.

a) Let $\xi \in F(R)$. Then ξ is called an *L-fuzzy ideal of R* if for all $x, y \in R$,
   (i) $\xi(x - y) \geq \xi(x) \wedge \xi(y)$,
   (ii) $\xi(xy) \geq \xi(x) \vee \xi(y)$.

b) Let $\mu \in F(M)$. Then μ is called an *L-fuzzy R-module of M* if for all $x, y \in M$ and for all $r \in R$,
   (i) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$,
   (ii) $\mu(rx) \geq \mu(x)$,
   (iii) $\mu(0_M) = 1$

Let $S(M)$ denote the set of all L-fuzzy R-modules of M and $I(R)$ the set of all L-fuzzy ideals of R. We note that when $R = M$, then $\mu \in S(M)$ if and only if $\mu(0_M) = 1$ and $\mu \in I(R)$.

An example of an L-fuzzy R-module M with $R = \mathbb{Z}$, $M = \mathbb{Z}_6$, is

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{3} & \text{if } x = 2, 4, \\ \frac{1}{4} & \text{if } x = 1, 3, 5. \end{cases}$$

The following are two basic operations which will be used to define an L-fuzzy prime submodule.

2.2. Definition. Let $\xi \in F(R)$ and $\mu \in F(M)$. Define the composition $\xi \circ \mu$, and product $\xi\mu$ respectively as follows: For all $w \in M$,

$$(\xi \circ \mu)(w) = \sup\{\xi(r) \wedge \mu(x) \mid r \in R,\ x \in M,\ w = rx\},$$

$$(\xi\mu)(w) = \sup\left\{ \inf_{i=1}^{n}\{\xi(r_i) \wedge \mu(x_i)\} \;\middle|\; r_i \in R,\ x_i \in M,\ n \in \mathbb{N},\ w = \sum_{i=1}^{n} r_i x_i \right\},$$

where as usual the supremum of an empty set is taken to be 0.

The product can be also expressed as

$$(\xi\mu)(w) = \sup\{\xi(r_1) \wedge \xi(r_2) \wedge \cdots \wedge \xi(r_n) \wedge \mu(x_1) \wedge \mu(x_2) \wedge \cdots \wedge \mu(x_n) \mid r_i \in R,\ x_i \in M,\ n \in \mathbb{N},\ w = \sum_{i=1}^{n} r_i x_i\}$$
$$= \bigvee \left\{ \bigwedge_{i=1}^{n} \{\xi(r_i) \wedge \mu(x_i)\} \,\middle|\, r_i \in R,\ x_i \in M,\ n \in \mathbb{N},\ w = \sum_{i=1}^{n} r_i x_i \right\}$$

Notice that $\xi \circ \mu$ is the case $n = 1$ in the definition of $\xi\mu$. Thus $\xi \circ \mu \subseteq \xi\mu$.

To give an example of the product of $\xi \in F(R)$ and $\mu \in F(M)$ with $R = \mathbb{Z}$ and $M = \mathbb{Z}_6$, let $\xi(r) = \begin{cases} \frac{1}{2} & \text{if } r \in 2\mathbb{Z}, \\ \frac{1}{5} & \text{otherwise} \end{cases}$ and $\mu(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{3} & \text{if } x = 2, 4, \\ \frac{1}{4} & \text{if } x = 1, 3, 5. \end{cases}$ Then:

$$\begin{aligned}
(\xi\mu)(0) &= \sup\{\underbrace{\xi(0) \wedge \mu(1)}_{0=0.1},\ \underbrace{\xi(2) \wedge \mu(3)}_{0=2.3},\ \underbrace{\xi(2) \wedge \mu(1) \wedge \xi(-1) \wedge \mu(2)}_{0=2.1-1.2}, \ldots\} \\
&= \sup\{\frac{1}{2} \wedge 1,\ \frac{1}{2} \wedge \frac{1}{4},\ \frac{1}{2} \wedge \frac{1}{4} \wedge \frac{1}{5}, \ldots\} = \frac{1}{2}, \\
(\xi\mu)(1) &= \sup\{\underbrace{\xi(1) \wedge \mu(1)}_{1=1.1},\ \underbrace{\xi(7) \wedge \mu(1)}_{1=7.1},\ \underbrace{\xi(2) \wedge \mu(2) \wedge \xi(-1) \wedge \mu(3)}_{1=2.2-1.3}, \ldots\} \\
&= \sup\{\frac{1}{4} \wedge \frac{1}{5},\ \frac{1}{5} \wedge \frac{1}{4},\ \frac{1}{3} \wedge \frac{1}{2} \wedge \frac{1}{5} \wedge \frac{1}{4}, \ldots\} = \frac{1}{5}, \\
(\xi\mu)(2) &= \sup\{\underbrace{\xi(2) \wedge \mu(1)}_{2=2.1},\ \underbrace{\xi(2) \wedge \mu(4)}_{2=2.4},\ \underbrace{\xi(2) \wedge \mu(2) \wedge \xi(-1) \wedge \mu(2)}_{2=2.2-1.2}, \ldots\} \\
&= \sup\{\frac{1}{2} \wedge 1,\ \frac{1}{2} \wedge \frac{1}{3},\ \frac{1}{2} \wedge \frac{1}{4} \wedge \frac{1}{5}, \ldots\} = \frac{1}{3}.
\end{aligned}$$

If we continue in this way we obtain $(\xi\mu)(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{3} & \text{if } x = 2, 4 \\ \frac{1}{5} & \text{if } x = 1, 3, 5. \end{cases}$

The following lemma can be found in [7,8], It gives the basic operations between L-fuzzy ideals and L-fuzzy modules where L is a complete lattice satisfying the infinite distributive law (completely distributive in the sense of Goguen).

2.3. Lemma. *Let $\xi \in I(R)$, $\nu, \mu \in S(M)$ and let L be a complete lattice satisfying the infinite distributive law. Then:*

1) *$\xi\mu \subseteq \nu$ if and only if $\xi \circ \mu \subseteq \nu$.*
2) *Let $r_t \in F(R)$, $x_s \in F(M)$ be fuzzy points. Then $r_t \circ x_s = r_t x_s = (rx)_{t \wedge s}$.*
3) *If $\xi(0_R) = 1$ then $\xi\nu \in S(M)$.*
4) *Let $r_t \in F(R)$ be a fuzzy point. Then for all $w \in M$,*

$$(r_t \circ \mu)(w) = \begin{cases} t \wedge \sup\{\mu(x) \mid x \in M,\ w = rx\} & \exists\, x \in M \text{ with } w = rx, \\ 0 & \text{otherwise.} \end{cases}$$

We give an example with $R = \mathbb{Z}$ and $M = \mathbb{Z}_6$ to illustrate $r_t \circ \mu$.

Let $2_{\frac{1}{2}} \in R$ and $\mu(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{3} & \text{if } x = 2, 4, \\ \frac{1}{4} & \text{if } x = 1, 3, 5. \end{cases}$ Then $\mu \in S(M)$ and

$$(2_{\frac{1}{2}} \circ \mu)(2) = \sup\{\underbrace{2_{\frac{1}{2}}(2) \wedge \mu(1)}_{2=2.1},\ \underbrace{2_{\frac{1}{2}}(2) \wedge \mu(4)}_{2=2.4},\ \underbrace{2_{\frac{1}{2}}(1) \wedge \mu(2)}_{2=1.2},\ \underbrace{2_{\frac{1}{2}}(5) \wedge \mu(4)}_{2=5.4} \ldots\}$$

$$= \sup\{\frac{1}{2} \wedge \frac{1}{4},\ \frac{1}{2} \wedge \frac{1}{3},\ 0 \wedge \frac{1}{3},\ 0 \wedge \frac{1}{3}, \ldots\} = \frac{1}{3}.$$

Thus $(2_{\frac{1}{2}} \circ \mu)(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0, \\ \frac{1}{3} & \text{if } x = 2, 4, \\ 0 & \text{if } x = 1, 3, 5 \end{cases}$ and $(2_{\frac{1}{2}} \circ \mu)(x) \notin S(M)$ since $(2_{\frac{1}{2}} \circ \mu)(0) \neq 1$.

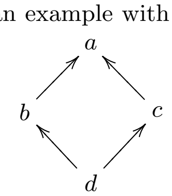
The following theorem gives a relation between L-fuzzy modules on M and submodules of M. It is a very practical method to construct an L-fuzzy module on M.

2.4. Theorem. [8] *Let $\mu \in F(M)$. Then μ is an L-fuzzy module if and only if for all $t \in L$ such that $\mu_t \neq \emptyset$, μ_t is an R-submodule of M.*

2.5. Definition. [4] For a non-constant $\xi \in I(R)$, ξ is called an *L-fuzzy prime ideal of R* if for any L-fuzzy points $x_r, y_s \in F(R)$,

$x_r y_s \in \xi$ implies that either $x_r \in \xi$ or $y_s \in \xi$.

We give an example with $R = \mathbb{Z}_3$ and $L = \{a, b, c, d\}$ where the ordering is given by the diagram:



Then $\xi(x) = \begin{cases} a & \text{if } x = 0, \\ c & \text{if } x = 1, 2 \end{cases}$ is an L-fuzzy prime ideal in R.

3. L-Fuzzy Prime Submodules

In this section, we will give a characterization of an L-fuzzy prime submodule of M.

3.1. Definition. [8] For $\mu, \nu \in S(M)$, ν is called an L-fuzzy submodule of μ if and only if $\nu \subset \mu$. In particular, if $\mu = \chi_M$, then we say ν is an L-fuzzy submodule of M.

3.2. Definition. Let ν be an L-fuzzy submodule of μ. ν is called an *L-fuzzy prime submodule of μ* if for $r_t \in F(R)$, $x_s \in F(M)$ $(r \in R,\ x \in M,\ s, t \in L)$,

$r_t x_s \in \nu$ implies that either $x_s \in \nu$ or $r_t \mu \subseteq \nu$.

In particular, taking $\mu = \chi_M$, if for $r_t \in F(R)$, $x_s \in F(M)$ we have

$r_t x_s \in \nu$ implies that either $x_s \in \nu$ or $r_t \chi_M \subseteq \nu$,

then ν is called an *L-fuzzy prime submodule* of M.

The following theorem says that L-fuzzy prime submodules and L-fuzzy prime ideals coincide when R is considered to be a module over itself.

3.3. Theorem. *If $M = R$, then $\nu \in F(R)$ is an L-fuzzy prime submodule of M if and only if $\nu \in F(R)$ is an L-fuzzy prime ideal.*

Proof. Let ν be an L-fuzzy prime submodule of M. Since $\nu \in S(M)$ and R is a commutative ring, $\nu \in I(R)$.

For arbitrary $a_s, b_t \in F(R)$, $a_s b_t \in \nu$ implies $a_s \in \nu$ or $b_t \chi_M \subseteq \nu$.

If $a_s \in \nu$, then ν is an L-fuzzy prime ideal.

If $b_t \chi_M \subseteq \nu$, then $b_t \chi_M(bm) \leq \nu(bm), \forall\, m \in M$. Since R has an identity $b = b1$, and $b_t \chi_M(b1) = t \leq \nu(b)$ implies that $t = b_t(b) \leq \nu(b)$, hence $b_t \in \nu$.

Conversely, let ν be an L-fuzzy prime ideal of R. Then $\nu \subset \chi_R$ and $\nu \in S(M)$. Now let $r_t x_s \in \nu$, for any $r_t \in F(R),\ x_s \in F(M)$.

If $x_s \in \nu$, then ν is an L-fuzzy prime submodule of M.

If $x_s \notin \nu$ then $r_t \in \nu \implies r_t \chi_M(rm) = t \leq \nu(r) \leq \nu(rm)$ by the definition of L-fuzzy ideal of R. Thus, $r_t \chi_M \subseteq \nu$. □

The following theorem, which relates fuzzy prime submodule to prime submodules of the module, will be needed in the proof of Theorem 3.6.

3.4. Theorem. *Let ν be an L-fuzzy prime submodule of μ. If $\nu_t \neq \mu_t$, $t \in L$, then ν_t is a prime submodule of μ_t.*

Proof. Let $\nu_t \neq \mu_t$ and $rx \in \nu_t$ for some $r \in R,\ x \in M$. If $rx \in \nu_t$, then $\nu(rx) \geq t \implies (rx)_t = r_t x_t \in \nu$, and since ν is an L-fuzzy prime submodule of μ, either $x_t \in \nu$ or $r_t \mu \subseteq \nu$.

case1: If $x_t \in \nu$ then $t \leq \nu(x)$, so $x \in \nu_t$.

case2: Let $r_t \mu \subseteq \nu$. Then for any $w \in r\mu_t$, $w = rz$, for some $z \in \mu_t$. So $\mu(z) \geq t$, and

$$t = t \wedge \mu(z) \leq \sup_{w=rx} \{t \wedge \mu(x)\} = r_t \mu(w) \leq \nu(w).$$

Thus $t \leq \nu(w)$, that is $w \in \nu_t$. Thereby $r\mu_t \subseteq \nu_t$. □

3.5. Corollary. *Let ν be an L-fuzzy prime submodule of M. Then*

$$\nu_* = \{x \in M \mid \nu(x) = \nu(0_M)\}$$

is a prime submodule of M.

Proof. Clear from Theorem 3.4. □

The following theorem is the main result of section 3. It generalizes the work in [3] from $[0,1]$ to a complete lattice L.

3.6. Theorem.

a) *Let N be a prime submodule of M and α a prime element in L. If μ is the fuzzy subset of M defined by*

$$\mu(x) = \begin{cases} 1 & \text{if } x \in N, \\ \alpha & \text{otherwise} \end{cases}$$

for all $x \in M$, then μ is an L-fuzzy prime submodule of M.

b) *Conversely, any L-fuzzy prime submodule can be obtained as in (a).*

Proof. a) Since N is a prime submodule of M, $N \neq M$, we have that μ is a non-constant L-fuzzy submodule of M. We show that μ is an L-fuzzy prime submodule of M.

Suppose $r_t \in F(R),\ x_s \in F(M)$ are such that $r_t x_s \in \mu$ and $x_s \notin \mu$.

If $x_s \notin \mu$ then $\mu(x) = \alpha$, hence $x \notin N$.

If $r_t x_s \in \mu$, then $(rx)_{t \wedge s}(rx) \leq \mu(rx) \implies t \wedge s \leq \mu(rx)$.

If $\mu(rx) = 1$, then $rx \in N$. Since $x \notin N$ and N is a prime submodule of M, we have $rM \subseteq N$. Hence $\mu(rm) = 1$, for all $m \in M$. Thus $r_t\chi_M(rm) = t \leq \mu(rm)$.

If $\mu(rx) = \alpha$, then $(t \wedge s) \leq \alpha$ and $s \nleq \alpha$ implies $t \leq \alpha$ because α is a prime element in L. So $r_t\chi_M(w) = t \leq \alpha \leq \mu(w)$, for all $w \in M$.

b) Let μ be an L-fuzzy prime submodule of M. We show that μ is of the form

$$\mu(x) = \begin{cases} 1 & \text{if } x \in N, \\ \alpha & \text{otherwise} \end{cases}$$

for a prime submodule N of M and for a prime α element in L.

Claim 1. $\mu_* = \{x \in M \mid \mu(x) = \mu(0_M)\}$ is a prime submodule of M.

Since μ is a nonconstant L-fuzzy prime submodule of M, $\mu_* \neq M$.

For all $r \in R, m \in M$, if $rm \in \mu_*$ implies that $(rm)_{\mu(0_M)} = r_{\mu(0_M)}m_{\mu(0_M)} \in \mu$, then $m_{\mu(0_M)} \in \mu$ or $r_{\mu(0_M)}\chi_M \subseteq \mu$.

Case 1: If $m_{\mu(0_M)} \in \mu$, then $\mu(0_M) \leq \mu(m)$ and $\mu(0_M) \geq \mu(m)$ (definition of fuzzy module). Hence $\mu(0_M) = \mu(m)$, so $m \in \mu_*$

Case 2: If $r_{\mu(0_M)}\chi_M \subseteq \mu$, then $\mu(0_M) \leq \mu(rm)$, thus $rm \in \mu_*$ for all $m \in M$.

$0_M \in N$ and $\mu(0_M) = 1$. For all $x \in \mu_*$, $\mu(0_M) = \mu(x) = 1$. Now, $\mu_* = N$.

Claim 2. μ has only two values.

Since μ_* is a prime submodule of M, $\mu_* \neq M$. Then there exists $z \in M \setminus \mu_*$. We will show that $\mu(y) = \mu(z) < \mu(0_M)$, for all $y \in M$ such that $y \notin \mu_*$. Then

$$z \notin \mu_* \implies \mu(z) < 1 = \mu(0_M),$$

so $z_1 \notin \mu$ and $z_{\mu(z)} = z_1 1_{\mu(z)} \in \mu$. Thus $1_{\mu(z)}\chi_M \subseteq \mu$, since $w = 1w$, for all $w \in M$, we have $\mu(z) \leq \mu(w)$.

Let $w = y$. Then, $\mu(z) \leq \mu(y)$. Similarly, $\mu(y) \leq \mu(z)$. Hence, $\mu(z) = \mu(y)$.

Claim 3. Let $\mu(z) = \alpha$, then α is a prime element in L.

First, let $t \wedge s \leq \alpha$ and $s \nleq \alpha$. Suppose $x \in M \setminus \mu_*$. Then $x_s \notin \mu$. Hence

$$1_t x_s = x_{t \wedge s} \in \mu \implies 1_t\chi_M \subseteq \mu,$$

and for all $w \in M$, $1_t\chi_M(w) \leq \mu(w)$. Let $w = x$. Then, $t = 1_t\chi_M(x) \leq \mu(x) = \alpha$.

Thus, every L-fuzzy prime submodule of M is of the form

$$\mu(x) = \begin{cases} 1 & \text{if } x \in N, \\ \alpha & \text{otherwise,} \end{cases}$$

where N is a prime submodule of M and α is a prime element in L. □

This theorem is particularly useful in deciding whether or not a fuzzy submodule is prime. The following example illustrates this.

3.7. Example. Let $M = \mathbb{Z}$ be a module over $R = \mathbb{Z}$. Then

$$\mu(x) = \begin{cases} 1 & \text{if } x \in 3\mathbb{Z}, \\ \frac{1}{4} & \text{otherwise} \end{cases}$$

is an L-fuzzy prime submodule of $\mathbb{Z}$ since $3\mathbb{Z}$ is prime submodule of $\mathbb{Z}$ and $\frac{1}{4}$ is a prime element in $[0, 1]$.

4. L-Fuzzy Prime Submodules of Homomorphic Modules

In this section, we investigate the behaviour of L-fuzzy prime submodules of M under an R-module epimorphism. Firstly, we recall the definition of image and inverse image of an L-fuzzy subset under a R-module homomorphism. From now on, M and M_1 are R-modules.

4.1. Definition. Let f be a R-module homomorphism from M to M_1, $\mu \in F(M)$ and $\nu \in F(M_1)$. Then $f(\mu) \in F(M_1)$ and $f^{-1}(\nu) \in F(M)$ are defined by

$$f(\mu)(w) = \begin{cases} \sup\limits_{m \in f^{-1}(w)} \mu(m) & \text{if } f^{-1}(w) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

and $f^{-1}(\nu)(m) = \nu(f(m))$, for all $w \in M_1,\ m \in M$.

In the next two theorems we show that both the image and the inverse image of an L-fuzzy prime submodule under a R-module epimorphism are again L-fuzzy prime submodules. Here we need to assume that the complete lattice L is distributive.

4.2. Theorem. *Let f be an R-modules epimorphism from M to M_1, and suppose that L is distributive. If μ is an L-fuzzy prime submodule of M such that $\chi_{\ker f} \subseteq \mu$, then $f(\mu)$ is an L-fuzzy prime submodule of M_1.*

Proof. We have $f(\mu)(w) = \sup\limits_{w=f(m)} \mu(m)$.

Claim 1: $f(\mu)$ is an L-fuzzy submodule of M_1.

(i) For all $\omega_1, \omega_2 \in M_1$,

$$\begin{aligned} f(\mu)(\omega_1) \wedge f(\mu)(\omega_2) &= [\sup_{\omega_1=f(m_1)} \mu(m_1)] \wedge [\sup_{\omega_2=f(m_2)} \mu(m_2)] \\ &= \sup_{\omega_1=f(m_1),\omega_2=f(m_2)} \{\mu(m_1) \wedge \mu(m_2)\} \\ &\leq \sup_{\omega_1=f(m_1),\omega_2=f(m_2)} \mu(m_1 - m_2) \\ &\leq \sup_{\omega_1-\omega_2=f(m_1-m_2)} \mu(m_1 - m_2) = f(\mu)(\omega_1 - \omega_2). \end{aligned}$$

(ii) For all $\omega_1 \in M_1$ and for all $r \in R$,

$$\begin{aligned} f(\mu)(\omega_1) &= \sup_{\omega_1=f(m)} \mu(m) \leq \sup_{\omega_1=f(m)} \mu(rm) = \sup_{r\omega_1=rf(m)=f(rm)} \mu(rm) \\ &= f(\mu)(r\omega_1). \end{aligned}$$

(iii) It is clear that $f(\mu)(0_{M_1}) = 1$. Thus $f(\mu)$ is an L-fuzzy submodule of M_1.

Claim 2: $f(\mu)$ is an L-fuzzy prime submodule of M_1.

Since μ is an L-fuzzy prime submodule of M, μ is of the form

$$\mu(x) = \begin{cases} 1 & \text{if } x \in N, \\ \alpha & \text{otherwise,} \end{cases}$$

where $N = \mu_*$ is a prime submodule of M and α is a prime element in L.

Subclaim: If μ_* is a prime submodule of M and $\chi_{\ker f} \subseteq \mu$, then $f(\mu_*)$ is a prime submodule of M_1.

Let $x \in \ker f$. Then

$$\chi_{\ker f}(x) = 1 \leq \mu(x) \implies \mu(x) = \mu(0_M) \implies x \in \mu_*.$$

Thus $\ker f \subseteq \mu_*$.

For all $r \in R,\ \omega \in M_1,\ r\omega \in f(\mu_*)$, there exists $z \in \mu_*$ such that $r\omega = f(z)$. Since f is an epimorphism there exists $m \in M$ such that $r\omega = rf(m) = f(z)$. Now $rm \in \mu_*$, and μ_* is a prime submodule of M, so either $m \in \mu_*$ or $rM \subseteq \mu_*$.

If $m \in \mu_*$, then $\omega = f(m) \in f(\mu_*)$.

If $rM \subseteq \mu_*$, then $rM_1 = f(rM) \subseteq f(\mu_*)$. Thus $f(\mu_*)$ is an L-fuzzy prime submodule of M_1, and α is a prime element in L, so by Theorem 3.6, for all $\omega \in M_1$,

$$f(\mu)(\omega) = \begin{cases} 1 & \text{if } \omega \in f(\mu_*), \\ \alpha & \text{otherwise.} \end{cases}$$

Hence $f(\mu)$ is an L-fuzzy prime submodule of M_1. □

4.3. Example. Let f be a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $f(x) = 2x$, and let

$$\mu(x) = \begin{cases} 1 & \text{if } x \in 3\mathbb{Z}, \\ \frac{1}{4} & \text{otherwise} \end{cases}$$

be an L-fuzzy prime submodule of $\mathbb{Z}$. Then:

$$\begin{aligned} &f(\mu)(0) = \sup\{\mu(n) \mid f(n) = 0\} = \mu(0) = 1, \\ &f(\mu)(1) = 0, \text{ since } f^{-1}(1) = \emptyset, \\ &f(\mu)(2) = \sup\{\mu(n) \mid f(n) = 2\} = \mu(1) = \tfrac{1}{4}, \\ &f(\mu)(3) = 0, \text{ since } f^{-1}(3) = \emptyset, \\ &f(\mu)(4) = \sup\{\mu(n) \mid f(n) = 4\} = \mu(2) = \tfrac{1}{4}, \\ &f(\mu)(5) = 0, \text{ since } f^{-1}(5) = \emptyset. \end{aligned}$$

If we continue this way we find that

$$f(\mu)(x) = \begin{cases} 1 & \text{if } x \in 6\mathbb{Z}, \\ \frac{1}{4} & \text{if } 0 \neq x \in 2\mathbb{Z} - 6\mathbb{Z}, \\ 0 & \text{if } 0 \neq x \in \mathbb{Z} - 2\mathbb{Z}, \end{cases}$$

is not an L-fuzzy prime submodule of $\mathbb{Z}$. This shows that the assumption that f be an epimorphism in Theorem 4.2 cannot be dropped.

4.4. Theorem. *Let f be a R-module epimorphism from M to M_1. If ν is an L-fuzzy prime submodule of M_1, then $f^{-1}(\nu)$ is anL-fuzzy prime submodule of M.*

Proof. Let ν be an L-fuzzy prime submodule of M_1. Then

$$\nu(x) = \begin{cases} 1 & \text{if } x \in \nu_*, \\ \alpha & \text{otherwise,} \end{cases}$$

where ν_* is a prime submodule of M_1 and α is a prime element in L.

Claim: $f^{-1}(\nu_*)$ is a prime submodule of M.

For all $r \in R$, $m \in M$, if

$$rm \in f^{-1}(\nu_*) \implies rf(m) \in \nu_*,$$

then $f(m) \in \nu_*$ or $rM_1 \subseteq \nu_*$.

If $f(m) \in \nu_*$, then $m \in f^{-1}(\nu_*)$.

If $rM_1 \subseteq \nu_*$, then

$$rf(M) = f(rm) \subseteq \nu_* \implies rM \subseteq f^{-1}(\nu_*).$$

Hence

$$f^{-1}(\nu)(x) = \begin{cases} 1 & \text{if } f(x) \in \nu_*, \\ \alpha & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } x \in f^{-1}(\nu_*), \\ \alpha & \text{otherwise,} \end{cases}$$

where $f^{-1}(\nu_*)$ is a prime submodule of M and α a prime element in L.

Thus, $f^{-1}(\nu)$ is an L-fuzzy prime submodule of M. □

Acknowledgement The author is grateful to the referee for several helpful suggestions.

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