Hacettepe Journal of Mathematics and Statistics
Volume 34 (2005), 33 – 38

UNIVERSAL MODULES ON $\mathbf{R} \otimes_{\mathbf{k}} \mathbf{S}$

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Received 24 : 05 : 2005 : Accepted 16 : 11 : 2005

Abstract

In this work we are basically interested in some structure related to the universal modules of high order derivations introduced and developed by H. Osborn.

Specifically, we have investigated universal modules on $R \otimes_k S$ and estimated the homological dimension of $\Omega_n(R \otimes_k S)$.

Keywords: Universal module, Projective module, Homological dimension.

2000 AMS Classification: 13 N 05

1. Introduction

Let R be a commutative algebra over an algebraically closed field k with characteristic zero. Let $\Omega_n(R)$ and $\delta_n : R \to \Omega_n(R)$ denote the universal module of n-th order derivations and the canonical n-th order k-derivation of R respectively.

The pair $(\delta_n, \Omega_n(R))$ has the universal mapping property that for any R-module N and any higher derivation $d : R \to N$ of order $\leq n$ there is a unique R-homomorphism $h : \Omega_n(R) \to N$ such that $d = h\delta_n$.

$\Omega_n(R)$ is generated by the set $\{\delta_n(r) : r \in R\}$. Hence if R is finitely generated k-algebra, $\Omega_n(R)$ will be a finitely generated R-module.

Let R and S be a commutative algebras over an algebraically closed field k with characteristic zero. Then $R \otimes_k S$ is a commutative ring with unit by defining

$$\left(\sum_i r_i \otimes s_i\right)\left(\sum_j r'_j \otimes s'_j\right) = \sum_i \sum_j r_i r'_j \otimes s_i s'_j,$$

where $r_i, r'_j \in R$ and $s_i, s'_j \in S$.

Let I and J be ideals of R and S respectively. If $R \to R/I$ and $S \to S/J$ are canonical homomorphism of k-algebras then there exists an k-algebra isomorphism

$$\frac{R \otimes_k S}{I \otimes_k S + R \otimes_k J} \simeq R/I \otimes_k S/J$$

(see Nortcott, [3]).

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2. Universal Modules

2.1. Theorem. *Consider affine k-algebras R and S. Let I be an ideal of R and $\delta_n : R \longrightarrow \Omega_n(R)$ the canonical n-th order k derivation of R. Suppose that N is the submodule of $\Omega_n(R)$ generated by all elements of the form $\delta_n(x),\ x \in I$. Then the sequence*

$$0 \longrightarrow \frac{N + I\Omega_n(R)}{I\Omega_n(R} \longrightarrow \frac{\Omega_n(R)}{I\Omega_n(R)} \longrightarrow \Omega_n(R/I) \longrightarrow 0$$

is an exact sequence of R/I modules.

Proof. See Nakai [2]. □

2.2. Proposition. *Let I and J be ideals of R and S respectively. Then there is an exact sequence*

$$0 \longrightarrow \operatorname{Ker}\theta \longrightarrow \Omega_n(R \otimes_k S) \xrightarrow{\theta} \Omega_n(R/I \otimes_k S/J) \longrightarrow 0$$

of $R \otimes_k S$ modules.

Proof. We have $\pi : R \otimes_k S \to R/J \otimes_k S/J$, the canonical homomorphism of the module $R \otimes_k S$. Let $\delta_n : R \otimes_k S \to \Omega_n(R \otimes_k S)$ and $\delta'_n : R/I \otimes_k S/J \to \Omega_n(R/I \otimes_k S/J)$ be the canonical n-th order k derivations of the modules $R \otimes_k S$ and $R/I \otimes_k S/J$, respectively. By the universal property of $\Omega_n(R \otimes_k S)$ there exists a unique homomorphism

$$\theta : \Omega_n(R \otimes_k S) \to \Omega_n(R/I \otimes_k S/J)$$

of $R \otimes_k S$ modules such that $\theta\delta_n = \delta'_n\pi$, i.e. the following diagram commutes.

$$\begin{array}{ccc}
R \otimes_k S & \xrightarrow{\quad\pi\quad} & R/I \otimes_k S/J \\
\downarrow{\scriptstyle \delta_n} & & \downarrow{\scriptstyle \delta'_n} \\
\Omega_n(R \otimes_k S) & \overset{\theta}{\dashrightarrow} & \Omega_n(R/I \otimes_k S/J) \simeq \Omega_n(\frac{R \otimes_k S}{I \otimes_k S + R \otimes_k J})
\end{array}$$

This homomorphism is onto, and we have that

$$0 \longrightarrow \operatorname{Ker}\theta \longrightarrow \Omega_n(R \otimes_k S) \xrightarrow{\theta} \Omega_n(R/I \otimes_k S/J) \longrightarrow 0$$

is an exact sequence of $R \otimes_k S$ modules. □

2.3. Theorem. *Consider affine k-algebras R and S. Let I and J be ideals of R and S respectively, and assume that $K = I \otimes_k S + R \otimes_k J$. Suppose that N is the submodule of $\Omega_n(R \otimes_k S)$ generated by all elements of the form $\delta_n(x),\ x \in K$, where $\delta_n : R \otimes_k S \longrightarrow \Omega_n(R \otimes_k S)$ is the canonical n-th order k derivation of $R \otimes_k S$. Then the sequence*

$$0 \longrightarrow \frac{N + K\Omega_n(R \otimes_k S)}{K\Omega_n(R \otimes_k S)} \longrightarrow \frac{\Omega_n(R \otimes_k S)}{K\Omega_n(R \otimes_k S)} \xrightarrow{\eta} \Omega_n(\frac{R \otimes_k S}{K}) \longrightarrow 0$$

is an exact sequence of $\frac{R \otimes_k S}{K}$ modules.

Proof. If we consider $R \otimes_k S$ instead of R and K instead of I in the following exact sequence of Theorem 2.1:

$$0 \longrightarrow \frac{N + I\Omega_n(R)}{I\Omega_n(R} \longrightarrow \frac{\Omega_n(R)}{I\Omega_n(R)} \longrightarrow \Omega_n(R/I) \longrightarrow 0,$$

then we obtain an exact sequence as required. □

2.4. Proposition. *Suppose that $R = k[x_1, \ldots, x_s]$, $S = k[y_1, \ldots, y_t]$ are polynomial algebras, let I and J be ideals generated by the elements $f_1, \ldots, f_k$ and the elements $g_1, \ldots, g_l$ of R and S, respectively, and let $K = I \otimes_k S + R \otimes_k J$. Then K is generated by the set*

$$\{f_i \otimes 1,\ 1 \otimes g_j : f_i \in I,\ g_j \in J\}.$$

Proof. Let $L = \{f_i \otimes 1,\ 1 \otimes g_j : f_i \in I,\ \ g_j \in J\}$. Then, $L \subseteq K$ is clear.

On the other hand, let $t \in K$ where $t = \sum(\alpha + \beta)$ for all $\alpha \in I \otimes_k S$ and $\beta \in R \otimes_k J$. Then

$$\alpha = \sum_{i,j} a_i f_i \otimes s_j = \sum_{i,j} (a_i \otimes s_j)(f_i \otimes 1),\ \beta = \sum_{i,j} r_i \otimes b_j g_j = \sum_{i,j} (r_i \otimes b_j)(1 \otimes g_j),$$

where $a_i, r_i \in R,\ b_j,\ s_j \in S$. Hence t is in the module generated by L. $\square$

2.5. Proposition. *Suppose that $R = k[x_1, \ldots, x_s]$, $S = k[y_1, \ldots, y_t]$ are polynomial algebras. Let I and J be ideals generated by the elements $f_1, \ldots, f_k$ and by the elements $g_1, \ldots, g_l$ of R and S, respectively, and let $K = I \otimes_k S + R \otimes_k J$. Consider L to be the submodule of $\Omega_n(R \otimes_k S)$ generated by*

$$\{\delta_n(x^\alpha f_i \otimes y^\beta),\ \delta_n(x^\mu \otimes y^\gamma g_j) \mid 0 \le \alpha + \beta < n,\ 0 \le \gamma + \mu < n,$$
$$i = 1, \ldots, k,\ j = 1, \ldots, l\}.$$

Then

$$(R \otimes_k S)\delta_n(K) \subseteq L + K\Omega_n(R \otimes_k S).$$

Proof. Since δ_n is a k-linear map we only need to prove that, for any $t \in (R \otimes_k S)$, $\delta_n(t(f_i \otimes 1))$ and $\delta_n(t(1 \otimes g_j))$ belong to $L + K\Omega_n(R \otimes_k S)$.

Let $t = \sum_\alpha a_\alpha x^\alpha \otimes \sum_\beta b_\beta y^\beta \in R \otimes_k S$, where $a_\alpha, b_\beta \in k$. Then:

$$\begin{aligned}\sum_\alpha a_\alpha x^\alpha \otimes \sum_\beta b_\beta y^\beta &= \sum_{\alpha,\beta} a_\alpha b_\beta (x^\alpha \otimes y^\beta) \\ &= \sum_{\alpha',\beta'} a_{\alpha'} b_{\beta'} (x^{\alpha'} \otimes y^{\beta'}) + \sum_{\alpha'',\beta''} a_{\alpha''} b_{\beta''} (x^{\alpha''} \otimes y^{\beta''}),\end{aligned}$$

where $a_{\alpha'}, a_{\alpha''}, b_{\beta'}, b_{\beta''} \in k$ are such that $|\alpha' + \beta'| \ge n$ and $|\alpha'' + \beta''| < n$. Hence,

$$\delta_n(t(f_i \otimes 1)) = \sum_{\alpha',\beta'} a_{\alpha'} b_{\beta'} \delta_n(f_i x^{\alpha'} \otimes y^{\beta'}) + \sum_{\alpha'',\beta''} a_{\alpha''} b_{\beta''} \delta_n(f_i x^{\alpha''} \otimes y^{\beta''})$$

The second part of the sum in the equality above is in L. As for the first part, we have

$$\delta_n(f_i x^{\alpha'} \otimes y^{\beta'}) = \sum_{\chi,\chi'} a_\chi b_{\chi'} \delta_n(f_i x^\chi \otimes y^{\chi'}) + (f_i \otimes 1) \sum_{\epsilon,\epsilon'} a_\epsilon b_{\epsilon'} \delta_n(x^\epsilon \otimes y^{\epsilon'}),$$

where $a_\chi, b_{\chi'}, a_\epsilon, b_{\epsilon'} \in R \otimes_k S$ are such that $|\chi + \chi'| < n$ and $|\epsilon + \epsilon'| < n$ since $\delta_n \in \operatorname{Der}^n(R \otimes_k S, \Omega_n(R \otimes_k S))$ using the definition of the derivation operator. By substituting in the last equality we get

$$\begin{aligned}\delta_n(t(f_i \otimes 1)) = \sum_{\chi,\chi'} a_\chi b_{\chi'} \delta_n(f_i x^\chi \otimes y^{\chi'}) + (f_i \otimes 1) \sum_{\epsilon,\epsilon'} a_\epsilon b_{\epsilon'} \delta_n(x^\epsilon \otimes y^{\epsilon'}) + \\ + \sum_{\alpha'',\beta''} a_{\alpha''} b_{\beta''} \delta_n(f_i x^{\alpha''} \otimes y^{\beta''}),\end{aligned}$$

which belongs to $L + K\Omega_n(R \otimes_k S)$.

Similarly, $\delta_n(t(1 \otimes g_j))$ is in $L + K\Omega_n(R \otimes_k S)$, therefore $(R \otimes_k S)\delta_n(K) \subseteq L + K\Omega_n(R \otimes_k S)$. $\square$

2.6. Corollary.

$$\frac{(R\otimes_k S)\delta_n(K)+K\Omega_n(R\otimes_k S)}{K\Omega_n(R\otimes_k S)}$$

is generated by

$$\{\delta_n(x^\alpha f_i\otimes y^\beta)+K\Omega_n(R\otimes_k S),\ \delta_n(x^\mu\otimes y^\gamma g_j)+K\Omega_n(R\otimes_k S):\\ 0\le\alpha+\beta<n,\ 0\le\gamma+\mu<n,\ i=1,\ldots,k\ j=1,\ldots,l\}$$

Proof. Let L be as above. Then $\frac{L+K\Omega_n(R\otimes_k S)}{K\Omega_n(R\otimes_k S)}$ is generated by

$$\{\overline{\delta_n(x^\alpha f_i\otimes y^\beta)}+K\Omega_n(R\otimes_k S),\ \ \overline{\delta_n(x^\mu\otimes y^\gamma g_j)}+K\Omega_n(R\otimes_k S):\\ 0\le\alpha+\beta<n,\ 0\le\gamma+\mu<n,\ i=1,\ldots,k\ j=1,\ldots,l\}.$$

By the last proposition $(R\otimes_k S)\delta_n(K)\subseteq L+K\Omega_n(R\otimes_k S)$. Therefore,

$$\frac{(R\otimes_k S)\delta_n(K)+K\Omega_n(R\otimes_k S)}{K\Omega_n(R\otimes_k S)}=\frac{L+K\Omega_n(R\otimes_k S)}{K\Omega_n(R\otimes_k S)},$$

as required. □

2.7. Corollary. *Let $\delta_n: R\otimes_k S\longrightarrow\Omega_n(R\otimes_k S)$ and $d_n:\frac{R\otimes_k S}{K}\longrightarrow\Omega_n(\frac{R\otimes_k S}{K})$ be the n-th order universal derivation operators. Then $\Omega_n(\frac{R\otimes_k S}{K})$ is generated by*

$$\{d_n(x^\alpha\otimes y^\beta+K):\quad 0\le|\alpha|+|\beta|\le n\}.$$

Proof. $\Omega_n(R\otimes_k S)$ is a free $R\otimes_k S$ module on the basis $\{\delta_n(x^\alpha\otimes y^\beta):|\alpha|+|\beta|\le n\}$. Hence $\frac{\Omega_n(R\otimes_k S)}{K\Omega_n(R\otimes_k S)}$ is a free $\frac{R\otimes_k S}{K}$-module with basis

$$\{\delta_n(\overline{x^\alpha\otimes y^\beta}):\ |\alpha|+|\beta|\le n\}.$$

Therefore $\Omega_n(\frac{R\otimes_k S}{K})$ is generated by $\{\eta(\overline{\delta_n(x^\alpha\otimes y^\beta)}):|\alpha|+|\beta|\le n\}$, which is

$$\{d_n(x^\alpha\otimes y^\beta+K):0\le\ |\alpha|+|\beta|\le n\},$$

as required. □

2.8. Theorem. *Consider the affine k-algebras R and S. Let I and J be ideals of R and S respectively, and assume that $K=I\otimes_k S+R\otimes_k J$. Suppose that N is the submodule of $\Omega_n(R\otimes_k S)$ generated by all elements of the form $\delta_n(x)$, $x\in K$, where $\delta_n: R\otimes_k S\longrightarrow\Omega_n(R\otimes_k S)$ is the canonical n-th order k derivation of $R\otimes_k S$. Then*

(i)

$$\operatorname{hd}\left(\frac{N+K\Omega_n(R\otimes_k S)}{K\Omega_n(R\otimes_k S)}\right)<\infty\iff\operatorname{hd}\left(\Omega_n\left(\frac{R\otimes_k S}{K}\right)\right)<\infty,$$

(ii)

$$\operatorname{hd}\left(\frac{N+K\Omega_n(R\otimes_k S)}{K\Omega_n(R\otimes_k S)}\right)=\infty\iff\operatorname{hd}\left(\Omega_n\left(\frac{R\otimes_k S}{K}\right)\right)=\infty.$$

Proof. From Theorem 2.3 we have the exact sequence

$$0\longrightarrow\frac{N+K\Omega_n(R\otimes_k S)}{K\Omega_n(R\otimes_k S)}\longrightarrow\frac{\Omega_n(R\otimes_k S)}{K\Omega_n(R\otimes_k S)}\longrightarrow\Omega_n\left(\frac{R\otimes_k S}{K}\right)\longrightarrow 0$$

of $R\otimes_k S$ modules. Known facts about the homological dimension now complete the proof. □

Now we give an example about estimating the homological dimension of the universal module $\Omega_n(R\otimes_k S)$.

2.9. Example. Consider the affine k-algebras $R = k[x,y]$ and $S = k[z,t]$. Let $I = (y^2 - x^3)$ and $J = (z^2 - t^3)$ be ideals of R and S respectively, and assume that $K = I \otimes_k S + R \otimes_k J$.

$\Omega_1(\dfrac{R \otimes_k S}{K})$: Let F be the free $R \otimes_k S$ module generated by

$$\{\delta_1(x \otimes 1),\ \delta_1(y \otimes 1),\ \delta_1(1 \otimes z),\ \delta_1(1 \otimes t)\},$$

and let N be the submodule of F generated by

$$\{\delta_1(f \otimes 1),\ \delta_1(1 \otimes g) : f = y^2 - x^3,\ g = z^2 - t^3\}.$$

By Corollary 2.7,

$$\Omega_1(\frac{R \otimes_k S}{K}) \cong \frac{F}{N}$$

and hence we have the exact sequence

$$0 \longrightarrow N \longrightarrow F \longrightarrow \Omega_1(\frac{R \otimes_k S}{K}) \longrightarrow 0$$

of $\frac{R \otimes_k S}{K}$ modules.

Since the rank of $\Omega_1(\frac{R \otimes_k S}{K})$ is 2 we have

$$\operatorname{rank} N = \operatorname{rank} F - \operatorname{rank} \Omega_1(\frac{R \otimes_k S}{K}) = 4 - 2 = 2.$$

So $\{\delta_1(f \otimes 1), \delta_1(1 \otimes g) : f = y^2 - x^3,\ g = z^2 - t^3\}$ must be a basis of N. Therefore we have a free resolution

$$0 \longrightarrow N \longrightarrow F \longrightarrow \Omega_1(\frac{R \otimes_k S}{K}) \longrightarrow 0$$

of $\Omega_1(\frac{R \otimes_k S}{K})$. Hence,

$$\operatorname{hd}(\Omega_1(\frac{R \otimes_k S}{K})) \leq 1.$$

$\Omega_2(\dfrac{R \otimes_k S}{K})$: Let F' be the free $R \otimes_k S$ module generated by

$$\{\delta_2(x \otimes 1),\ \delta_2(y \otimes 1),\ \delta_2(1 \otimes z),\ \delta_2(1 \otimes t),\ \delta_2(x \otimes z),\ \delta_2(x \otimes t),\ \delta_2(y \otimes z),$$
$$\delta_2(y \otimes t),\ \delta_2(x^2 \otimes 1),\ \delta_2(y^2 \otimes 1),\ \delta_2(1 \otimes z^2),\ \delta_2(1 \otimes t^2),\ \delta_2(xy \otimes 1),\ \delta_2(1 \otimes zt)\}$$

and let N' be be the submodule of F' generated by

$$\{\delta_2(f \otimes 1), \delta_2(1 \otimes g),\ \delta_2(fx \otimes 1),\ \delta_2(fy \otimes 1)\, \delta_2(1 \otimes zg),\ \delta_2(1 \otimes tg),$$
$$\delta_2(f \otimes z),\ \delta_2(f \otimes t),\ \delta_2(x \otimes g),\ \delta_2(y \otimes g) : f = y^2 - x^3,\ g = z^2 - t^3\}.$$

From Corollary 2.7,

$$\Omega_2(\frac{R \otimes_k S}{K}) \cong \frac{F'}{N'},$$

and hence we have the exact sequence

$$0 \longrightarrow N' \longrightarrow F' \longrightarrow \Omega_2(\frac{R \otimes_k S}{K}) \longrightarrow 0$$

of $\frac{R \otimes_k S}{K}$ modules.

Since the rank of $\Omega_2(\frac{R \otimes_k S}{K})$ is 5, and the rank of F' is 14, we have that the rank of N' is $14 - 5 = 9$.

The result $\operatorname{hd}(\Omega_2(\dfrac{R \otimes_k S}{K})) \leq 2$ is proved by Erdoğan and Çimen in [1].

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