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A RELATED FIXED POINT THEOREM FOR TWO PAIRS OF MAPPINGS ON TWO COMPLETE METRIC SPACES

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Abstract

A related fixed point theorem for two pairs of mappings on two complete metric spaces is obtained.

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1. Introduction

In the following, we give a new related fixed point theorem. The first related fixed point theorem was the following, see [1].

1.1. Theorem. *Let* (X, d_1) *and* (Y, d_2) *be complete metrics spaces. If* T *is a mapping of* X *into* Y *and* S *is a mapping of* Y *into* X *satisfying the inequalities*

$$d_2(Tx, TSy) \le c \max\{d_1(x, Sy), d_2(y, Tx), d_2(y, TSy)\},$$
$$d_1(Sy, STx) \le c \max\{d_2(y, Tx), d_1(x, Sy), d_1(x, STx)\},$$

for all x *in* X *and* y *in* Y, *where* $0 \le c < 1$, *then* ST *has a unique fixed point* z *in* X *and* TS *has a unique fixed point* w *in* Y. *Further,* $Tz = w$ *and* $Sw = z$.

Related fixed point theorems were later extended to two pairs of mappings on metric spaces, see for example [2], where the following related fixed point theorem was proved.

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1.2. Theorem. *Let (X, d) and (Y, ρ) be complete metric spaces, let A, B be mappings of X into Y and let S, T be mappings of Y into X satisfying the inequalities*

$$d(SAx, TBx') \leq c \max\{d(x, x'), d(x, SAx), d(x', TBx'), \rho(Ax, Bx')\},$$
$$\rho(BSy, ATy') \leq c \max\{\rho(y, y'), \rho(y, BSy), \rho(y', ATy'), d(Sy, Ty')\},$$

for all x, x' in X and y, y' in Y, where $0 \leq c < 1$. If one of the mappings A, B, S and T is continuous, then SA and TB have a unique common fixed point u in X and BS and AT have a unique common fixed point v in Y. Further, $Au = Bu = v$ and $Sv = Tv = u$.

For further related fixed point theorems see [3] to [7].

2. Main Results

We prove now the following related fixed point theorem.

2.1. Theorem. *Let (X, d) and (Y, ρ) be complete metric spaces, let A, B be mappings of X into Y and let S, T be mappings of Y into X satisfying the inequalities*

$$d(SAx, TBx') \leq c \frac{f(x, x', y, y')}{h(x, x', y, y')}, \tag{2.1}$$

$$\rho(BSy, ATy') \leq c \frac{g(x, x', y, y')}{h(x, x', y, y')}, \tag{2.2}$$

for all x, x' in X and y, y' in Y for which $h(x, x', y, y') \neq 0$, where

$$f(x, x', y, y') = \max\{d(Sy, Ty')\rho(Ax, BSy), d(Sy, TBx')d(x, Sy), d(x, x')d(SAx, Ty'), d(x, Ty')\rho(y, ATy')\},$$
$$g(x, x', y, y') = \max\{d(x, Sy)\rho(y, y'), d(x', TBx')\rho(y', Ax), d(SAx, Ty')\rho(Ax, Bx'), \rho(Ax, ATy')d(SAx, Sy)\},$$
$$h(x, x', y, y') = \max\{\rho(Ax, BSy), d(x, SAx), d(Sy, TBx'), \rho(Bx', ATy')\},$$

and $0 \leq c < 1$. If one of the mappings A, B, S and T is continuous, then SA and TB have a unique common fixed point u in X and BS and AT have a unique common fixed point v in Y. Further, $Au = Bu = v$ and $Sv = Tv = u$.

Proof. Let x_0 be an arbitrary point in X, let

$$Ax_0 = y_1, \ Sy_1 = x_1, \ Bx_1 = y_2, \ Ty_2 = x_2 \text{ and } Ax_2 = y_3,$$

and in general let

$$Sy_{2n-1} = x_{2n-1}, \ Bx_{2n-1} = y_{2n}, \ Ty_{2n} = x_{2n} \text{ and } Ax_{2n} = y_{2n+1},$$

for $n = 1, 2, \ldots$.

We will first of all suppose that for some n

$$\begin{aligned} h(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) &= \max\{\rho(Ax_{2n}, BSy_{2n-1}), d(x_{2n}, SAx_{2n}), \\ &\qquad d(Sy_{2n-1}, TBx_{2n-1}), \rho(Bx_{2n-1}, ATy_{2n})\} \\ &= \max\{\rho(y_{2n+1}, y_{2n}), d(x_{2n}, x_{2n+1}), \\ &\qquad d(x_{2n-1}, x_{2n}), \rho(y_{2n}, y_{2n+1})\} \\ &= 0. \end{aligned}$$

Then putting $x_{2n-1} = x_{2n} = x_{2n+1} = u$ and $y_{2n} = y_{2n+1} = v$, we see that

$$Au = BSv = v, \ SAu = u, \ Sv = TBu = u \text{ and } Bu = ATv = v,$$

from which it follows that

$$Bu = v,\ Tv = u \text{ and } ATv = v.$$

Similarly, $h(x_{2n},\ x_{2n+1},\ y_{2n+1}$ and $y_{2n}) = 0$ for some n implies that there exists u in X and v in Y such that

$$SAu = TBu = u,\ BSv = ATv = v,\ Au = Bu = v \text{ and } Sv = Tv = u. \tag{2.3}$$

We will now suppose that

$$h(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) \neq 0 \neq h(x_{2n}, x_{2n+1}, y_{2n+1}, y_{2n})$$

for all n. Applying inequality (2.1) we get

$$\begin{aligned} d(x_{2n+1}, x_{2n}) &= d(SAx_{2n}, TBx_{2n-1}) \\ &\leq c\frac{f(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n})}{h(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n})} \\ &= cd(x_{2n-1}, x_{2n})\frac{\max\{\rho(y_{2n+1}, y_{2n}), d(x_{2n-1}, x_{2n}), d(x_{2n+1}, x_{2n})\}}{\max\{\rho(y_{2n+1}, y_{2n}), d(x_{2n+1}, x_{2n}), d(x_{2n-1}, x_{2n})\}}, \end{aligned}$$

from which it follows that

$$d(x_{2n+1}, x_{2n}) \leq c\max\{d(x_{2n-1}, x_{2n}), \rho(y_{2n+1}, y_{2n})\}. \tag{2.4}$$

Using inequality (2.1) again, we get

$$\begin{aligned} d(x_{2n-1}, x_{2n}) &= d(SAx_{2n-2}, TBx_{2n-1}) \\ &\leq c\frac{f(x_{2n-2}, x_{2n-1}, y_{2n-1}, y_{2n-2})}{h(x_{2n-2}, x_{2n-1}, y_{2n-1}, y_{2n-2})} \\ &= cd(x_{2n-1}, x_{2n-2})\frac{\max\{\rho(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n}), d(x_{2n-2}, x_{2n-1})\}}{\max\{\rho(y_{2n-1}, y_{2n}), d(x_{2n-2}, x_{2n-1}), d(x_{2n-1}, x_{2n})\}}, \end{aligned}$$

from which it follows that

$$d(x_{2n-1}, x_{2n}) \leq c\max\{d(x_{2n-2}, x_{2n-1}), \rho(y_{2n-1}, y_{2n})\}. \tag{2.5}$$

Similarly, on using inequality (2.2) we have

$$\begin{aligned} \rho(y_{2n}, y_{2n+1}) &= d(BSy_{2n-1}, ATy_{2n}) \\ &\leq c\frac{g(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n})}{h(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n})}, \end{aligned}$$

where

$$\begin{aligned} g(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) = \max\{&d(x_{2n-1}, x_{2n})\rho(y_{2n-1}, y_{2n}), \\ &d(x_{2n-1}, x_{2n})\rho(y_{2n+1}, y_{2n}), \\ &d(x_{2n+1}, x_{2n})\rho(y_{2n+1}, y_{2n})\}. \end{aligned}$$

We then have either

$$g(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) = d(x_{2n-1}, x_{2n})\max\{\rho(y_{2n-1}, y_{2n}), \rho(y_{2n+1}, y_{2n})\},$$

or

$$g(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) = \rho(y_{2n+1}, y_{2n})\max\{d(x_{2n-1}, x_{2n}), d(x_{2n+1}, x_{2n})\}.$$

Further,

$$\begin{aligned} h(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) &= \max\{\rho(y_{2n+1}, y_{2n}), d(x_{2n+1}, x_{2n}), d(x_{2n-1}, x_{2n})\} \\ &= \max\{\rho(y_{2n+1}, y_{2n}), d(x_{2n-1}, x_{2n})\} \end{aligned}$$

on using inequality (2.4). It follows that either

$$\rho(y_{2n}, y_{2n+1}) \leq c\max\{\rho(y_{2n-1}, y_{2n}), \rho(y_{2n}, y_{2n+1})\} = c\rho(y_{2n-1}, y_{2n})$$

or

$$\rho(y_{2n}, y_{2n+1}) \leq c \max\{d(x_{2n-1}, x_{2n})\}, d(x_{2n+1}, x_{2n})\} = cd(x_{2n-1}, x_{2n}),$$

and so

$$\rho(y_{2n}, y_{2n+1}) \leq c \max\{d(x_{2n-1}, x_{2n}), \rho(y_{2n-1}, y_{2n})\}. \tag{2.6}$$

Using inequality (2.2) again, we get

$$\begin{aligned}\rho(y_{2n}, y_{2n-1}) &= \rho(BSy_{2n-1}, ATy_{2n-2}) \\ &\leq c\frac{g(x_{2n-2}, x_{2n-1}, y_{2n-1}, y_{2n-2})}{h(x_{2n-2}, x_{2n-1}, y_{2n-1}, y_{2n-2})},\end{aligned}$$

where

$$\begin{aligned}g(x_{2n-2}, x_{2n-1}, y_{2n-1}, y_{2n-2}) = \max\{&d(x_{2n-2}, x_{2n-1})\rho(y_{2n-1}, y_{2n-2}), \\ &d(x_{2n-1}, x_{2n})\rho(y_{2n-2}, y_{2n-1}), \\ &d(x_{2n-1}, x_{2n-2})\rho(y_{2n-1}, y_{2n})\}.\end{aligned}$$

We then have either

$$\begin{aligned}&g(x_{2n-2}, x_{2n-1}, y_{2n-1}, y_{2n-2}) = \\ &\qquad d(x_{2n-2}, x_{2n-1}) \max\{\rho(y_{2n-1}, y_{2n-2}), \rho(y_{2n-1}, y_{2n})\}\end{aligned}$$

or

$$\begin{aligned}&g(x_{2n-2}, x_{2n-1}, y_{2n-1}, y_{2n-2}) = \\ &\qquad \rho(y_{2n-1}, y_{2n-2}) \max\{d(x_{2n-2}, x_{2n-1}), d(x_{2n-1}, x_{2n})\}.\end{aligned}$$

Further

$$\begin{aligned}h(x_{2n-2}, x_{2n-1}, y_{2n-1}, y_{2n-2}) &= \max\{\rho(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n}), \\ &\qquad d(x_{2n-1}, x_{2n-2})\} \\ &= \max\{\rho(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n-2})\}\end{aligned}$$

on using inequality (2.5). It follows that either

$$\rho(y_{2n}, y_{2n-1}) \leq c \max\{\rho(y_{2n-1}, y_{2n}), \rho(y_{2n-1}, y_{2n-2})\} = c\rho(y_{2n-1}, y_{2n-2})$$

or

$$\rho(y_{2n}, y_{2n-1}) \leq c \max\{d(x_{2n-2}, x_{2n-1})\}, d(x_{2n-1}, x_{2n})\} = cd(x_{2n-1}, x_{2n-2}),$$

and so

$$\rho(y_{2n}, y_{2n-1}) \leq c \max\{d(x_{2n-2}, x_{2n-1}), \rho(y_{2n-1}, y_{2n-2})\}. \tag{2.7}$$

From inequalities (2.4) to (2.7), we obtain

$$d(x_n, x_{n+1}) \leq c^n \max\{d(x_1, x_2), \rho(y_0, y_1)\}, \tag{2.8}$$

$$\rho(y_n, y_{n+1}) \leq c^n \max\{d(x_0, x_1), \rho(y_0, y_1)\}. \tag{2.9}$$

Since $0 < c < 1$, it follows from inequalities (2.8) and (2.9) that $\{x_n\}$ is a Cauchy sequence in X with a limit u and $\{y_n\}$ is a Cauchy sequence in Y with a limit v.

Now, suppose that A is continuous. Then

$$v = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Ax_{2n} = Au, \tag{2.10}$$

and so

$$\lim_{n\to\infty} f(u, x_{2n-1}, v, y_{2n}) = \max\{d(Sv,u)\rho(v,BSv), d^2(Sv,u)\}, \tag{2.11}$$

$$\lim_{n\to\infty} g(u, x_{2n-1}, v, y_{2n}) = 0, \text{ and} \tag{2.12}$$

$$\lim_{n\to\infty} h(u, x_{2n-1}, v, y_{2n}) = \max\{\rho(v,BSv), d(u,Sv)\}. \tag{2.13}$$

If

$$\max\{\rho(v,BSv), d(u,Sv)\} = 0, \tag{2.14}$$

then

$$BSv = v,\ Sv = u \text{ and } Bu = v. \tag{2.15}$$

If

$$\max\{\rho(v,BSv), d(u,Sv)\} \neq 0, \tag{2.16}$$

then we have, on using inequality (2.1) and equations (2.11) and (2.13),

$$\begin{aligned} d(Sv,u) &= \lim_{n\to\infty} d(SAu, TBx_{2n-1}) \\ &\le \lim_{n\to\infty} c\frac{f(u,x_{2n-1},v,y_{2n})}{h(u,x_{2n-1},v,y_{2n})} \\ &= c\frac{\max\{d(Sv,u)\cdot\rho(v,BSv), d^2(Sv,u)\}}{\max\{\rho(v,BSv), d(u,Sv)\}} \\ &\le cd(Sv,u), \end{aligned}$$

and so $Sv = u$, since $c < 1$.

Further, using inequality (2.2) and equations (2.12) and (2.13), we get

$$\begin{aligned} \rho(BSv,v) &= \lim_{n\to\infty} \rho(BSv, ATy_{2n}) \\ &\le \lim_{n\to\infty} c\frac{g(u,x_{2n-1},v,y_{2n})}{h(u,x_{2n-1},v,y_{2n})} \\ &= 0, \end{aligned}$$

and so $BSv = v$, contracting equation (2.15). Therefore equations (2.14) and (2.16) must hold.

Now suppose that $Tv \neq u$. Then

$$\lim_{n\to\infty} f(x_{2n}, u, v, v) = d(u,Tv)\rho(v,ATv), \tag{2.17}$$

$$\begin{aligned} \lim_{n\to\infty} h(x_{2n}, u, v, v) &\le \max\{d(u,Tv), \rho(v,ATv)\} \\ &\neq 0. \end{aligned} \tag{2.18}$$

Using inequality (2.1) and equations (2.17) and (2.18), we have

$$\begin{aligned} d(u,Tv) &= \lim_{n\to\infty} d(SAx_{2n}, TBu) \\ &\le \lim_{n\to\infty} c\frac{f(x_{2n},u,v,v)}{h(x_{2n},u,v,v)} \\ &\le \frac{cd(u,Tv)\rho(v,ATv)}{\max\{d(u,Tv),\rho(v,ATv)\}} \end{aligned}$$

and so $Tv = u$, giving a contradiction. Hence $Tv = u$ and equations (2.3) again follow.

It follows in a similar way that the same results hold if one of the mappings B, S or T is continuous instead of A.

To prove the uniqueness, suppose that SA and TB have a second distinct common fixed point u' so that $Au \neq Bu'$. Then,

$$f(u, u', v, v) = 0, \tag{2.19}$$

$$h(u, u', v, v) = \max\{d(u, u'), \rho(Au, Bu'\} \neq 0. \tag{2.20}$$

Using inequality (2.1) and equations (2.19) and (2.20), we get

$$\begin{aligned} d(u, u') &= d(SAu, TBu') \\ &\leq c\frac{f(u, u', v, v)}{h(u, u', v, v)} \\ &= 0, \end{aligned}$$

a contradiction. Therefore u is unique. It can be proved similarly that v is the unique common fixed point of BS and AT. This completes the proof of the theorem. □

2.2. Corollary. *Let A, B, S and T be self mappings on the complete metric space (X, d) satisfying the inequalities*

$$d(SAx, TBy) \leq c\frac{f(x, , y)}{h(x, y)},$$

$$d(BSx, ATy) \leq c\frac{g(x, y)}{h(x, y)}$$

for all x, y in X for which $h(x, y) \neq 0$, where

$$\begin{aligned} f(x, y) = \max\{&d(Sx, Ty)d(Ax, BSx), d(Sx, TBy)d(x, Sx), \\ &d(x, y)d(SAx, Ty), d(x, Ty)d(x, ATy)\}, \\ g(x, y) = \max\{&d(x, Sx)d(x, y), d(y, TBy)d(y, Ax), \\ &d(SAx, Ty)d(Ax, By), d(Ax, ATy)d(SAx, Sx)\}, \\ h(x, y) = \max\{&d(Ax, BSx), d(x, SAx), d(Sx, TBy), d(By, ATy)\}, \end{aligned}$$

and $0 \leq c < 1$. If one of the mappings A, B, S or T is continuous, then SA and TB have a unique common fixed point u and BS and AT have a unique common fixed point v. Further, $Au = Bu = v$ and $Sv = Tv = u$.

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