Hacettepe Journal of Mathematics and Statistics
Volume 44 (6) (2015), 1315 – 1332

On the fine spectrum of the generalized difference operator defined by a double sequential band matrix over the sequence space ℓ_p, $(1 < p < \infty)$

Feyzi Başar*† and Ali Karaisa‡

Abstract

The main purpose of this paper is to determine the fine spectrum with respect to the Goldberg's classification of the operator $B(\widetilde{r}, \widetilde{s})$ defined by a double sequential band matrix over the sequence space ℓ_p, where $1 < p < \infty$. These results are more general than the spectrum of the generalized difference operator $B(r, s)$ over ℓ_p of Bilgiç and Furkan [Nonlinear Anal. **68**(3)(2008), 499–506].

Keywords: Spectrum of an operator, double sequential band matrix, spectral mapping theorem, the sequence space ℓ_p, Goldberg's classification.

2000 AMS Classification: Primary: 47A10, Secondary: 47B37

Received : 13.08.2013 *Accepted* : 25.09.2014 *Doi* : 10.15672/HJMS.2015449669

1. Introduction

Let X and Y be Banach spaces, and $T : X \to Y$ also be a bounded linear operator. By $R(T)$, we denote the range of T, i.e.,

$$R(T) = \{y \in Y : y = Tx, \ x \in X\}.$$

By $B(X)$, we also denote the set of all bounded linear operators on X into itself. If X is any Banach space and $T \in B(X)$ then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^* f)(x) = f(Tx)$ for all $f \in X^*$ and $x \in X$.

Given an operator $T \in B(X)$, the set

$$\rho(T) := \{\lambda \in \mathbb{C} : T_\lambda = \lambda I - T \text{ is a bijection}\}$$

is called the *resolvent set* of T and its complement with respect to the complex plain

$$\sigma(T) := \mathbb{C} \backslash \rho(T)$$

*Department of Mathematics, Faculty of Arts and Sciences, Fatih University, The Hadımköy Campus, Büyükçekmece, 34500–İstanbul, Turkey, Email: fbasar@fatih.edu.tr
†Corresponding Author.
‡Department of Mathematics-Computer Science, Necmettin Erbakan University Meram Campus, Meram, Konya, Turkey, Email: alikaraisa@hotmail.com, akaraisa@konya.edu.tr

is called the *spectrum* of T. By the closed graph theorem, the inverse operator

$$R(\lambda;T) := (\lambda I - T)^{-1},\ (\lambda \in \rho(T)) \tag{1.1}$$

is always bounded and is usually called *resolvent operator* of T at λ.

2. Subdivisions of the spectrum

In this section, we give the definitions of the parts point spectrum, continuous spectrum, residual spectrum, approximate point spectrum, defect spectrum and compression spectrum of the spectrum. There are many different ways to subdivide the spectrum of a bounded linear operator. Some of them are motivated by applications to physics, in particular, quantum mechanics.

2.1. The point spectrum, continuous spectrum and residual spectrum. Associated with each complex number λ is the perturbed operator $T_\lambda = \lambda I - T$, defined on the same domain $D(T)$ as T. The *spectrum* $\sigma(T,X)$ consist of those $\lambda \in \mathbb{C}$ for which T_λ is not invertible, and the *resolvent* is the mapping from the complement $\sigma(T,X)$ of the spectrum into the algebra of bounded linear operators on X defined by $\lambda \mapsto T_\lambda^{-1}$. The name *resolvent* is appropriate, since T_λ^{-1} helps to solve the equation $T_\lambda x = y$. Thus, $x = T_\lambda^{-1} y$ provided T_λ^{-1} exists. More important, the investigation of properties of T_λ^{-1} will be basic for an understanding of the operator T itself. Naturally, many properties of T_λ and T_λ^{-1} depend on λ, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all λ's in the complex plane such that T_λ^{-1} exists. Boundedness of T_λ^{-1} is another property that will be essential. We shall also ask for what λ's the domain of T_λ^{-1} is dense in X, to name just a few aspects. A *regular value* λ of T is a complex number such that T_λ^{-1} exists and bounded and whose domain is dense in X. For our investigation of T, T_λ and T_λ^{-1}, we need some basic concepts in spectral theory which are given as follows (see [30, pp. 370-371]):

The *resolvent set* $\rho(T,X)$ of T is the set of all regular values λ of T. Furthermore, the spectrum $\sigma(T,X)$ is partitioned into three disjoint sets as follows:

The *point (discrete) spectrum* $\sigma_p(T,X)$ is the set such that T_λ^{-1} does not exist. An $\lambda \in \sigma_p(T,X)$ is called an *eigenvalue* of T.

The *continuous spectrum* $\sigma_c(T,X)$ is the set such that T_λ^{-1} exists and is unbounded and the domain of T_λ^{-1} is dense in X.

The *residual spectrum* $\sigma_r(T,X)$ is the set such that T_λ^{-1} exists (and may be bounded or not) but the domain of T_λ^{-1} is not dense in X.

Therefore, these three subspectras form a disjoint subdivisions

$$\sigma(T,X) = \sigma_p(T,X) \cup \sigma_c(T,X) \cup \sigma_r(T,X). \tag{2.1}$$

To avoid trivial misunderstandings, let us say that some of the sets defined above, may be empty. This is an existence problem which we shall have to discuss. Indeed, it is well-known that $\sigma_c(T,X) = \sigma_r(T,X) = \emptyset$ and the spectrum $\sigma(T,X)$ consists of only the set $\sigma_p(T,X)$ in the finite dimensional case.

2.2. The approximate point spectrum, defect spectrum and compression spectrum. In this subsection, following Appell et al. [9], we define the three more subdivisions of the spectrum called as the *approximate point spectrum*, *defect spectrum* and *compression spectrum*.

Given a bounded linear operator T in a Banach space X, we call a sequence (x_k) in X as a *Weyl sequence* for T if $\|x_k\| = 1$ and $\|Tx_k\| \to 0$, as $k \to \infty$.

In what follows, we call the set

$$\sigma_{ap}(T,X) := \{\lambda \in \mathbb{C} : \text{there exists a Weyl sequence for } \lambda I - T\} \tag{2.2}$$

the *approximate point spectrum of* T. Moreover, the subspectrum

(2.3) $$\sigma_\delta(T,X) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\}$$

is called *defect spectrum* of T.

The two subspectra given by (2.2) and (2.3) form a (not necessarily disjoint) subdivisions

$$\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_\delta(T,X)$$

of the spectrum. There is another subspectrum,

$$\sigma_{co}(T,X) = \{\lambda \in \mathbb{C} : \overline{R(\lambda I - T)} \neq X\}$$

which is often called *compression spectrum* in the literature. The compression spectrum gives rise to another (not necessarily disjoint) decomposition

$$\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_{co}(T,X)$$

of the spectrum. Clearly, $\sigma_p(T,X) \subseteq \sigma_{ap}(T,X)$ and $\sigma_{co}(T,X) \subseteq \sigma_\delta(T,X)$. Moreover, comparing these subspectra with those in (2.1) we note that

$$\begin{aligned} \sigma_r(T,X) &= \sigma_{co}(T,X)\backslash\sigma_p(T,X), \\ \sigma_c(T,X) &= \sigma(T,X)\backslash[\sigma_p(T,X) \cup \sigma_{co}(T,X)]. \end{aligned}$$

Sometimes it is useful to relate the spectrum of a bounded linear operator to that of its adjoint. Building on classical existence and uniqueness results for linear operator equations in Banach spaces and their adjoints.

2.1. Proposition. [9, Proposition 1.3, p. 28] *Spectra and subspectra of an operator* $T \in B(X)$ *and its adjoint* $T^* \in B(X^*)$ *are related by the following relations:*

(a) $\sigma(T^*,X^*) = \sigma(T,X)$.
(b) $\sigma_c(T^*,X^*) \subseteq \sigma_{ap}(T,X)$.
(c) $\sigma_{ap}(T^*,X^*) = \sigma_\delta(T,X)$.
(d) $\sigma_\delta(T^*,X^*) = \sigma_{ap}(T,X)$.
(e) $\sigma_p(T^*,X^*) = \sigma_{co}(T,X)$.
(f) $\sigma_{co}(T^*,X^*) \supseteq \sigma_p(T,X)$.
(g) $\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_p(T^*,X^*) = \sigma_p(T,X) \cup \sigma_{ap}(T^*,X^*)$.

The relations (c)–(f) show that the approximate point spectrum is in a certain sense dual to defect spectrum, and the point spectrum dual to the compression spectrum.

The equality (g) implies, in particular, that $\sigma(T,X) = \sigma_{ap}(T,X)$ if X is a Hilbert space and T is normal. Roughly speaking, this shows that normal (in particular, self-adjoint) operators on Hilbert spaces are most similar to matrices in finite dimensional spaces (see [9]).

2.3. Goldberg's classification of spectrum. If X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$:

(A) $R(T) = X$.
(B) $R(T) \neq \overline{R(T)} = X$.
(C) $\overline{R(T)} \neq X$.

and

(1) T^{-1} exists and is continuous.
(2) T^{-1} exists but is discontinuous.
(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3. If an operator is in state C_2 for example, then $\overline{R(T)} \neq X$ and T^{-1} exist but is discontinuous (see [22]).

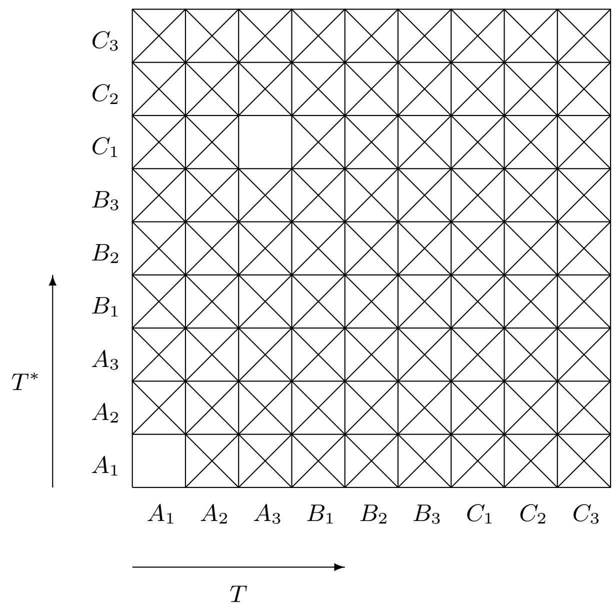

Table 1.1: State diagram for $B(X)$ and $B(X^*)$ for a non-reflective Banach space X

If λ is a complex number such that $T_\lambda = \lambda I - T \in A_1$ or $T_\lambda = \lambda I - T \in B_1$, then $\lambda \in \rho(T, X)$. All scalar values of λ not in $\rho(T, X)$ comprise the spectrum of T. The further classification of $\sigma(T, X)$ gives rise to the fine spectrum of T. That is, $\sigma(T, X)$ can be divided into the subsets $A_2\sigma(T, X) = \emptyset$, $A_3\sigma(T, X)$, $B_2\sigma(T, X)$, $B_3\sigma(T, X)$, $C_1\sigma(T, X)$, $C_2\sigma(T, X)$, $C_3\sigma(T, X)$. For example, if $T_\lambda = \lambda I - T$ is in a given state, C_2 (say), then we write $\lambda \in C_2\sigma(T, X)$.

By the definitions given above, we can illustrate the subdivisions (2.1) in the following table:

		1	2	3
		T_λ^{-1} exists and is bounded	T_λ^{-1} exists and is unbounded	T_λ^{-1} does not exist
A	$R(\lambda I - T) = X$	$\lambda \in \rho(T, X)$	–	$\lambda \in \sigma_p(T, X)$ $\lambda \in \sigma_{ap}(T, X)$
B	$\overline{R(\lambda I - T)} = X$	$\lambda \in \rho(T, X)$	$\lambda \in \sigma_c(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$	$\lambda \in \sigma_p(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$
C	$\overline{R(\lambda I - T)} \neq X$	$\lambda \in \sigma_r(T, X)$ $\lambda \in \sigma_\delta(T, X)$ $\lambda \in \sigma_{co}(T, X)$	$\lambda \in \sigma_r(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$ $\lambda \in \sigma_{co}(T, X)$	$\lambda \in \sigma_p(T, X)$ $\lambda \in \sigma_{ap}(T, X)$ $\lambda \in \sigma_\delta(T, X)$ $\lambda \in \sigma_{co}(T, X)$

Table 1.2: Subdivisions of spectrum of a linear operator

Observe that the case in the first row and second column cannot occur in a Banach space X, by the closed graph theorem. If we are not in the third column, i.e., if λ is not an eigenvalue of T, we may always consider the resolvent operator T_λ^{-1} (on a possibly "thin" domain of definition) as "algebraic" inverse of $\lambda I - T$.

From now on, we should note that the index p has different meanings in the notation of the spaces ℓ_p, ℓ_p^* and the point spectrums $\sigma_p[B(\widetilde{r}, \widetilde{s}), \ell_p]$, $\sigma_p[B(\widetilde{r}, \widetilde{s})^*, \ell_p^*]$ which occur in theorems given in Section 3.

By a *sequence space*, we understand a linear subspace of the space $\omega = \mathbb{C}^{\mathbb{N}_1}$ of all complex sequences which contains ϕ, the set of all finitely non–zero sequences, where $\mathbb{N}_1$ denotes the set of positive integers. We write ℓ_∞, c, c_0 and bv for the spaces of all bounded, convergent, null and bounded variation sequences which are the Banach spaces with the sup-norm $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|$ and $\|x\|_{bv} = \sum_{k=0}^\infty |x_k - x_{k+1}|$ while ϕ is not a Banach space with respect to any norm, respectively, where $\mathbb{N} = \{0, 1, 2, \dots\}$. Also by ℓ_p, we denote the space of all p–absolutely summable sequences which is a Banach space with the norm $\|x\|_p = \left(\sum_{k=0}^\infty |x_k|^p\right)^{1/p}$, where $1 \leq p < \infty$.

Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $n, k \in \mathbb{N}$, and write

$$(Ax)_n = \sum_k a_{nk} x_k \;; \quad (n \in \mathbb{N},\; x \in D_{00}(A)), \tag{2.4}$$

where $D_{00}(A)$ denotes the subspace of w consisting of $x \in w$ for which the sum exists as a finite sum. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞ and we shall use the convention that any term with negative subscript is equal to naught. More generally if μ is a normed sequence space, we can write $D_\mu(A)$ for the $x \in w$ for which the sum in (2.4) converges in the norm of μ. We write

$$(\lambda : \mu) = \{A : \lambda \subseteq D_\mu(A)\}$$

for the space of those matrices which send the whole of the sequence space λ into μ in this sense.

We give a short survey concerning with the spectrum and the fine spectrum of the linear operators defined by some particular triangle matrices over certain sequence spaces. The fine spectrum of the Cesàro operator of order one on the sequence space ℓ_p studied by Gonzàlez [23], where $1 < p < \infty$. Also, weighted mean matrices of operators on ℓ_p investigated by Cartlidge [15]. The spectrum of the Cesàro operator of order one on the sequence spaces bv_0 and bv investigated by Okutoyi [32, 33]. The spectrum and fine spectrum of the Rhally operators on the sequence spaces c_0, c and ℓ_p examined by Yıldırım [41, 42, 43, 44]. The fine spectrum of the difference operator Δ over the sequence spaces c_0 and c studied by Altay and Başar [5]. The same authors also worked the fine spectrum of the generalized difference operator $B(r,s)$ over c_0 and c, in [6].

The fine spectra of Δ over ℓ_1 and bv studied by Kayaduman and Furkan [29]. The fine spectra of the difference operator Δ over the sequence spaces ℓ_p and bv_p studied by Akhmedov and Başar [2, 3], where bv_p is the space of p-bounded variation sequences and introduced by Başar and Altay [10] with $1 \leq p < \infty$. The fine spectrum of $B(r,s,t)$ over the sequence spaces c_0 and c studied by Furkan et al. [20]. de Malafosse [31] studied the spectrum and the fine spectrum of the difference operator on the sequence spaces s_r, where s_r denotes the Banach space of all sequences $x = (x_k)$ normed by $\|x\|_{s_r} = \sup_{k\in\mathbb{N}} \frac{|x_k|}{r^k}$, $(r > 0)$. Altay and Karakuş [7] determined the fine spectrum of the Zweier matrix which is a band matrix as an operator over the sequence spaces ℓ_1 and bv. Farés and de Malafosse [19] studied the spectra of the difference operator on the sequence spaces $\ell_p(\alpha)$, where (α_n) denotes the sequence of positive reals and $\ell_p(\alpha)$ is the Banach space of all sequences $x = (x_n)$ normed by $\|x\|_{\ell_p(\alpha)} = \left[\sum_{n=1}^{\infty} (|x_n|/\alpha_n)^p\right]^{1/p}$ with $p \geq 1$. The fine spectrum of the operator $B(r,s)$ over ℓ_p and bv_p studied by Bilgiç and Furkan [11]. Besides, the fine spectrum with respect to the Goldberg's classification of the operator $B(r,s,t)$ defined by a triple band matrix over the sequence spaces ℓ_p and bv_p with $1 < p < \infty$ studied by Furkan et al. [21]. In 2010, Srivastava and Kumar [36] determined the spectra and the fine spectra of generalized difference operator Δ_ν on ℓ_1, where Δ_ν is defined by $(\Delta_\nu)_{nn} = \nu_n$ and $(\Delta_\nu)_{n+1,n} = -\nu_n$ for all $n \in \mathbb{N}$, under certain conditions on the sequence $\nu = (\nu_n)$ and they generalized these results by the generalized difference operator Δ_{uv} defined by $\Delta_{uv}x = (u_n x_n + v_{n-1}x_{n-1})_{n\in\mathbb{N}}$ for all $n \in \mathbb{N}$, (see [38]).

Recently, Altun [8] have obtained the fine spectra of the Toeplitz operators, which are represented by upper and lower triangular n-band infinite matrices, over the sequence spaces c_0 and c. Later, Karaisa [26, 25] have determined the fine spectrum of the generalized difference operator $A(\widetilde{r}, \widetilde{s})$, defined as an upper triangular double-band matrix with the convergent sequences $\widetilde{r} = (r_k)$ and $\widetilde{s} = (s_k)$ having certain properties, over the sequence space ℓ_p, where $(1 \leq p < \infty)$.

Quite recently, Akhmedov and El-Shabrawy [4], and El-Shabrawy [28] have obtained the fine spectrum of the generalized difference operator $\Delta_{a,b}$, defined as a double band matrix with the convergent sequences $\widetilde{a} = (a_k)$ and $\widetilde{b} = (b_k)$ having certain properties, over the sequence spaces c and c_0. Karaisa and Başar [13, 14, 27] have determined the fine spectrum of the upper triangular triple band matrix $A(r,s,t)$ over some sequence spaces. Yeşilkayagil and Başar [40] have computed the fine spectrum with respect to Goldberg's classification of the operator defined by the lambda matrix over the sequence spaces c_0 and c. Finally, Dündar and Başar [18] have studied the fine spectrum of the matrix operator Δ^+ defined by an upper triangle double band matrix acting on the sequence space c_0 with respect to the Goldberg's classification. At this stage, the following table may be useful:

$\sigma(A,\lambda)$	$\sigma_p(A,\lambda)$	$\sigma_c(A,\lambda)$	$\sigma_r(A,\lambda)$	refer to:
$\sigma(C_1^p,c)$	-	-	-	[39]
$\sigma(W,c)$	-	-	-	[35]
$\sigma(C_1,c_0)$	-	-	-	[34]
$\sigma(C_1,c_0)$	$\sigma_p(C_1,c_0)$	$\sigma_c(C_1,c_0)$	$\sigma_r(C_1,c_0)$	[1]
$\sigma(C_1,bv)$	-	-	-	[33]
$\sigma(C_1^p,c_0)$	-	-	-	[16]
$\sigma(\Delta,s_r)$	-	-	-	[31]
$\sigma(\Delta,c_0)$	-	-	-	[31]
$\sigma(\Delta,c)$	-	-	-	[31]
$\sigma(\Delta^{(1)},c)$	$\sigma_p(\Delta^{(1)},c)$	$\sigma_c(\Delta^{(1)},c)$	$\sigma_r(\Delta^{(1)},c)$	[5]
$\sigma(\Delta^{(1)},c_0)$	$\sigma_p(\Delta^{(1)},c_0)$	$\sigma_c(\Delta^{(1)},c_0)$	$\sigma_r(\Delta^{(1)},c_0)$	[5]
$\sigma(B(r,s),\ell_p)$	$\sigma_p(B(r,s),\ell_p)$	$\sigma_c(B(r,s),\ell_p)$	$\sigma_r(B(r,s),\ell_p)$	[12]
$\sigma(B(r,s),bv_p)$	$\sigma_p(B(r,s),bv_p)$	$\sigma_c(B(r,s),bv_p)$	$\sigma_r(B(r,s),bv_p)$	[12]
$\sigma(B(r,s,t),\ell_p)$	$\sigma_p(B(r,s,t),\ell_p)$	$\sigma_c(B(r,s,t),\ell_p)$	$\sigma_r(B(r,s,t),\ell_p)$	[21]
$\sigma(B(r,s,t),bv_p)$	$\sigma_p(B(r,s,t),bv_p)$	$\sigma_c(B(r,s,t),bv_p)$	$\sigma_r(B(r,s,t),bv_p)$	[21]
$\sigma(\Delta_{a,b},c)$	$\sigma_p(\Delta_{a,b},c)$	$\sigma_c(\Delta_{a,b},c)$	$\sigma_r(\Delta_{a,b},c)$	[4]
$\sigma(\Delta_\nu,\ell_1)$	$\sigma_p(\Delta_\nu,\ell_1)$	$\sigma_c(\Delta_\nu,\ell_1)$	$\sigma_r(\Delta_\nu,\ell_1)$	[36]
$\sigma(\Delta_{uv}^2,c_0)$	$\sigma_p(\Delta_{uv}^2,c_0)$	$\sigma_c(\Delta_{uv}^2,c_0)$	$\sigma_r(\Delta_{uv}^2,c_0)$	[37]
$\sigma(\Delta_{uv},\ell_1)$	$\sigma_p(\Delta_{uv},\ell_1)$	$\sigma_c(\Delta_{uv},\ell_1)$	$\sigma_r(\Delta_{uv},\ell_1)$	[38]
$\sigma(\Lambda,c_0)$	$\sigma_p(\Lambda,c_0)$	$\sigma_c(\Lambda,c_0)$	$\sigma_r(\Lambda,c_0)$	[40]
$\sigma(\Delta^+,c_0)$	$\sigma_p(\Delta^+,c_0)$	$\sigma_c(\Delta^+,c_0)$	$\sigma_r(\Delta^+,c_0)$	[18]

Table 1.3: Spectrum and fine spectrum of some triangle matrices in certain sequence spaces.

In this paper, we study the fine spectrum of the generalized difference operator $B(\widetilde{r},\widetilde{s})$ defined by a double sequential band matrix acting on the sequence space ℓ_p with respect to the Goldberg's classification, where $1<p<\infty$. Additionally, we give the approximate point spectrum, defect spectrum and compression spectrum of the matrix operator $B(\widetilde{r},\widetilde{s})$ over the space ℓ_p.

We quote some lemmas which are needed in proving the theorems given in Section 3.

2.2. Lemma. [17, p. 253, Theorem 34.16] *The matrix $A=(a_{nk})$ gives rise to a bounded linear operator $T\in B(\ell_1)$ from ℓ_1 to itself if and only if the supremum of ℓ_1 norms of the columns of A is bounded.*

2.3. Lemma. [17, p. 245, Theorem 34.3] *The matrix $A=(a_{nk})$ gives rise to a bounded linear operator $T\in B(\ell_\infty)$ from ℓ_∞ to itself if and only if the supremum of ℓ_1 norms of the rows of A is bounded.*

2.4. Lemma. [17, p. 254, Theorem 34.18] *Let $1<p<\infty$ and $A\in(\ell_\infty:\ell_\infty)\cap(\ell_1:\ell_1)$. Then, $A\in(\ell_p:\ell_p)$.*

Let $\widetilde{r}=(r_k)$ and $\widetilde{s}=(s_k)$ be sequences whose entries either constants or distinct none-zero real numbers satisfying the following conditions:

$$\lim_{k\to\infty} r_k = r,$$
$$\lim_{k\to\infty} s_k = s\neq 0$$
$$|r_k-r|\neq|s|.$$

Then, we define the sequential generalized difference matrix $B(\widetilde{r},\widetilde{s})$ by

$$B(\widetilde{r},\widetilde{s}) = \begin{bmatrix} r_0 & 0 & 0 & 0 & \dots \\ s_0 & r_1 & 0 & 0 & \dots \\ 0 & s_1 & r_2 & 0 & \dots \\ 0 & 0 & s_2 & r_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Therefore, we introduce the operator $B(\widetilde{r},\widetilde{s})$ from ℓ_p to itself by

$$B(\widetilde{r},\widetilde{s})x = (r_k x_k + s_{k-1}x_{k-1})_{k=0}^{\infty} \text{ with } x_{-1} = 0, \text{ where } x = (x_k) \in \ell_p.$$

3. The Fine spectrum of the Operator $B(\widetilde{r},\widetilde{s})$ on the Operator sequence space ℓ_p

3.1. Theorem. *The operator $B(\widetilde{r},\widetilde{s}) : \ell_p \to \ell_p$ is a bounded linear operator and*

$$(|r_k|^p + |s_k|^p)^{1/p} \le \|B(\widetilde{r},\widetilde{s}\|_p \le \|\widetilde{s}\|_\infty + \|\widetilde{r}\|_\infty. \tag{3.1}$$

Proof. Since the linearity of the operator $B(\widetilde{r},\widetilde{s})$ does not hard, we omit the detail.

Now we prove that (3.1) holds for the operator $B(\widetilde{r},\widetilde{s})$ on the space ℓ_p. It is trivial that $B(\widetilde{r},\widetilde{s})e^{(k)} = (0, 0, \dots, r_k, s_k, 0, 0, \dots)$ for $e^{(k)} \in \ell_p$. Therefore, we have

$$\frac{\|B(\widetilde{r},\widetilde{s})e^{(k)}\|_p}{\|e^{(k)}\|_p} = (|r_k|^p + |s_k|^p)^{1/p}$$

which implies that

$$(|r_k|^p + |s_k|^p)^{1/p} \le \|B(\widetilde{r},\widetilde{s})\|_p. \tag{3.2}$$

Let $x = (x_k) \in \ell_p$, where $p > 1$. Then, since $(s_{k-1}x_{k-1}), (r_k x_k) \in \ell_p$ it is easy to see by Minkowski's inequality that

$$\begin{aligned} \|B(\widetilde{r},\widetilde{s})x\|_p &= \left(\sum_{k=0}^{\infty} |s_{k-1}x_{k-1} + r_k x_k|^p\right)^{1/p} \\ &\le \left(\sum_{k=0}^{\infty} |s_{k-1}x_{k-1}|^p\right)^{1/p} + \left(\sum_{k=0}^{\infty} |r_k x_k|^p\right)^{1/p} \\ &\le (\|\widetilde{s}\|_\infty + \|\widetilde{r}\|_\infty)\|x\|_p \end{aligned}$$

which leads us to the the result that

$$\|B(\widetilde{r},\widetilde{s})\|_p \le \|\widetilde{s}\|_\infty + \|\widetilde{r}\|_\infty. \tag{3.3}$$

Therefore, by combining the inequalities in (3.2) and (3.3) we have (3.1), as desired. □

3.2. Theorem. *Let $\mathcal{A} = \{\alpha \in \mathbb{C} : |r - \alpha| \le |s|\}$ and $\mathcal{B} = \{r_k : k \in \mathbb{N}, |r - r_k| > |s|\}$. Then, the set $\mathcal{B}$ is finite and $\sigma[B(\widetilde{r},\widetilde{s}), \ell_p] = \mathcal{A} \cup \mathcal{B}$.*

Proof. We firstly prove that

$$\sigma[B(\widetilde{r},\widetilde{s}), \ell_p] \subseteq \mathcal{A} \cup \mathcal{B} \tag{3.4}$$

which is equivalent to show that $\alpha \in \mathbb{C}$ such that $|r - \alpha| > |s|$ and $\alpha \ne r_k$ for all $k \in \mathbb{N}$ implies $\alpha \notin \sigma[B(\widetilde{r},\widetilde{s}), \ell_p]$. It is easy to see that $\mathcal{B}$ is finite and $\{r_k \in \mathbb{C} : k \in \mathbb{N}\} \subseteq \mathcal{A} \cup \mathcal{B}$.

It is immediate that $B(\widetilde{r},\widetilde{s})-\alpha I$ is a triangle and so has an inverse. Let $y=(y_k)\in\ell_1$. Then, by solving the equation

$$\begin{aligned}[B(\widetilde{r},\widetilde{s})-\alpha I]x &= \begin{bmatrix} r_0-\alpha & 0 & 0 & \dots \\ s_0 & r_1-\alpha & 0 & \dots \\ 0 & s_1 & r_2-\alpha & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} \\ &= \begin{bmatrix} (r_0-\alpha)x_0 \\ s_0x_0+(r_1-\alpha)x_1 \\ s_1x_1+(r_2-\alpha)x_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix}\end{aligned}$$

for $x=(x_k)$ in terms of y, we obtain

$$\begin{aligned} x_0 &= \frac{y_0}{r_0-\alpha}, \\ x_1 &= \frac{y_1}{r_1-\alpha}+\frac{-s_0y_0}{(r_1-\alpha)(r_0-\alpha)}, \\ x_2 &= \frac{y_2}{r_2-\alpha}+\frac{-s_1y_1}{(r_2-\alpha)(r_1-\alpha)}+\frac{s_0s_1y_0}{(r_2-\alpha)(r_1-\alpha)(r_0-\alpha)}, \\ &\vdots \\ x_k &= \frac{(-1)^k s_0s_1s_2\cdots s_{k-1}y_0}{(r_0-\alpha)(r_1-\alpha)(r_2-\alpha)\cdots(r_k-\alpha)}+\cdots-\frac{s_{k-1}y_{k-1}}{(r_k-\alpha)(r_{k-1}-\alpha)}+\frac{y_k}{r_k-\alpha}, \\ &\vdots \end{aligned}$$

Therefore, we obtain $B=(b_{nk})=[B(\widetilde{r},\widetilde{s})-\alpha I]^{-1}$ as follows:

$$(b_{nk})=\begin{bmatrix} \frac{1}{r_0-\alpha} & 0 & 0 & \dots \\ \frac{-s_0}{(r_1-\alpha)(r_0-\alpha)} & \frac{1}{r_1-\alpha} & 0 & \dots \\ \frac{s_0s_1}{(r_0-\alpha)(r_1-\alpha)(r_2-\alpha)} & \frac{-s_1}{(r_2-\alpha)(r_1-\alpha)} & \frac{1}{r_2-\alpha} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then, $\sum_k |x_k|\le\sum_k S^k|y_k|$, where

$$\begin{aligned} S^k &= \left|\frac{1}{r_k-\alpha}\right|+\left|\frac{s_k}{(r_k-\alpha)(r_{k+1}-\alpha)}\right|+\left|\frac{s_ks_{k+1}}{(r_k-\alpha)(r_{k+1}-\alpha)(r_{k+2}-\alpha)}\right|+\cdots. \\ S_n^k &= \left|\frac{1}{r_k-\alpha}\right|+\left|\frac{s_k}{(r_k-\alpha)(r_{k+1}-\alpha)}\right|+\left|\frac{s_ks_{k+1}}{(r_k-\alpha)(r_{k+1}-\alpha)(r_{k+2}-\alpha)}\right|+\cdots \\ &\quad+\left|\frac{s_ks_{k+1}\cdots s_{n+k}}{(r_k-\alpha)(r_{k+1}-\alpha)(r_{k+2}-\alpha)\cdots(r_{k+n+1}-\alpha)}\right| \quad \text{for all } k,n\in\mathbb{N}. \end{aligned}$$

Then, since

$$S_n=\lim_{k\to\infty}S_n^k=\left|\frac{1}{r-\alpha}\right|+\left|\frac{s}{(r-\alpha)^2}\right|+\left|\frac{s^2}{(r-\alpha)^3}\right|+\cdots+\left|\frac{s^{n+1}}{(r-\alpha)^{n+2}}\right|,$$

we have

$$S=\lim_{n\to\infty}S_n=\left|\frac{1}{r-\alpha}\right|\left(1+\left|\frac{s}{r-\alpha}\right|+\left|\frac{s}{r-\alpha}\right|^2+\cdots\right)<\infty, \tag{3.5}$$

since $|r-\alpha|>|s|$. Then we have

$$\lim_{n\to\infty}\lim_{k\to\infty}S_n^k=\lim_{k\to\infty}\lim_{n\to\infty}S_n^k=S$$

and $(S^k)_k \in c$. Thus,

$$\sum_k |x_k| \leq \sum_k S^k |y_k| \leq \|(S^k)\|_\infty \sum_k |y_k| < \infty,$$

since $y \in \ell_1$. This shows that $[B(\widetilde{r},\widetilde{s}) - \alpha I]^{-1} \in (\ell_1 : \ell_1)$.

Suppose that $y = (y_k) \in \ell_\infty$. By solving the equation $[B(\widetilde{r},\widetilde{s}) - \alpha I]x = y$, for $x = (x_k)$ in terms of y, we get

$$|x_k| \leq S_k \left(\sup_{k\in\mathbb{N}} |y_k| \right),$$

where;

$$\begin{aligned} S_k \quad = \quad & \left|\frac{1}{r_k - \alpha}\right| + \left|\frac{s_{k-1}}{(r_{k-1} - \alpha)(r_k - \alpha)}\right| + \left|\frac{s_{k-1}s_{k-2}}{(r_{k-2} - \alpha)(r_{k-1} - \alpha)(r_k - \alpha)}\right| \\ & + \cdots + \left|\frac{s_0 s_1 \ldots s_{k-1}}{(r_0 - \alpha)(r_1 - \alpha)\cdots(r_k - \alpha)}\right|. \end{aligned}$$

Now, we prove that $(S_k) \in \ell_\infty$. Since $\lim_{k\to\infty} |s_k/(r_k - \alpha)| = |s/(r - \alpha)| = p < 1$, then there exists $k_0 \in \mathbb{N}$ such that $|s_k/(r_k - \alpha)| < p_0$ with $p_0 < 1$, for all $k \geq k_0 + 1$,

$$\begin{aligned} S_k \quad = \quad & \frac{1}{|r_k - \alpha|}\left[1 + \left|\frac{s_{k-1}}{r_{k-1} - \alpha}\right| + \left|\frac{s_{k-1}s_{k-2}}{(r_{k-1} - \alpha)(r_{k-2} - \alpha)}\right|\right. \\ & \left. + \cdots + \left|\frac{s_{k-1}s_{k-2}\ldots s_{k_0+1}s_{k_0}\ldots s_0}{(r_{k-1} - \alpha)(r_{k-2} - \alpha)\cdots(r_{k_0+1} - \alpha)(r_{k_0} - \alpha)\cdots(r_0 - \alpha)}\right|\right] \\ \leq \quad & \frac{1}{|r_k - \alpha|}\left[1 + p_0 + p_0^2 + \cdots + p_0^{k-k_0} + p_0^{k-k_0}\frac{|s_{k_0-1}|}{|r_{k_0-1} - \alpha|}\right. \\ & \left. + \cdots + p_0^{k-k_0}\left|\frac{s_{k_0-1}s_{k_0-2}\ldots s_0}{(r_{k_0-1} - \alpha)(r_{k_0-2} - \alpha)\cdots(r_0 - \alpha)}\right|\right]. \end{aligned}$$

Therefore;

$$S_k \leq \frac{1}{|r_k - \alpha|}\left(1 + p_0 + p_0^2 + \cdots p_0^{k-k_0} + p_0^{k-k_0} M k_0\right),$$

where

$$Mk_0 = 1 + \left|\frac{s_{k_0-1}}{r_{k_0-1} - \alpha}\right| + \left|\frac{s_{k_0-1}s_{k_0-2}}{(r_{k_0-1} - \alpha)(r_{k_0-2} - \alpha)}\right| + \cdots + \left|\frac{s_{k_0-1}s_{k_0-2}\ldots s_0}{(r_{k_0-1} - \alpha)(r_{k_0-2} - \alpha)\cdots(r_0 - \alpha)}\right|.$$

Then, $Mk_0 \geq 1$ and so

$$S_k \leq \frac{Mk_0}{|r_k - \alpha|}\left(1 + p_0 + p_0^2 + \cdots + p_0^{k-k_0}\right).$$

But there exists $k_1 \in \mathbb{N}$ and a real number p_1 such that $\frac{1}{|r_k - \alpha|} < p_1$ for all $k \geq k_1$. Then, $S_k \leq (Mp_1k_0)/(1-p_0)$ for all $k > max\{k_0, k_1\}$. Hence, $\sup_{k\in\mathbb{N}} S_k < \infty$. This shows that $\|x\|_\infty \leq \|(S_k)\|_\infty \|y\|_\infty < \infty$ which means $[B(\widetilde{r},\widetilde{s}) - \alpha I]^{-1} \in (\ell_\infty : \ell_\infty)$. By Lemma 2.4, we have

$$(3.6) \qquad [B(\widetilde{r},\widetilde{s}) - \alpha I]^{-1} \in (\ell_p : \ell_p) \quad \text{for} \quad |r - \alpha| > |s| \quad \text{and} \quad \alpha \neq r_k.$$

Hence,

$$(3.7) \qquad \sigma[B(\widetilde{r},\widetilde{s}), \ell_p] \subseteq \mathcal{A} \cup \mathcal{B}.$$

Now we show that $\mathcal{A} \cup \mathcal{B} \subseteq \sigma[B(\widetilde{r},\widetilde{s}), \ell_p]$. We assume that $\alpha \neq r_k$ for all $k \in \mathbb{N}$ and $\alpha \in \mathbb{C}$ with $|r - \alpha| \leq |s|$. Clearly, $B(\widetilde{r},\widetilde{s}) - \alpha I$ is a triangle and so, $[B(\widetilde{r},\widetilde{s}) - \alpha I]^{-1}$ exists. For $e^{(0)} = (1, 0, 0, \ldots) \in \ell_p$, $[B(\widetilde{r},\widetilde{s}) - \alpha I]^{-1} e^{(0)} = S^0 \notin \ell_p$, and so $[B(\widetilde{r},\widetilde{s}) - \alpha I]^{-1} \notin$

$B(\ell_p)$. Then, $\alpha \in \sigma[B(\widetilde{r},\widetilde{s}),\ell_p]$. Case $r_k = \alpha$ for some k. We then have either $\alpha = r$ or $\alpha = r_k \neq r$ for some k. We have

$$\begin{aligned}[B(\widetilde{r},\widetilde{s}) - r_k I]x &= \begin{bmatrix} r_0 - r_k & 0 & 0 & \cdots \\ s_0 & r_1 - r_k & 0 & \cdots \\ 0 & s_1 & r_2 - r_k & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} \\ &= \begin{bmatrix} (r_0 - r_k)x_0 \\ s_0x_0 + (r_1 - r_k)x_1 \\ s_1x_1 + (r_2 - r_k)x_2 \\ \vdots \\ s_{k-2}x_{k-2} + (r_{k-1} - r_k)x_{k-1} \\ s_{k-1}x_{k-1} + (r_k - r_k)x_k \\ s_kx_k + (r_{k+1} - r_k)x_{k+1} \\ \vdots \end{bmatrix}.\end{aligned}$$

Let $\alpha = r_k = r$ for all k and solving the equation $[B(\widetilde{r},\widetilde{s}) - \alpha I]x = \theta$ we obtain $x_0 = x_1 = x_2 = \cdots = 0$ which shows that $B(\widetilde{r},\widetilde{s}) - \alpha I$ is one to one but its range $R[B(\widetilde{r},\widetilde{s}) - \alpha I] = \{y = (y_k) \in \omega : y \in \ell_p,\ y_1 = 0\}$ is not dense in ℓ_p and $\alpha = r \in \sigma[B(\widetilde{r},\widetilde{s}),\ell_p]$. Now let $\alpha = r_k$ for some k. Then the equation $[B(\widetilde{r},\widetilde{s}) - \alpha I]x = \theta$ yields

$$x_0 = x_1 = x_2 = \cdots = x_{k-1} = 0 \ \text{ and } \ x_n = \frac{s_{n-1}}{r_k - r_n}x_{n-1} \ \text{ for all } \ n \geq k+1.$$

This shows that $B(\widetilde{r},\widetilde{s}) - \alpha I$ is not injective for $\alpha = r_k$ such that $|\alpha - r| > |s|$. Therefore $[B(\widetilde{r},\widetilde{s}) - \alpha I]^{-1}$ does not exist. So $r_k \in \sigma[B(\widetilde{r},\widetilde{s}),\ell_p]$ for all $k \in \mathbb{N}$. Thus,

$$\mathcal{A} \cup \mathcal{B} \subseteq \sigma[B(\widetilde{r},\widetilde{s}),\ell_p]. \tag{3.8}$$

Combining the inclusions (3.7) and (3.8), we get $\sigma[B(\widetilde{r},\widetilde{s}),\ell_p] = \mathcal{A} \cup \mathcal{B}$.

This completes the proof. □

Throughout the paper, by $\mathcal{C}$ and $\mathcal{SD}$ we denote the set of constant sequences and the set of sequences of distinct none-zero real numbers, respectively.

3.3. Theorem. $\sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p] = \begin{cases} \emptyset &, \quad \widetilde{r},\widetilde{s} \in \mathcal{C}, \\ \mathcal{B} &, \quad \widetilde{r},\widetilde{s} \in \mathcal{SD}, \end{cases}$

Proof. We prove the theorem by dividing into two parts.

Part 1. Assume that $\widetilde{r},\widetilde{s} \in \mathcal{C}$. Consider $B(\widetilde{r},\widetilde{s})x = \alpha x$ for $x \neq \theta = (0,0,0,\ldots)$ in ℓ_p. Then, by solving the system of linear equations

$$\begin{aligned} rx_0 &= \alpha x_0 \\ sx_0 + rx_1 &= \alpha x_1 \\ sx_1 + rx_2 &= \alpha x_2 \\ &\vdots \\ sx_{k-1} + rx_k &= \alpha x_k \\ &\vdots \end{aligned}$$

Case $\alpha = r$. Let x_{n_0} is the first non zero entry of the sequence $x = (x_n)$ and $\alpha = r$, then we get $sx_{n_0} + rx_{n_0+1} = \alpha x_{n_0+1}$ which implies $x_{n_0} = 0$ which contradicts the assumption $x_{n_0} \neq 0$. Hence, the equation $B(\widetilde{r},\widetilde{s})x = \alpha x$ has no solution $x \neq \theta$.

Part 2. Assume that $\widetilde{r},\widetilde{s} \in \mathcal{SD}$. Then, by solving the equation $B(\widetilde{r},\widetilde{s})x = \alpha x$ for $x \neq \theta = (0,0,0,\ldots)$ in ℓ_p we obtain $(r_0 - \alpha)x_0 = 0$ and $(r_{k+1} - \alpha)x_{k+1} + s_kx_k = 0$ for all

$k \in \mathbb{N}$. Hence, for all $\alpha \notin \{r_k : k \in \mathbb{N}\}$, we have $x_k = 0$ for all $k \in \mathbb{N}$, which contradicts our assumption. So, $\alpha \notin \sigma_p[B(\widetilde{r}, \widetilde{s}), \ell_p]$. This shows that $\sigma_p[B(\widetilde{r}, \widetilde{s}), \ell_p] \subseteq \{r_k : k \in \mathbb{N}\}\backslash\{r\}$. Now, we prove that

$$\alpha \in \sigma_p[B(\widetilde{r}, \widetilde{s}), \ell_p] \text{ if and only if } \alpha \in \mathcal{B}.$$

Let $\alpha \in \sigma_p[B(\widetilde{r}, \widetilde{s}), \ell_p]$. We consider the case $\alpha = r_0$ and $\alpha = r_k$ for some $k \geq 1$. Then, by solving the equation $B(\widetilde{r}, \widetilde{s})x = \alpha x$ for $x \neq \theta = (0, 0, 0, \ldots)$ in ℓ_p with $\alpha = r_0$

$$x_k = \frac{s_0 s_1 s_2 \ldots s_{k-1}}{(r_0 - r_k)(r_0 - r_{k-1})(r_0 - r_{k-2}) \cdots (r_0 - r_1)} x_0 \quad \text{for all} \quad k \geq 1$$

which can expressed by the recursion relation

$$x_k = \frac{s_{k-1}}{r_0 - r_k} x_{k-1} \quad \text{for all} \quad k \in \mathbb{N}_1.$$

Therefore, since

$$\lim_{k \to \infty} \left| \frac{x_k}{x_{k-1}} \right|^p = \lim_{k \to \infty} \left| \frac{s_{k-1}}{r_k - r_0} \right|^p = \left| \frac{s}{r - r_0} \right|^p \leq 1,$$

But $\left| \frac{s}{r - r_0} \right|^p \neq 1$. Then $\alpha = r_0 \in \{r_k : k \in \mathbb{N}, |r_k - r| > |s|\} = \mathcal{B}$.
If we choose $\alpha = r_k \neq r$ for all $k \in \mathbb{N}_1$, then we get $x_0 = x_1 = x_2 = \cdots = x_{k-1} = 0$ and

$$x_{n+1} = \frac{s_n s_{n-1} s_{n-2} \ldots s_k}{(r_k - r_{n+1})(r_k - r_n)(r_k - r_{n-1}) \cdots (r_k - r_{k+1})} x_k \quad \text{for all} \quad n \geq k$$

which can also be expressed by the recursion relation

$$x_{n+1} = \frac{s_n}{r_k - r_{n+1}} x_n \quad \text{for all} \quad n \geq k.$$

Therefore, we have

$$\lim_{n \to \infty} \left| \frac{x_{n+1}}{x_n} \right|^p = \lim_{n \to \infty} \left| \frac{s_n}{r_{n+1} - r_k} \right|^p = \left| \frac{s}{r - r_k} \right|^p \leq 1.$$

But $\left| \frac{s}{r - r_k} \right| \neq 1$. Then $\alpha = r_k \in \{r_k : k \in \mathbb{N}, |r_k - r| > |s|\} = \mathcal{B}$. Thus $\sigma_p[B(\widetilde{r}, \widetilde{s}), \ell_p] \subseteq \mathcal{B}$.
Conversely, let $\alpha \in \mathcal{B}$. Then, there exists $k \in \mathbb{N}$, $\alpha = r_k \neq r$ and

$$\lim_{n \to \infty} \left| \frac{s_n}{r_{n+1} - r_k} \right| = \left| \frac{s}{r - r_k} \right| < 1,$$

so we have $x \in \ell_p$. Thus $\mathcal{B} \subseteq \sigma_p[B(\widetilde{r}, \widetilde{s}), \ell_p]$. This completes the proof. □

3.4. Theorem. $\sigma_p[B(\widetilde{r}, \widetilde{s})^*, \ell_p^*] = \begin{cases} \{\alpha \in \mathbb{C} : |r - \alpha| < |s|\} & , \quad \widetilde{r}, \widetilde{s} \in \mathcal{C}, \\ \{\alpha \in \mathbb{C} : |r - \alpha| \leq |s|\} \cup \mathcal{B} & , \quad \widetilde{r}, \widetilde{s} \in \mathcal{SD}. \end{cases}$

Proof. By solving the equation $B(\widetilde{r}, \widetilde{s})^* f = \alpha f$ for $\theta \neq f \in \ell_p^* \cong \ell_q$, we derive the system of linear equations

$$\begin{array}{rcl} r_0 f_0 + s_0 f_1 & = & \alpha f_0 \\ r_1 f_1 + s_1 f_2 & = & \alpha f_1 \\ r_2 f_2 + s_2 f_3 & = & \alpha f_2 \\ & \vdots & \\ r_{k-1} f_{k-1} + s_{k-1} f_k & = & \alpha f_{k-1} \\ & \vdots & \end{array}$$

This gives $f_k = \left(\frac{\alpha - r_{k-1}}{s_{k-1}}\right) f_{k-1}$ for all $k \geq 1$. Therefore, we have

$$|f_k| = \left|\frac{\alpha - r_{k-1}}{s_{k-1}}\right| |f_{k-1}| \text{ for all } k \in \mathbb{N}_1. \tag{3.9}$$

We also prove this theorem by dividing into two parts.

Part 1. Assume that $\widetilde{r}, \widetilde{s} \in \mathcal{C}$ with $r_k = r$ and $s_k = s$ for all $k \in \mathbb{N}$. Using (3.9), we get

$$f_k = \left(\frac{\alpha - r}{s}\right)^k f_0 \text{ for all } k \in \mathbb{N}_1.$$

Then, since

$$\lim_{k\to\infty} \left|\frac{f_{k+1}}{f_k}\right|^q = \left|\frac{\alpha - r}{s}\right|^q < 1 \text{ provided } \left|\frac{r-\alpha}{s}\right| < 1$$

the series $\sum_{k=1}^{\infty} |f_k|^q = \sum_{k=1}^{\infty} |(\alpha - r)/s|^{q(k-1)} |f_0|$ converges by the ratio test, i.e., $f \in \ell_q$.

If $\alpha \in \mathbb{C}$ with $|\alpha - r| = |s|$, then the ratio test fails. But, since $\lim_{k\to\infty} |f_k| = |f_0| \neq 0$ the series $\sum_{k=0}^{\infty} |f_k|^q$ is divergent. This means that $f \in \ell_q$ if and only if $f_0 \neq 0$ and $|r - \alpha| < |s|$. Hence, $\sigma_p[B(\widetilde{r}, \widetilde{s})^*, \ell_p^*] = \{\alpha \in \mathbb{C} : |r - \alpha| < |s|\}$.

Part 2. Let $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$. It is clear that for all $k \in \mathbb{N}$, the vector $f = (f_0, f_1, \ldots\ldots, f_k, 0, 0, \ldots)$ is an eigenvector of the operator $B(\widetilde{r}, \widetilde{s})^*$ corresponding to the eigenvalue $\alpha = r_k$, where $f_0 \neq 0$ and $f_n = \left(\frac{\alpha - r_{n-1}}{s_{n-1}}\right) f_{n-1}$ for all $k \in \{1, 2, 3, \ldots, n\}$. Thus $\mathcal{B} \subseteq \sigma_p[B(\widetilde{r}, \widetilde{s})^*, \ell_p^*]$. If $|r - \alpha| < |s|$ and $\alpha = r_k$, by taking into account (3.9), since

$$\lim_{k\to\infty} \left|\frac{f_k}{f_{k-1}}\right|^q = \lim_{k\to\infty} \left|\frac{\alpha - r_{k-1}}{s_{k-1}}\right|^q = \left|\frac{r-\alpha}{s}\right|^q < 1,$$

the ratio test gives that $f \in \ell_q$. If $\alpha \in \mathbb{C}$ with $|r - \alpha| = |s|$, the ratio test fails. But one can easily find a decreasing sequence of positive real numbers $f = (f_k) \in \ell_q$ such that $\lim_{k\to\infty}(|f_k/f_{k-1}|) = 1$, for example $f = (f_k) = (1/k^2)$. Hence, $|r - \alpha| \leq s$ implies $f \in \ell_q$.

Conversely, we have to show that $f \in \ell_q$ implies $|r - \alpha| \leq s$. If the condition $|r - \alpha| \leq |s|$ does not hold, then $|r - \alpha| > |s|$ which implies that $\sum_{k=0}^{\infty} |f_k|^q$ is divergent. This means that $f \in \ell_q$ if and only if $f_0 \neq 0$ and $|r - \alpha| \leq |s|$. Hence,

$$\sigma_p[B(\widetilde{r}, \widetilde{s})^*, \ell_p^*] = \{\alpha \in \mathbb{C} : |r - \alpha| \leq |s|\} \cup \mathcal{B}.$$

This completes the proof. □

3.5. Lemma. [22, p. 59] *T has a dense range if and only if T^* is one to one.*

3.6. Lemma. [22, p. 60] *The adjoint operator T^* of T is onto if and only if T is a bounded operator.*

3.7. Theorem. $\sigma_r[B(\widetilde{r}, \widetilde{s}), \ell_p] = \begin{cases} \{\alpha \in \mathbb{C} : |r - \alpha| < |s|\} & , \quad \widetilde{r}, \widetilde{s} \in \mathcal{C}, \\ \{\alpha \in \mathbb{C} : |r - \alpha| \leq |s|\} & , \quad \widetilde{r}, \widetilde{s} \in \mathcal{SD}. \end{cases}$

Proof. We prove the theorem by dividing into two parts.

Part 1. Let $\widetilde{r}, \widetilde{s} \in \mathcal{C}$. We show that the operator $B(\widetilde{r}, \widetilde{s}) - \alpha I$ has an inverse and $\overline{R(B(\widetilde{r}, \widetilde{s}) - \alpha I)} \neq \ell_p$ for α satisfying $|r - \alpha| < |s|$. For $\alpha \neq r$ $B(\widetilde{r}, \widetilde{s}) - \alpha I$ is triangle so has an inverse. For $\alpha = r$, the operator $B(\widetilde{r}, \widetilde{s}) - \alpha I$ is one to one by Theorem 3.3. So it has a inverse. By Theorem 3.4, the operator $[B(\widetilde{r}, \widetilde{s}) - \alpha I)]^* = B(\widetilde{r}, \widetilde{s})^* - \alpha I$ is not one to one for $\alpha \in \mathbb{C}$ such that $|r - \alpha| < |s|$. Hence the range of the operator $B(\widetilde{r}, \widetilde{s}) - \alpha I$ is not dense in ℓ_p by Lemma 3.5. So, $\sigma_r[B(\widetilde{r}, \widetilde{s}), \ell_p] = \{\alpha \in \mathbb{C} : |r - \alpha| < |s|\}$.

Part 2. Let $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$ with $r_k \longrightarrow r$ and $s_k \longrightarrow s$ as $k \longrightarrow \infty$ for $\alpha \in \mathbb{C}$ such that $|r - \alpha| \leq |s|$. Then, the operator $B(\widetilde{r}, \widetilde{s}) - \alpha I$ is triangle with $\alpha \neq r_k$ for all $k \in \mathbb{N}$. So, the operator $B(\widetilde{r}, \widetilde{s}) - \alpha I$ has an inverse. By Theorem 3.3 the operator $B(\widetilde{r}, \widetilde{s}) - \alpha I$ is one to one for $\alpha = r_k$ for all $k \in \mathbb{N}$. Thus, $[B(\widetilde{r},\widetilde{s}) - \alpha I]^{-1}$ exists. But by Theorem 3.4, $[B(\widetilde{r},\widetilde{s}) - \alpha I]^* = B(\widetilde{r},\widetilde{s})^* - \alpha I$ is not one to one with $\alpha \in \mathbb{C}$ such that $|r - \alpha| \leq |s|$. Hence, the range of the operator $B(\widetilde{r},\widetilde{s}) - \alpha I$ is not dense in ℓ_p, by Lemma 3.5. So, $\sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p] = \{\alpha \in \mathbb{C} : |r - \alpha| \leq |s|\}$.

This completes the proof. □

3.8. Theorem. $\sigma_c[B(\widetilde{r},\widetilde{s}),\ell_p] = \begin{cases} \{\alpha \in \mathbb{C} : |r-\alpha| = |s|\} & , \quad \widetilde{r},\widetilde{s} \in \mathcal{C}, \\ \emptyset & , \quad \widetilde{r},\widetilde{s} \in \mathcal{SD}. \end{cases}$

Proof. We prove the theorem by dividing into two parts.

Part 1. Let $\widetilde{r},\widetilde{s} \in \mathcal{C}$ for $\alpha \in \mathbb{C}$ such that $|r - \alpha| = |s|$. Since $\sigma[B(\widetilde{r},\widetilde{s}),\ell_p]$ is the disjoint union of the parts $\sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p]$, $\sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p]$ and $\sigma_c[B(\widetilde{r},\widetilde{s}),\ell_p]$, we must have $\sigma_c[B(\widetilde{r},\widetilde{s}),\ell_p] = \{\alpha \in \mathbb{C} : |r - \alpha| = |s|\}$.

Part 2. Let $\widetilde{r},\widetilde{s} \in \mathcal{SD}$. It is known that $\sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p]$, $\sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p]$ and $\sigma_c[B(\widetilde{r},\widetilde{s}),\ell_p]$ are mutually disjoint sets and their union is $\sigma[B(\widetilde{r},\widetilde{s}),\ell_p]$. Therefore, it is immediate from Theorems 3.2, 3.3 and 3.7 that $\sigma[B(\widetilde{r},\widetilde{s}),\ell_p] = \sigma_p[B(\widetilde{r},\widetilde{s}),\ell_p] \cup \sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p]$ and hence $\sigma_c[B(\widetilde{r},\widetilde{s}),\ell_p] = \emptyset$.

This completes the proof. □

3.9. Theorem. *When* $|r - \alpha| > |s|$ *for* $\alpha \neq r_k$, $[B(\widetilde{r},\widetilde{s}) - \alpha I] \in A_1$.

Proof. We show that the operator $B(\widetilde{r},\widetilde{s}) - \alpha I$ is bijective and has a continuous inverse for $\alpha \in \mathbb{C}$ such that $|r - \alpha| > |s|$. Since $\alpha \neq r_k$, then $B(\widetilde{r},\widetilde{s}) - \alpha I$ is a triangle. So, it has an inverse. The inverse of the operator $B(\widetilde{r},\widetilde{s}) - \alpha I$ is continuous for $\alpha \in \mathbb{C}$ such that $|r - \alpha| > |s|$, by equation (3.6). Thus for every $y \in \ell_p$, we can find that $x \in \ell_p$ such that

$$[B(\widetilde{r},\widetilde{s}) - \alpha I]x = y, \quad \text{since} \quad [B(\widetilde{r},\widetilde{s}) - \alpha I]^{-1} \in (\ell_p : \ell_p).$$

This shows that the operator $B(\widetilde{r},\widetilde{s}) - \alpha I$ is onto and so $B(\widetilde{r},\widetilde{s}) - \alpha I \in A_1$. □

3.10. Theorem. *Let* $\widetilde{r},\widetilde{s} \in \mathcal{C}$ *with* $r_k = r$ *and* $s_k = s$ *for all* $k \in \mathbb{N}$. *Then,* $r \in \sigma[B(\widetilde{r},\widetilde{s}),\ell_p]C_1$.

Proof. We have $\sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p] = \{\alpha \in \mathbb{C} : |r - \alpha| < |s|\}$, by Theorem 3.7. Clearly, $r \in \sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p]$. It is sufficient to show that the operator $[B(\widetilde{r},\widetilde{s}) - rI]^{-1}$ is continuous. By Lemma 3.6, it is enough to show that $[B(\widetilde{r},\widetilde{s}) - Ir]^*$ is onto and for given $y = (y_k) \in \ell_p^* = \ell_q$, we have to find $x = (x_k) \in \ell_q$ such that $[B(\widetilde{r},\widetilde{s}) - Ir]^* x = y$. Solving the system of linear equations

$$\begin{aligned} s_0 x_1 &= y_0 \\ s_1 x_2 &= y_1 \\ s_2 x_3 &= y_2 \\ &\vdots \\ s_{k-1} x_k &= y_{k-1} \\ &\vdots \end{aligned}$$

one can easily observe that $sx_k = y_{k-1}$ for all $k \geq 1$ which implies that $(x_k) \in \ell_q$, since $y = (y_k) \in \ell_q$. This shows that $[B(\widetilde{r},\widetilde{s}) - Ir]^*$ is onto. Hence, $r \in \sigma[B(\widetilde{r},\widetilde{s}),\ell_p]C_1$. □

3.11. Theorem. *Let* $\widetilde{r},\widetilde{s} \in \mathcal{C}$ *with* $r_k = r$ *and* $s_k = s$ *for all* $k \in \mathbb{N}$ *and* $\alpha \in \sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p]$ *for all* $r \neq \alpha$. *Then,* $\alpha \in \sigma[B(\widetilde{r},\widetilde{s}),\ell_p]C_2$.

Proof. It is sufficient to show that the operator $[B(\widetilde{r},\widetilde{s}) - I\alpha]^{-1}$ is discontinuous for $r \neq \alpha$ and $\alpha \in \sigma_r[B(\widetilde{r},\widetilde{s}),\ell_p]$. It is obvious that the operator $[B(\widetilde{r},\widetilde{s}) - I\alpha]^{-1}$ is discontinuous for $r \neq \alpha$ and $\alpha \in \mathbb{C}$ such that $|r - \alpha| < |s|$ with $r_k \neq \alpha$, by (3.5). □

3.12. Theorem. *If* $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$ *and* $\alpha \in \sigma_r[B(\widetilde{r}, \widetilde{s}), \ell_p]$*, then* $\alpha \in \sigma[B(\widetilde{r}, \widetilde{s}), \ell_p]C_2$*.*

Proof. It is sufficient to show that the operator $[B(\widetilde{r}, \widetilde{s}) - I\alpha]^{-1}$ is discontinuous for $\alpha \in \sigma_r[B(\widetilde{r}, \widetilde{s}), \ell_p]$. By (3.5), the operator $[B(\widetilde{r}, \widetilde{s}) - I\alpha]^{-1}$ is discontinuous for $r_k \neq \alpha$ and $\alpha \in \mathbb{C}$ with $|r - \alpha| \leq |s|$. □

3.13. Theorem. *Let* $\widetilde{r}, \widetilde{s} \in \mathcal{C}$ *with* $r_k = r$*,* $s_k = s$ *for all* $k \in \mathbb{N}$*. Then, the following statements hold:*

(i) $\sigma_{ap}[B(\widetilde{r}, \widetilde{s}), \ell_p] = \mathcal{A} \setminus \{r\}$.
(ii) $\sigma_\delta[B(\widetilde{r}, \widetilde{s}), \ell_p] = \mathcal{A}$.
(iii) $\sigma_{co}[B(\widetilde{r}, \widetilde{s}), \ell_p] = \mathcal{A}^\circ$.

Proof. (i) From Table 1.2, we get

$$\sigma_{ap}[B(\widetilde{r}, \widetilde{s}), \ell_p] = \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] \setminus \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] C_1.$$

We have by Theorem 3.10 and Theorem 3.2 that

$$\sigma_{ap}[B(\widetilde{r}, \widetilde{s}), \ell_p] = (\mathcal{A} \cup \mathcal{B}) \setminus \{r\} = \mathcal{A} \setminus \{r\}.$$

(ii) Since the following equality

$$\sigma_\delta[B(\widetilde{r}, \widetilde{s}), \ell_p] = \sigma[B(\widetilde{r}, \widetilde{s}), \ell_p] \setminus \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] A_3$$

holds from Table 1.2, we derive by Theorem 3.2 and Theorem 3.3 that $\sigma_\delta[B(\widetilde{r}, \widetilde{s}), \ell_p] = \mathcal{A}$.

(iii) From Table 1.2, we have

$$\sigma_\delta[B(\widetilde{r}, \widetilde{s}), \ell_p] = \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] C_1 \cup \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] C_2 \cup \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] C_3$$

and since $\sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] C_3 = \emptyset$ by Theorem 3.3 it is immediate that $\sigma_{co}[B(\widetilde{r}, \widetilde{s}), \ell_p] = \sigma_r\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right]$. Therefore, we obtain by Theorem 3.11 that $\sigma_{co}[B(\widetilde{r}, \widetilde{s}), \ell_p] = \mathcal{A}^\circ$. □

3.14. Theorem. *Let* $\widetilde{r}, \widetilde{s} \in \mathcal{SD}$*. Then*

$$\sigma_{ap}[B(\widetilde{r}, \widetilde{s}), \ell_p] = \sigma_\delta[B(\widetilde{r}, \widetilde{s}), \ell_p] = \sigma_{co}[B(\widetilde{r}, \widetilde{s}), \ell_p] = \mathcal{A} \cup \mathcal{B}.$$

Proof. We have by Theorem 3.4 and Part (e) of Proposition 2.1 that

$$\sigma_p[B^*(\widetilde{r}, \widetilde{s}), \ell_p^*] = \sigma_{co}[B(\widetilde{r}, \widetilde{s}), \ell_p] = \{\alpha \in \mathbb{C} \ : \ |r - \alpha| \leq |s|\}.$$

Furthermore, because of $\sigma_p[B(\widetilde{r}, \widetilde{s}), \ell_p] = \{r_k\}$ by Theorem 3.3 and the subdivisions in Goldberg's classification are disjoint, we must have

$$\sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] A_3 = \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] B_3 = \emptyset.$$

Hence, $\sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] C_3 = \{r_k\}$. Additionally, since $\sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] C_1 = \emptyset$ by Theorem 3.7 and Theorem 3.12, we have

$$\sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] = \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] C_2 \cup \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] C_3.$$

Therefore, we derive from Table 1.2 that

$$\begin{aligned}
\sigma_{ap}[B(\widetilde{r}, \widetilde{s}), \ell_p] &= \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] \setminus \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] C_1 = \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] \\
\sigma_\delta[B(\widetilde{r}, \widetilde{s}), \ell_p] &= \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] \setminus \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] A_3 = \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] \\
\sigma_\delta[B(\widetilde{r}, \widetilde{s}), \ell_p] &= \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] C_2 \cup \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right] C_3 = \sigma\left[B(\widetilde{r}, \widetilde{s}), \ell_p\right].
\end{aligned}$$

□

4. Conclusion

In the present work, as a natural continuation of Akhmedov and El-Shabrawy [4] and, Srivastava and Kumar [38], we have determined the spectrum and the fine spectrum of the double sequential band matrix $B(\widetilde{r},\widetilde{s})$ on the space ℓ_p. Many researchers determine the spectrum and fine spectrum of a matrix operator in some sequence spaces. In addition to this, we add the definition of some new divisions of spectrum called as approximate point spectrum, defect spectrum and compression spectrum of the matrix operator and give the related results for the matrix operator $B(\widetilde{r},\widetilde{s})$ on the space ℓ_p which is a new development for this type works giving the fine spectrum of a matrix operator on a sequence space with respect to the Goldberg's classification.

Finally, we should note that in the case $r_k = r$ and $s_k = s$ for all $k \in \mathbb{N}$ since the operator $B(\widetilde{r},\widetilde{s})$ defined by a double sequential band matrix reduces to the operator $B(r,s)$ defined by the generalized difference matrix our results are more general and more comprehensive than the corresponding results obtained by Furkan et al. [12] and Bilgiç and Furkan [11], respectively. We record from now on that our next paper will be devoted to the investigation of the fine spectrum of the matrix operator $B(\widetilde{r},\widetilde{s})$ on the space bv_p.

Acknowledgement

The authors would like to express their pleasure to Professor Mustafa Yıldırım, Department of Mathematics, Faculty of Sciences, Cumhuriyet University, 58140 Sivas, Turkey, for his careful reading and making some useful corrections which improved the presentation of the paper. The authors also thank to Professor Bilâl Altay, Department of Mathematical Education, Faculty of Education, İnönü University, 44280 Malatya, Turkey, for many helpful suggestions and interesting comments on the main results of the earlier version of the manuscript.

Finally, the authors have benefited much from the constructive reports of the anonymous referees, and they are grateful for their valuable comments and corrections on the first draft of this paper which improved the presentation and readability.

References

[1] A.M. Akhmedov, F. Başar, *On the fine spectrum of the Cesàro operator in c_0*, Math. J. Ibaraki Univ. **36** (2004), 25–32.

[2] A.M. Akhmedov, F. Başar, *On the spectra of the difference operator Δ over the sequence space ℓ_p*, Demonstratio Math. **39** (3) (2006), 585–595.

[3] A.M. Akhmedov, F. Başar, *On the fine spectra of the difference operator Δ over the sequence space bv_p*, $(1 \leq p < \infty)$, Acta Math. Sin. Eng. Ser. **23** (10) (2007), 1757–1768.

[4] A.M. Akhmedov, S.R. El-Shabrawy, *On the fine spectrum of the operator $\Delta_{a,b}$ over the sequence space c*, Comput. Math. Appl. **61** (10) (2011), 2994–3002.

[5] B. Altay, F. Başar, *On the fine spectrum of the difference operator Δ on c_0 and c*, Inform. Sci. **168** (2004), 217–224.

[6] B. Altay, F. Başar, *On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces c_0 and c*, Int. J. Math. Math. Sci. 2005: **18** (2005), 3005–3013.

[7] B. Altay, M. Karakuş, *On the spectrum and fine spectrum of the Zweier matrix as an operator on some sequence spaces*, Thai J. Math. **3** (2005), 153–162.

[8] M. Altun, *On the fine spectra of triangular Toeplitz operators*, Appl. Math. Comput. **217** (20) (2011), 8044–8051.

[9] J. Appell, E. Pascale, A. Vignoli, *Nonlinear Spectral Theory*, de Gruyter Series in Nonlinear Analysis and Applications 10, Walter de Gruyter · Berlin · New York, 2004.

[10] F. Başar, B. Altay, *On the space of sequences of p-bounded variation and related matrix mappings*, Ukrainian Math. J. **55** (1) (2003), 136–147.

[11] H. Bilgiç, H. Furkan, *On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces ℓ_p and bv_p* $(1 < p < \infty)$, Nonlinear Anal. **68** (3) (2008), 499–506.
[12] H. Bilgiç, H. Furkan, *On the fine spectrum of the operator $B(r,s,t)$ over the sequence spaces ℓ_1 and bv*, Math. Comput. Modelling **45** (2007), 883–891.
[13] A. Karaisa, F. Başar, *Fine spectra of upper triangular triple-band matrices over the sequence space ℓ_p,* $(0 < p < \infty)$, Abstr. Appl. Anal. **2013** Article ID 342682, 10 pages, 2013. doi:10.1155/2013/342682.
[14] A. Karaisa, F. Başar, *Fine spectra of upper triangular triple-band matrix over the sequence space ℓ_p where* $(0 < p < 1)$, AIP Conference Proceedings **1470** (2012), 134–137.
[15] P.J. Cartlidge, Weighted mean matrices as operators on ℓ^p, Ph.D. Dissertation, Indiana University, 1978.
[16] C. Coşkun, *The spectra and fine spectra for p-Cesàro operators*, Turkish J. Math. **21** (1997), 207–212.
[17] B. Choudhary, S. Nanda, *Functional Analysis with Applications*, John Wiley & Sons Inc. New York · Chichester · Brisbane · Toronto · Singapore, 1989.
[18] E. Dündar, F. Başar, *On the fine spectrum of the upper triangle double band matrix Δ^+ on the sequence space c_0*, Math. Commun. **18** (2013), 337–348.
[19] A. Farés, B. de Malafosse, *Spectra of the operator of the first difference in s_α, s_α^0, $s_\alpha^{(c)}$ and $l_p(\alpha)$ $(1 \le p < \infty)$ and application to matrix transformations*, Demonstratio Math. **41** (3) (2008), 661–676.
[20] H. Furkan, H. Bilgiç, B. Altay, *On the fine spectrum of the operator $B(r,s,t)$ over c_0 and c*, Comput. Math. Appl. **53** (2007), 989–998.
[21] H. Furkan, H. Bilgiç, F. Başar, *On the fine spectrum of the operator $B(r,s,t)$ over the sequence spaces ℓ_p and bv_p,* $(1 < p < \infty)$, Comput. Math. Appl. **60** (7) (2010), 2141–2152.
[22] S. Goldberg, *Unbounded Linear Operators*, Dover Publications, Inc. New York, 1985.
[23] M. Gonzàlez, *The fine spectrum of the Cesàro operator in ℓ_p* $(1 < p < \infty)$, Arch. Math. **44** (1985), 355–358.
[24] M. Imaninezhad, M.R. Miri, *The dual space of the sequence space bv_p, $(1 \le p < \infty)$*, Acta Math. Univ. Comen. **79** (1) (2010), 143–149.
[25] A. Karaisa, *Spectrum and fine spectrum generalized diffrence over the sequence space ℓ_1*, Math. Sci. Lett. **3** (3) (2014), 215–221.
[26] A. Karaisa, *Fine spectra of upper triangular double-band matrices over the sequence space ℓ_p,* $(1 < p < \infty)$, Discrete Dyn. Nature Soc. **2012** Article ID 381069 (2012).
[27] F. Başar, A. Karaisa, *Spectrum and fine spectrum of the upper triangular triple-band matrix over some sequence spaces*, AIP Conference Proceedings **1676** (2015), 020085: 1-10, doi.org/10.1063/1.4930511.
[28] Saad R. El-Shabrawy, *Spectra and fine spectra of certain lower triangular double-band matrices as operators on c_0* J. Inequal. Appl. **2014** (2014), 1–9.
[29] K. Kayaduman, H. Furkan, *The fine specta of the difference operator Δ over the sequence spaces ℓ_1 and bv*, Int. Math. Forum **1** (24) (2006), 1153–1160.
[30] E. Kreyszig,*Introductory Functional Analysis with Applications*, John Wiley & Sons Inc. New York · Chichester · Brisbane · Toronto, 1978.
[31] B. de Malafosse, *Properties of some sets of sequences and application to the spaces of bounded difference sequences of order μ*, Hokkaido Math. J. **31** (2002), 283–299.
[32] J.I. Okutoyi, *On the spectrum of C_1 as an operator on bv_0*, J. Austral. Math. Soc. Ser. A. **48** (1990), 79–86.
[33] J.I. Okutoyi, *On the spectrum of C_1 as an operator on bv*, Commun. Fac. Sci. Univ. Ank. Ser. A_1 **41** (1992), 197–207.
[34] J.B. Reade, *On the spectrum of the Cesaro operator*, Bull. Lond. Math. Soc. **17** (1985), 263–267.
[35] B.E. Rhoades, *The fine spectra for weighted mean operators*, Pacific J. Math. **104** (1) (1983), 219–230.
[36] P.D. Srivastava, S. Kumar, *Fine spectrum of the generalized difference operator Δ_ν on sequence space ℓ_1*, Thai J. Math. **8**(2)(2010), 7–19.
[37] B.L. Panigrahi, P.D. Srivastava, *Spectrum and fine spectrum of generalized second order difference operator Δ^2_{uv} on sequence space c_0*, Thai J. Math. **9** (1) (2011), 57–74.

[38] P.D. Srivastava, S. Kumar, *Fine spectrum of the generalized difference operator* Δ_{uv} *on sequence space* l_1, Appl. Math. Comput. **218** (11) (2012), 6407–6414.
[39] R.B. Wenger, *The fine spectra of Hölder summability operators*, Indian J. Pure Appl. Math. **6** (1975), 695–712.
[40] M. Yeşilkayagil, F. Başar, *On the fine spectrum of the operator defined by a lambda matrix over the sequence spaces of null and convergent sequences*, Abstr. Appl. Anal. **2013**, Article ID 687393, 13 pages, 2013. doi: 10.1155/2013/687393.
[41] M. Yıldırım, *On the spectrum and fine spectrum of the compact Rhally operators*, Indian J. Pure Appl. Math. **27** (8) (1996), 779–784.
[42] M. Yıldırım, *On the spectrum of the Rhaly operators on* c_0 *and* c, Indian J. Pure Appl. Math. **29** (1998), 1301–1309.
[43] M. Yıldırım, *On the spectrum of the Rhaly operators on* ℓ_p, Indian J. Pure Appl. Math. **32** (2001), 191–198.
[44] M. Yıldırım, *On the fine spectrum of the Rhaly operators on* ℓ_p, East Asian Math. J. **20** (2004), 153–160.