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GCED and reciprocal GCED matrices

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Abstract

We have given structure theorems for a GCED (greatest common exponential divisor) and Reciprocal GCED matrix. We have also calculated the value of the determinant of these matrices. The formulae for the inverse and determinant of GCED and Reciprocal GCED matrices defined on an exponential divisor closed set have been determined.

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1. Introduction

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a finite ordered set of distinct positive integers. The matrix (S) where $s_{ij} = (x_i, x_j)$ =greatest common divisor of x_i and x_j, is called the greatest common divisor(GCD) matrix on the set S. A set $S = \{x_1, x_2, \ldots, x_n\}$ is said to be factor closed if for every $x_i \in S$, and $d \mid x_i$ then $d \in S$.

In 1876, H.J. Smith [7] proved that the determinant of a GCD matrix on $S = \{1, 2, \ldots, n\}$ is equal to $\varphi(1)\varphi(2)\cdots\varphi(n)$ where φ is Euler's totient function. The result holds if S is a factor closed set. The structure theorems for Reciprocal GCD matrices and LCM (least common multiple) matrices were determined by S.J. Beslin [2]. The structures of Power GCD matrix, Power LCM matrix, Reciprocal LCM matrix, GCD Reciprocal LCM matrix, GCUD (greatest common unitary divisor) Reciprocal LCUM (least common unitary multiple) matrices have been determined [1, 3, 5, 9]. Research has also been extended to divisibility properties of such matrices and their applications [4, 6]. It is worth to note that the structures of most of the above mentioned matrices have been determined on factor closed sets, gcd closed sets, lcm closed sets or unitary divisor closed sets or on sets contained in factor closed sets. This has motivated the authors to follow the same direction.

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We recall that an integer $d = \prod_{i=1}^{t} {p_i}^{a_i}$ is said to be an exponential divisor of $m = \prod_{i=1}^{t} {p_i}^{b_i}$, if $a_i | b_i$ for every $1 \leq i \leq t$ and is denoted by $d|_e m$. This notion was introduced by M. V. Subrarao [8]. Note that unlike divisor and unitary divisor, 1 is not an exponential divisor for every $m > 1$. By convention $1|_e 1$. The smallest exponential divisor of $m > 1$ is its square free kernel $\kappa(m) = \prod_{i=1}^{r} p_i$ [10].

Two integers n and m have common exponential divisor if and only if they have the same prime factors. Two integers $m = \prod_{i=1}^{r} {p_i}^{b_i}$ and $n = \prod_{i=1}^{r} {p_i}^{c_i}$ are exponentially co-prime if $(b_i, c_i) = 1$ for every $1 \leq i \leq r$. We denote the GCED (greatest common exponential divisor) of two integers m and n by $(m, n)_e$. By convention $(1, 1)_{(e)} = 1$ and $(1, m)_{(e)}$ does not exist for every $m > 1$.

A set $S = \{x_1, x_2, x_3, \ldots, x_n\}$ is said to be an exponential divisor closed set if the exponential divisor of every element of S belongs to S. For example the set $\{12, 18, 36\}$ is not an exponential divisor closed set. But, $\{6, 12, 18, 36\}$ is an exponential divisor closed set.

Similarly, a set $S = \{x_1, x_2, x_3, \ldots, x_n\}$ is said to be GCED closed if $(x_i, x_j)_{(e)} \in S$ for every $x_i, x_j \in S$. Note that $\{6, 12, 18, 36\}$ is also a GCED closed set.

The exponential convolution of two arithmetic functions f and g is given as

$$(f \odot g)(n) = \sum_{k_1 l_1 = m_1} \cdots \sum_{k_r l_r = m_r} f({p_1}^{k_1} {p_2}^{k_2} \ldots {p_r}^{k_r}) g({p_1}^{l_1} {p_2}^{l_2} \ldots {p_r}^{l_r}),$$

where $n = {p_1}^{m_1} {p_2}^{m_2} \ldots {p_r}^{m_r}$.

The inverse with respect to $\odot$ of the constant function 1 is called the exponential analogue of Möbius function and is denoted by $\mu^{(e)}$. It should be noted that the sets considered in section 2 are such that the GCED of every two elements exists.

2. Structure of GCED matrix

Let $T = \{x_1, x_2, x_3, \ldots, x_n\}$ be an ordered set of distinct positive integers greater than 1. The $n \times n$ matrix $T_{(e)} = (t_{ij})_{(e)}$ having $t_{ij} = (x_i, x_j)_{(e)}$ as its ij^{th} entry is referred as the GCED (greatest common exponential divisor) matrix on the set T, where $(x_i, x_j)_{(e)}$ is the greatest common exponential divisor of x_i and x_j. Let $R = \{y_1, y_2, y_3, \ldots, y_m\}$ which is ordered by $y_1 < y_2 < y_3 < \ldots < y_m$ be a minimal exponential divisor-closed set containing T. We refer R the exponential closure of the set T. It is easy to see that GCED matrices are symmetric. We always assume that $x_1 < x_2 < x_3 < \cdots < x_n$ in T.

We define arithmetic function $g(n)$ as follows:

$$g(n) = \sum_{a_1 b_1 = c_1} \sum_{a_2 b_2 = c_2} \cdots \sum_{a_r b_r = c_r} {p_1}^{a_1} {p_2}^{a_2} \cdots {p_r}^{a_r} \mu^{(e)}({p_1}^{b_1} {p_2}^{b_2} \cdots {p_r}^{b_r}), \tag{2.1}$$

where $n = {p_1}^{c_1} {p_2}^{c_2} \cdots {p_r}^{c_r}$.

2.1. Theorem. *Let $R = \{y_1, y_2, \ldots, y_m\}$ be the exponential closure of the set $T = \{x_1, x_2, \ldots, x_n\}$, where $y_1 < y_2 < y_3 < \cdots < y_m$ and $x_1 < x_2 < x_3 < \cdots < x_n$.*

Define the $n \times m$ matrix $C = (c_{ij})$ by

$$c_{ij} = \begin{cases} 1, & y_j |_e x_i \\ 0, & \textit{otherwise} \end{cases}$$

and the $m \times m$ diagonal matrix by

$$\Psi = diag(g(y_1), g(y_2), \ldots, g(y_m)).$$

Then,

$$T_{(e)} = C \Psi C^t.$$

Proof. The ij^{th} entry of $C\Psi C^t$ is equal to

$$(C\Psi C^t)_{ij} = \sum_{k=1}^{n} c_{ik} g(y_k) c_{jk} = \sum_{y_k |_e x_i, y_k |_e x_j} g(y_k) = \sum_{y_k |_e (x_i, x_j)_{(e)}} g(y_k),$$

where the function g is defined in Equation 2.1.
By Möbius Inversion Exponential formula, we have,

$$\sum_{d|_e n} g(d) = n.$$

Finally, we get,

$$(C\Psi C^t)_{ij} = (x_i, x_j)_{(e)}.$$

2.2. Theorem. *Let $R = \{y_1, y_2, \dots, y_m\}$ be the exponential closure of the set $T = \{x_1, x_2, \dots, x_n\}$ where $y_1 < y_2 < y_3 < \cdots < y_m$ and $x_1 < x_2 < x_3 < \cdots < x_n$. Then*

$$\det T_{(e)} \quad = \sum_{1\le k_1<k_2<\ldots<k_n\le m} (\det C_{(k_1,k_2,\ldots,k_n)})^2 g(y_{k_1})g(y_{k_2})\ldots g(y_{k_n}),$$

where $C_{(k_1,k_2,\ldots,k_n)}$ is the sub matrix of C consisting of the ${k_1}^{th}$, ${k_2}^{th}, \ldots, {k_n}^{th}$ columns of C.

Proof. By Theorem 2.1, we have, $T_{(e)} = (C\Psi^{\frac{1}{2}})(C\Psi^{\frac{1}{2}})^t$. Thus we can write $E = C\Psi^{\frac{1}{2}}$ which leads to $T_{(e)} = EE^t$. By applying Cauchy-Binet formula, we get

$$\begin{aligned}\det(T)_{(e)} &= \sum_{1\le k_1<k_2<\ldots<k_n\le m} \det E_{(k_1,k_2,\ldots,k_n)} \det {E^t}_{(k_1,k_2,\ldots,k_n)}\\ &= \sum_{1\le k_1<k_2<\ldots<k_n\le m} (\det E_{(k_1,k_2,\ldots,k_n)})^2,\end{aligned}$$

where $E_{(k_1,k_2,\ldots,k_n)}$ is the sub matrix of E consisting of the ${k_1}^{th}, {k_2}^{th}, \ldots, {k_n}^{th}$ columns of E.

$$\det E_{(k_1,k_2,\ldots,k_n)} = \sqrt{g(y_{k_1})g(y_{k_2})\ldots g(y_{k_n})}\,\det C_{(k_1,k_2,\ldots,k_n)}.$$

Hence,

$$\det T_{(e)} = \sum_{1\le k_1<k_2<\ldots<k_n\le m} (\det C_{(k_1,k_2,\ldots,k_n)})^2 g(y_{k_1})g(y_{k_2})\ldots g(y_{k_n}).$$

2.3. Corollary. *Let $T = \{x_1, x_2, \dots, x_n\}$ be a finite ordered set of distinct positive integers. If $T = R$, then the determinant of GCED matrix $T_{(e)}$ defined on T is given as:*

$$\det T_{(e)} = \prod_{k=1}^{n} g(x_k).$$

Proof. Note that C is a lower triangular matrix with diagonal $(1, 1, \dots, 1)_n$. This implies that $\det C = 1$. Since the determinant of a diagonal matrix is equal to the product of its diagonal entries, hence the desired outcome achieved.

2.4. Corollary. *If $T_{(e)}$ is an $n \times n$ GCED matrix on a finite ordered set of distinct integers denoted by $T = \{x_1, x_2, \dots, x_n\}$, then the trace is given as:*

$$trT_{(e)} = \sum_{k=1}^{n} x_i.$$

2.5. Lemma. *Let $T_{(e)} = (t_{ij})_{(e)}$ is an $n \times n$ GCED matrix defined on an exponential divisor closed set T. Consider $n \times n$ matrix $C = (c_{ij})$ as defined in Theorem 2.1. Then, the $n \times n$ matrix $W = (w_{ij})$ defined by*

$$w_{ij} = \begin{cases} \mu^{(e)}(\frac{x_i}{x_j}), & x_j|_e x_i \\ 0, & otherwise \end{cases}$$

is the inverse of C.

Proof. The ij^{th} entry of CW is given by

$$(CW)_{ij} = \sum_{k=1}^{n} c_{ik} w_{kj} = \sum_{x_k|_e x_i, x_j|_e x_k} \mu^{(e)}(\frac{x_k}{x_j}) = \sum_{x_d|_e \frac{x_i}{x_j}} \mu^{(e)}(x_d) = \begin{cases} 1, & if \quad x_i = x_j \\ 0, & otherwise \end{cases}$$

If $\frac{x_i}{x_j}$ is not an integer then no x_d divides $\frac{x_i}{x_j}$. If $x_i = x_j$ then, $1|_e 1$ and $\mu^{(e)}(1) = 1$.

2.6. Theorem. *Let $T_{(e)}$ be an $n \times n$ GCED matrix on an exponential divisor closed set. Then, its inverse matrix $(A)_{(e)} = (a_{ij})_{(e)}$ is given as*

$$(a_{ij})_{(e)} = \sum_{x_i|_{(e)} x_k, x_j|_{(e)} x_k} \frac{\mu^{(e)} \frac{x_d}{x_i} \mu^{(e)} \frac{x_d}{x_j}}{g(x_d)}.$$

Proof. Since $T_{(e)} = (C\Psi C^t)$ and Lemma 2.5 suggests that, $C^{-1} = W$, therefore

$$(T)_{(e)}^{-1} = (C\Psi C^t)^{-1} = W^t \Psi^{-1} W,$$

where ij^{th} entry of $(T)_{(e)}^{-1}$ is given as

$$(a_{ij})_{(e)} = \sum_{x_i|_{(e)} x_d, x_j|_{(e)} x_d} \frac{\mu^{(e)} \frac{x_d}{x_i} \mu^{(e)} \frac{x_d}{x_j}}{g(x_d)}.$$

Hence, the required result.

3. Structure of Reciprocal GCED matrix

Let $T = \{x_1, x_2, x_3, \ldots, x_n\}$ be an ordered set of positive integers greater than 1. The $n \times n$ matrix $\overline{T}_{(e)} = (t_{ij})_{(e)}$ having $t_{ij} = \frac{1}{(x_i, x_j)_{(e)}}$ as its ij^{th} entry on T is called a Reciprocal GCED matrix. It is easy to note that Reciprocal GCED matrices are symmetric. We always assume that $x_1 < x_2 < x_3 < \cdots < x_n$.

We define arithmetic function $f(n)$ as follows:

$$f(n) = \sum_{a_1 b_1 = c_1} \sum_{a_2 b_2 = c_2} \cdots \sum_{a_r b_r = c_r} \frac{1}{{p_1}^{a_1} {p_2}^{a_2} \cdots {p_r}^{a_r}} \mu^{(e)}({p_1}^{b_1} {p_2}^{b_2} \cdots {p_r}^{b_r}), \tag{3.1}$$

where $n = {p_1}^{c_1} {p_2}^{c_2} \cdots {p_r}^{c_r}$.

3.1. Theorem. *Let $R = \{y_1, y_2, \ldots, y_m\}$ be an exponential closure of the set $T = \{x_1, x_2, \ldots, x_n\}$, where $y_1 < y_2 < y_3 < \cdots < y_m$ and $x_1 < x_2 < x_3 < \cdots < x_n$. Define the $n \times m$ matrix $C = (c_{ij})$ by*

$$c_{ij} = \begin{cases} 1, & y_j|_e x_i \\ 0, & otherwise \end{cases}$$

and the $m \times m$ diagonal matrix by

$$\Xi = diag(f(y_1), f(y_2), \ldots, f(y_m)).$$

Then,

$$\overline{T}_{(e)} = C \Xi C^t.$$

Proof. The ij^{th} entry of $C\Xi C^t$ is equal to

$$(C\Xi C^t)_{ij} = \sum_{k=1}^{n} c_{ik} f(y_k) c_{jk} = \sum_{y_k|_e x_i, y_k|_e x_j} f(y_k) = \sum_{y_k|_e (x_i, x_j)_{(e)}} f(y_k),$$

where f is defined in Equation 3.1. By Möbius Inversion Exponential formula,

$$\sum_{d|_e n} g(d) = \frac{1}{n}.$$

Finally we get,

$$(C\Xi C^t)_{ij} = \frac{1}{(x_i, x_j)_{(e)}}.$$

3.2. Theorem. *Let $R = \{y_1, y_2, \dots, y_m\}$ be an exponential closure of the set $T = \{x_1, x_2, \dots, x_n\}$, where $y_1 < y_2 < y_3 < \cdots < y_m$ and $x_1 < x_2 < x_3 < \cdots < x_n$. Then*

$$\det \overline{T}_{(e)} = \sum_{1 \le k_1 < k_2 < \cdots < k_n \le m} (\det C_{(k_1, k_2, \dots, k_n)})^2 f(y_{k_1}) f(y_{k_2}) \dots f(y_{k_n}),$$

where $C_{(k_1, k_2, \dots, k_n)}$ is the sub matrix of C consisting of the ${k_1}^{th}, {k_2}^{th}, \dots, {k_n}^{th}$ columns of C.

Proof. The proof can be done on similar lines as Theorem 2.2.

3.3. Corollary. *Let $T = \{x_1, x_2, \dots, x_n\}$ be a finite ordered set of distinct positive integers. If $T = R$, then the determinant of Reciprocal GCED matrix $\overline{T}_{(e)}$ defined on T is given as:*

$$\det \overline{T}_{(e)} = \prod_{k=1}^{n} f(x_k).$$

Proof. Note that C is a lower triangular matrix with diagonal $(1, 1, \dots, 1)_n$. This implies that $\det C = 1$. The result is further proved by using the fact that the determinant of a diagonal matrix is equal to the product of its diagonal entries.

3.4. Corollary. *If $\overline{T}_{(e)}$ is an $n \times n$ Reciprocal GCED matrix on a set $T = \{x_1, x_2, \dots, x_n\}$, then the trace is given as:*

$$tr\overline{T}_{(e)} = \sum_{k=1}^{n} \frac{1}{x_i}.$$

3.5. Theorem. *Let $\overline{T}_{(e)}$ be an $n \times n$ Reciprocal GCED matrix on an exponential divisor closed set T. Then, its inverse matrix $\overline{A}_{(e)} = (a_{ij})_{(e)}$ is given as:*

$$(a_{ij})_{(e)} = \sum_{x_i|_{(e)} x_k, x_j|_{(e)} x_k} \frac{\mu^{(e)} \frac{x_d}{x_i} \mu^{(e)} \frac{x_d}{x_j}}{f(x_d)}.$$

Proof. Since $\overline{T}_{(e)} = (C\Xi C^t)$ and by Lemma 2.5, $C^{-1} = W$, therefore

$$(T)_{(e)}^{-1} = (C\Xi C^t)^{-1} = W^t \Xi^{-1} W,$$

where ij^{th} entry of $(T)_{(e)}^{-1}$ is given as

$$(a_{ij})_{(e)} = \sum_{x_i|_{(e)} x_k, x_j|_{(e)} x_k} \frac{\mu^{(e)} \frac{x_d}{x_i} \mu^{(e)} \frac{x_d}{x_j}}{f(x_d)}.$$

Hence, the required result.

4. Examples

4.1. Example. Let $T = \{12, 18, 36\}$. The GCED matrix $T_{(e)}$ on T is given as:

$$T_{(e)} = \begin{bmatrix} 12 & 6 & 12 \\ 6 & 18 & 18 \\ 12 & 18 & 36 \end{bmatrix}.$$

Note that $T=\{12,18,36\}$ is not an exponential divisor closed set. Its exponential closure is $R=\{6,12,18,36\}$. The 3×4 matrix $(C)_{(e)}$ is

$$C=\begin{bmatrix}1&1&0&0\\1&0&1&0\\1&1&1&1\end{bmatrix}.$$

By Theorem 2.2, we know that,

$$\det T_{(e)}=\sum_{1\le k_1<k_2<\ldots<k_n\le m}(\det C_{k_1,k_2,\ldots,k_n})^2 g(y_{k_1})g(y_{k_2})\ldots g(y_{k_n}).$$

So,

$$\det T_{(e)}=\begin{vmatrix}1&1&0\\1&0&1\\1&1&1\end{vmatrix}^2 g(6)g(12)g(18)+\begin{vmatrix}1&1&0\\1&0&0\\1&1&1\end{vmatrix}^2 g(6)g(12)g(36)+$$
$$\begin{vmatrix}1&0&0\\1&1&0\\1&1&1\end{vmatrix}^2 g(6)g(18)g(36)+\begin{vmatrix}1&0&0\\0&1&0\\1&1&1\end{vmatrix}^2 g(12)g(18)g(36)$$

where, $g(6)=6, g(12)=6, g(18)=12$ and $g(36)=12$.
Hence, the determinant is given as:

$$\det T_{(e)}=(6)(6)(12)+(6)(6)(12)+(6)(12)(12)+(6)(12)(12)=2592.$$

The Reciprocal GCED matrix $\overline{T}_{(e)}$ on T is given as:

$$\overline{T}_{(e)}=\begin{bmatrix}\frac{1}{12}&\frac{1}{6}&\frac{1}{12}\\ \frac{1}{6}&\frac{1}{18}&\frac{1}{18}\\ \frac{1}{12}&\frac{1}{18}&\frac{1}{36}\end{bmatrix}.$$

By Theorem 3.2,

$$\det \overline{T}_{(e)}=\sum_{1\le k_1<k_2<\ldots<k_n\le m}(\det C_{k_1,k_2,\ldots,k_n})^2 f(y_{k_1})f(y_{k_2})\ldots f(y_{k_n}).$$

So,

$$\det \overline{T}_{(e)}=\begin{vmatrix}1&1&0\\1&0&1\\1&1&1\end{vmatrix}^2 f(6)f(12)f(18)+\begin{vmatrix}1&1&0\\1&0&0\\1&1&1\end{vmatrix}^2 f(6)f(12)f(36)+$$
$$\begin{vmatrix}1&0&0\\1&1&0\\1&1&1\end{vmatrix}^2 f(6)f(18)f(36)+\begin{vmatrix}1&0&0\\0&1&0\\1&1&1\end{vmatrix}^2 f(12)f(18)f(36),$$

where,
$f(6)=\frac{1}{6}\mu^{(e)}((2)(3))=\frac{1}{6}$, $f(12)=\frac{1}{(2)(3)}\mu^{(e)}((2^2)(3))+\frac{1}{(2^2)(3)}\mu^{(e)}((2)(3))=\frac{-1}{12}$
$f(18)=\frac{1}{(2)(3)}\mu^{(e)}((3^2)(2))+\frac{1}{(2)(3^2)}\mu^{(e)}((2)(3))=\frac{-1}{9}$ and
$f(36)=\frac{1}{(2)(3)}\mu^{(e)}((2^2)(3^2))+\frac{1}{(2^2)(3)}\mu^{(e)}((3^2)(2))+\frac{1}{(2)(3^2)}\mu^{(e)}((2^2)(3))+$
$\frac{1}{(2^2)(3^2)}\mu^{(e)}((2)(3))=\frac{1}{18}$. Hence,

$$\det \overline{T}_{(e)}=\frac{1}{3888}.$$

4.2. Example. Let $T = \{2, 4, 16\}$. This set is an exponential divisor closed, so we apply the Corollary to Theorem 2.2 directly to calculate the determinant. The GCED matrix defined on T is

$$T_{(e)} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 16 \end{bmatrix},$$

where, $g(2) = 2\mu^{(e)}(2) = 2, g(4) = 2\mu^{(e)}(2^2) + 2^2\mu^{(e)}(2) = 2$, and $g(16) = 2\mu^{(e)}(2^4) + 2^2\mu^{(e)}(2^2) + 2^4\mu^{(e)}(2) = 2(0) + 4(-1) + 16 = 12$. Thus,

$$\det T_{(e)} = \prod_{k=1}^{3} g(x_k) = g(2)g(4)g(16) = (2)(12)(12) = 48.$$

The Reciprocal GCED matrix $\overline{T}_{(e)}$ on T is given as

$$\overline{T}_{(e)} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{16} \end{bmatrix},$$

where, $f(2) = \frac{1}{2}, f(4) = \frac{-1}{4}$ and $f(16) = \frac{-3}{16}$. Thus,

$$\det \overline{T}_{(e)} = \prod_{k=1}^{3} f(x_k) = (\frac{1}{2})(\frac{-1}{4})(\frac{-3}{16}) = \frac{3}{128}.$$

4.3. Example. Let $T = \{2, 4, 16\}$. The 3×3 GCED matrix $T_{(e)}$ defined on T is

$$T_{(e)} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 16 \end{bmatrix}.$$

By Theorem 2.6, we know that $(T)^{-1}{}_{(e)} = (a_{ij})$ where,
$a_{11} = \sum_{2|_e x_k} \frac{\mu^{(e)}(\frac{x_k}{2})\mu^{(e)}(\frac{x_k}{2})}{g(x_k)} = \frac{\mu^{(e)}(2^2)\mu^{(e)}(2^2)}{g(2)} + \frac{\mu^{(e)}(2^2)\mu^{(e)}(2^2)}{g(4)} + \frac{\mu^{(e)}(2^4)\mu^{(e)}(2^4)}{g(16)} = 1$,
$a_{12} = \frac{\mu^{(e)}(2^2)\mu^{(e)}(2)}{g(4)} + \frac{\mu^{(e)}(2^4)\mu^{(e)}(2^2)}{g(16)} = \frac{-1}{2}$, and $a_{13} = \frac{\mu^{(e)}(2^4)\mu^{(e)}(2)}{g(16)} = 0$.
Similarly, one can calculate and verify the following values
$a_{22} = \frac{7}{12}, a_{23} = \frac{-1}{12}$ and $a_{33} = \frac{1}{12}$. So, the inverse of the GCED matrix $T_{(e)}$ is

$$(T)^{-1}{}_{(e)} = \begin{bmatrix} 1 & \frac{-1}{2} & 0 \\ \frac{-1}{2} & \frac{7}{12} & \frac{-1}{12} \\ 0 & \frac{-1}{12} & \frac{1}{12} \end{bmatrix}.$$

The 3×3 Reciprocal GCED matrix on T is given as

$$\overline{T}_{(e)} = \begin{bmatrix} \frac{1}{12} & \frac{1}{6} & \frac{1}{12} \\ \frac{1}{6} & \frac{1}{18} & \frac{1}{18} \\ \frac{1}{12} & \frac{1}{18} & \frac{1}{36} \end{bmatrix}.$$

The inverse of the Reciprocal GCED matrix $\overline{T}_{(e)}$ is calculated to be

$$(\overline{T})^{-1}_{(e)} = \begin{bmatrix} -2 & 4 & 0 \\ 4 & \frac{-28}{3} & \frac{16}{3} \\ 0 & \frac{16}{3} & \frac{-16}{3} \end{bmatrix}.$$

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