Hacettepe Journal of Mathematics and Statistics
Volume 34 S (2005), 91 – 100

Doğan Çoker
Memorial Issue

CONCERNING FUZZY GRILLS: SOME APPLICATIONS

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Received 05 : 04 : 2005 : Accepted 21 : 10 : 2005

Abstract

In the first part of this article, we develop to some extent the theory of fuzzy grills, while in the latter part our intention is to obtain some applications of the obtained results to the study of fuzzy compactness, and fuzzy almost compactness, in particular.

Keywords: Prefilter, Fuzzy grill, Fuzzy compact, Fuzzy almost compact, Θ-conjoint fuzzy grill.

2000 AMS Classification: Primary 54 A 40, Secondary 54 D 20, 54 D 30.

1. Introduction and Preliminaries

Like nets and filters, the concept of grills, first introduced by Choquet [5], is known to be a pretty useful tool for studying certain topological concepts; its applications to the theories of proximity spaces and compactifications are especially noteworthy.

The idea of fuzzy grills for fuzzy topological spaces was initiated by Azad [1] basically for the study and characterization of proximities in a fuzzy setting. Subsequently, Srivastava and Gupta [17] investigated the fuzzy basic proximity and the fuzzy LO-proximity, in particular, by use of fuzzy grills. In the process, they showed that certain important known properties of grills are no longer valid for fuzzy grills. Such fuzzy grills have also been studied and utilized as a convenient appliance in recent investigations also (e.g. see [4] and [8]) concerning proximities.

In this paper, our aim is to study fuzzy grills once again in some detail to ultimately apply the results obtained to the investigation of two well known types of fuzzy covering property, viz. fuzzy compactness and almost compactness. In the next section, we take up the proposed study of fuzzy grills with the sole motivation of implicating the results in studying and characterizing fuzzy compactness, and especially fuzzy almost compactness, in Section 3. To make the exposition self-contained as for as practicable, we now go on

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†The first author is grateful to the CSIR, New Delhi, for sponsoring this work.

to clarify the requisite concepts and notations to be adhered to in the course of the subsequent deliberation.

In this article, by an fts X we mean a fuzzy topological space (X, τ), as initiated by Chang [3].

For two fuzzy sets [18] A, B in X, i.e., $A, B \in I^X$ ($I = [0,1]$), we write $A \leq B$ if $A(x) \leq B(x)$ for each $x \in X$, while the notation $A\,q\,B$ means that A is q-coincident [15] with B, i.e., $A(x) + B(x) > 1$ for some $x \in X$. The negations of these statements are denoted by $A \nleq B$ and $A\,\bar{q}\,B$, respectively.

For $A, B \in I^X$, A is called a q-nbd of B [15] if for some fuzzy open set U in X, $B\,q\,U \leq A$; if in addition, A itself is fuzzy open, then it is called an open q-nbd of B.

The notation σ will be used to denote the class of all fuzzy closed sets in (X, τ). The two constant fuzzy sets, taking the constant values 0 and 1 respectively at each point of X are denoted, as usual, by 0_X and 1_X. A fuzzy point [15] with singleton support x and value α ($0 < \alpha \leq 1$) is denoted by x_α. For a fuzzy set A in an fts X, the fuzzy closure, interior and complement of A in X are written as $\operatorname{cl} A$, $\operatorname{int} A$ and $1 - A$, respectively.

The fuzzy θ-closure of A [13] (where $A \in I^X$), to be denoted by $\theta\text{-}\operatorname{cl} A$, is the union of all fuzzy points x_α such that for each open q-nbd U of x_α, $\operatorname{cl} U\,q\,A$. The fuzzy set A is called θ-closed in X [13] if $A = \theta\text{-}\operatorname{cl} A$.

The fuzzy set $A \in I^X$ is called fuzzy regular open (regular closed) if $A = \operatorname{int}\operatorname{cl} A$ (resp. $A = \operatorname{cl}\operatorname{int} A$) [2].

The fuzzy version of filterbases has been studied in [10, 14]. A collection $\mathcal{F}$ of fuzzy sets in an fts X is called a prefilterbase on X [10] if

(i) $0_X \notin \mathcal{F}$, and
(ii) $A, B \in \mathcal{F} \implies \exists\, C \in \mathcal{F}$ such that $C \leq A \cap B$.

If, in addition,

(iii) $A \in \mathcal{F}$ and $A \leq B \in I^X \implies B \in \mathcal{F}$

holds, then $\mathcal{F}$ is called a prefilter on X [10].

A collection $\mathcal{F}$ of fuzzy open sets in X is called an open prefilter on X if the above conditions (i), (ii) hold and in addition

(iii)$'$ $A \in \mathcal{F}$ and $A \leq B \in \tau \implies B \in \mathcal{F}$

is satisfied. Similarly, one defines a closed prefilter.

A prefilter (an open prefilter) $\mathcal{F}$ is called a fuzzy ultrafilter [9] (open fuzzy ultrafilter) if $\mathcal{F}$ is not properly contained in any prefilter (resp. open prefilter) on X.

A prefilter or an open or closed prefilter on an fts X is said to adhere (θ-adhere) at a fuzzy point x_α in X if for each open q-nbd U of x_α and each $F \in \mathcal{F}$, it follows that $F\,q\,U$ (resp. $F\,q\operatorname{cl} U$) [10]. It is easy to see that a prefilterbase (or an open or closed prefilterbase) $\mathcal{F}$ adheres at a fuzzy point x_α if and only if $x_\alpha \leq \bigcap\{\operatorname{cl} F : F \in \mathcal{F}\}$.

2. Fuzzy Grills

We start by recalling the following definition of a fuzzy grill from [1].

2.1. Definition. A non-void collection $\mathcal{G}$ of fuzzy sets in an fts X is called a *fuzzy grill on* X if

(i) $0_X \notin \mathcal{G}$,
(ii) $A \in \mathcal{G}$, $B \in I^X$ and $A \leq B \implies B \in \mathcal{G}$, and
(iii) $A, B \in I^X$ and $A \cup B \in \mathcal{G} \implies A \in \mathcal{G}$ or $B \in \mathcal{G}$.

2.2. Remark. It is known [9] that if $\mathcal{F}$ is a fuzzy ultrafilter on an fts X, then $(A, B \in I^X$ and $A \cup B \in \mathcal{F}) \implies (A \in \mathcal{F}$ or $B \in \mathcal{F})$. It then follows that every fuzzy ultrafilter on an fts X is a fuzzy grill on X.

2.3. Definition. For a fuzzy grill or a prefilter Ω, we define

$$\operatorname{Sec}\Omega = \{A \in I^X : A\, q\, G, \text{ for each } G \in \Omega\}.$$

2.4. Theorem. *Let X be an fts.*

(a) *If Ω is a fuzzy grill or a prefilter on X, then $A \in \operatorname{Sec}\Omega \iff 1 - A \notin \Omega$.*
(b) *If Ω is a fuzzy grill (prefilter) on X, then $\operatorname{Sec}\Omega$ is a prefilter (fuzzy grill) on X.*

Proof. (a) Indeed, $A \in \operatorname{Sec}\Omega \implies A\, q\, G$, for each $G \in \Omega$. But $A\, \bar{q}\, (1 - A)$ so that $1 - A \notin \Omega$. Conversely, let $A \notin \operatorname{Sec}\Omega$. Then $A\, \bar{q}\, G$ for some $G \in \Omega \implies G \leq 1 - A$. Since Ω is a fuzzy grill (prefilter) and $G \in \Omega$, it follows that $1 - A \in \Omega$.

(b) (i) Let Ω be a fuzzy grill. Clearly $0_X \notin \operatorname{Sec}\Omega$. Let $A \in \operatorname{Sec}\Omega$ and $B \in I^X$ with $A \leq B$. Then, obviously $B \in \operatorname{Sec}\Omega$. Let $A, B \in \operatorname{Sec}\Omega$. Then $1 - A, 1 - B \notin \Omega \implies (1 - A) \cup (1 - B) \notin \Omega \implies 1 - A \cap B \notin \Omega \implies A \cap B \in \operatorname{Sec}\Omega$. Hence $\operatorname{Sec}\Omega$ is a prefilter.

(ii) Let Ω be a prefilter. Clearly $0_X \notin \operatorname{Sec}\Omega$. Let $A \in \operatorname{Sec}\Omega$, $B \in I^X$ and $A \leq B$. Then $B \in \operatorname{Sec}\Omega$. Let $A, B \in I^X$ and $A \cup B \in \operatorname{Sec}\Omega$. Then by (a), $1 - (A \cup B) = (1 - A) \cap (1 - B) \notin \Omega$ so that $1 - A \notin \Omega$ or $1 - B \notin \Omega$, and hence $A \in \operatorname{Sec}\Omega$ or $B \in \operatorname{Sec}\Omega$. Thus $\operatorname{Sec}\Omega$ is a fuzzy grill on X. □

2.5. Definition. Let $\mathcal{G}$ be a fuzzy grill on an fts X, and x_α a fuzzy point. Then $\mathcal{G}$ is said to

(a) *Adhere* (Θ-*adhere*) at x_α if for each open q-nbd U of x_α and each $G \in \mathcal{G}$, $U\, q\, G$ (resp. $\operatorname{cl} U\, q\, G$),
(b) *Converge* (Θ-*converge*) to x_α if for each open q-nbd U of x_α, $G \leq U$ (resp. $G \leq \operatorname{cl} U$), for some $G \in \mathcal{G}$.

2.6. Remark. From Definition2.5 (b), it becomes clear that a fuzzy grill $\mathcal{G}$ on an fts X, Θ-converges to some fuzzy point x_α in X iff $\{\operatorname{cl} U : U$ is an open q-nbd of $x_\alpha\} \subseteq \mathcal{G}$, and converges to some fuzzy point x_α in X iff $\{U : U$ is an open q-nbd of $x_\alpha\} \subseteq \mathcal{G}$.

The following example shows that a fuzzy grill may converge and Θ-converge to a fuzzy point x_α, but may neither adhere nor Θ-adhere at x_α.

2.7. Example. Let $X = \{a, b\}$ and $\tau = \{0_x, 1_X, A, B\}$, where $A(a) = 0.2$, $A(b) = B(a) = B(b) = 0.5$. Then (X, τ) is an fts. Let $\mathcal{G}$ consist of 1_X and all fuzzy sets G, where $0.2 \leq G(a) \leq 1$ and $0 \leq G(b) \leq 1$. Then $\mathcal{G}$ is a fuzzy grill on X. Also, $\mathcal{G}$ Θ-converges to $a_{0.9}$, because A and B (together with 1_X) are the open q-nbds of $a_{0.9}$, and $\operatorname{cl} A = \operatorname{cl} B = B$. Then $\mathcal{G}$ converges as well as Θ-converges to $a_{0.9}$. But $\mathcal{G}$ neither adheres nor Θ-adheres at $a_{0.9}$.

The next example shows that a fuzzy grill may have a Θ-adherent point x_α such that the fuzzy grill does not Θ-converge to x_α.

2.8. Example. Consider the fts (X, τ) of Example 2.7. Define $\mathcal{G}$ to be the collection consisting of 1_X and all fuzzy sets G such that $G(a) \geq 0.6$ and $0 \leq G(b) \leq 1$. Then $\mathcal{G}$ is a fuzzy grill on X which Θ-adheres at $a_{0.9}$, but does not Θ-converge to $a_{0.9}$. Indeed, A and B (together with 1_X) are the open q-nbds of $a_{0.9}$ and $\operatorname{cl} A = \operatorname{cl} B = B$. Then $B\, q\, G$, for each $G \in \mathcal{G}$, but there does not exist any $G \in \mathcal{G}$ such that $G \leq B$.

The following example shows that the adherence of a fuzzy grill $\mathcal{G}$ at a fuzzy point x_α does not imply that $\mathcal{G}$ converges to x_α.

2.9. Example. Let $X = \{a\}$ and $\tau = \{0_X, 1_X, B\}$ where $B(a) = 0.5$. Then (X, τ) is an fts. Define the collection $\mathcal{G}$ which consists of 1_X and all fuzzy sets G such that $G(a) \geq 0.6$. Then $\mathcal{G}$ is a fuzzy grill on X which adheres at $a_{0.9}$, but does not converge to $a_{0.9}$. Indeed, B (apart from 1_X) is the only open q-nbd of $a_{0.9}$ and $B\,q\,G$, for all $G \in \mathcal{G}$; but there does not exist any $G \in \mathcal{G}$ such that $G \leq B$.

2.10. Theorem. *For each fuzzy point x_α in an fts X, the collection*

$$\mathcal{G}(\Theta, x_\alpha) = \{A \in I^X : x_\alpha \leq \Theta\text{-cl}\,A\}$$

is a fuzzy grill on X.

Proof. Straightforward. □

2.11. Definition. For a given fuzzy point x_α in an fts X, the fuzzy grill $\mathcal{G}(\Theta, x_\alpha)$ (as described above) is called the *Θ-adherence fuzzy grill on X at x_α*.

2.12. Theorem. *A fuzzy grill $\mathcal{G}$ on an fts X, Θ-adheres at some fuzzy point x_α in X iff $\mathcal{G} \subseteq \mathcal{G}(\Theta, x_\alpha)$.*

Proof. A fuzzy grill $\mathcal{G}$ on X Θ-adheres at a fuzzy point x_α in X $\iff$ $\text{cl}\,U\,q\,G$, for each open q-nbd U of x_α and each $G \in \mathcal{G}$ $\iff$ $x_\alpha \leq \Theta\text{-cl}\,G$, for each $G \in \mathcal{G}$ $\iff$ $G \in \mathcal{G}(\Theta, x_\alpha)$, for each $G \in \mathcal{G}$ $\iff$ $\mathcal{G} \subseteq \mathcal{G}(\Theta, x_\alpha)$. □

2.13. Remark. The above theorem justifies the terminology "Θ-adherence fuzzy grill at a fuzzy point x_α" in the sense that such a Θ-adherence fuzzy grill at x_α must Θ-adhere at x_α.

2.14. Theorem. *A fuzzy grill $\mathcal{G}$ on an fts X Θ-converges to some fuzzy point x_α in X iff* $\text{Sec}\,\mathcal{G}(\Theta, x_\alpha) \subseteq \mathcal{G}$.

Proof. First suppose that the fuzzy grill $\mathcal{G}$ on X Θ-converges to a fuzzy point x_α in X. Then,

(1) for any open q-nbd U of x_α there exists some $G \in \mathcal{G}$ such that $G \leq \text{cl}\,U$, and hence $\text{cl}\,U \in \mathcal{G}$.

Now, $B \in \text{Sec}\,\mathcal{G}(\Theta, x_\alpha) \implies 1 - B \notin \mathcal{G}(\Theta, x_\alpha) \implies x_\alpha \nleq \Theta\text{-cl}\,(1 - B) \implies \text{cl}\,U\,\bar{q}\,(1 - B)$ for some open q-nbd U of $x_\alpha \implies \text{cl}\,U \leq B$, where U is an open q-nbd of $x_\alpha \implies B \in \mathcal{G}$ (using (1)). Hence $\text{Sec}\,\mathcal{G}(\Theta, x_\alpha) \subseteq \mathcal{G}$.

Conversely, let V be an open q-nbd of x_α. As $\text{cl}\,V\,\bar{q}\,(1 - \text{cl}\,V)$, $x_\alpha \nleq \Theta\text{-cl}\,(1 - \text{cl}\,V)$. Thus $1 - \text{cl}\,V \notin \mathcal{G}(\Theta, x_\alpha)$ and consequently, $\text{cl}\,V \in \text{Sec}\,\mathcal{G}(\Theta, x_\alpha) \subseteq \mathcal{G}$, and this is true for each open q-nbd V of x_α. Hence $\mathcal{G}$, Θ-converges to x_α. □

2.15. Definition. A non-empty subcollection $\mathcal{G}$ of σ (resp. of τ) in an fts (X, τ) is said to form a *closed* (resp. *open*) *fuzzy grill on X* if

(a) $0_X \notin \mathcal{G}$;
(b) $A \in \mathcal{G}$ and $C \in \sigma$ (resp. $C \in \tau$) with $A \leq C \implies C \in \mathcal{G}$; and
(c) $A, B \in \sigma$ (resp. $A, B \in \tau$) with $A \cup B \in \mathcal{G} \implies A \in \mathcal{G}$ or $B \in \mathcal{G}$.

2.16. Remark. It is quite evident from the above definition that if $\mathcal{G}$ is a fuzzy grill on X, then $\mathcal{G} \cap \tau$ and $\mathcal{G} \cap \sigma$ are open and closed fuzzy grills respectively.

2.17. Theorem. *Let (X, τ) be an fts.*

(a) *If $\mathcal{G}$ is a closed (resp. open) fuzzy grill on X, then* $\text{Sec}\,\mathcal{G} \cap \tau$ (*resp.* $\text{Sec}\,\mathcal{G} \cap \sigma$) *is an open (resp. a closed) prefilter on X.*
(b) *If $\mathcal{F}$ is an open (closed) prefilter on X, then* $\text{Sec}\,\mathcal{F} \cap \sigma$ (*resp.* $\text{Sec}\,\mathcal{F} \cap \tau$) *is a closed (resp. open) fuzzy grill on X.*

Proof. Straightforward. □

2.18. Theorem. *In an fts X, every fuzzy grill converges iff every open fuzzy grill converges.*

Proof. We first assume that every fuzzy grill on X converges in X and let $\mathcal{G}$ be any open fuzzy grill on X. Then by Theorem 2.17 (a), $\mathrm{Sec}\,\mathcal{G} \cap \sigma$ is a closed prefilter on X and hence a prefilterbase on X. Let $\mathcal{F}$ be the prefilter generated by the prefilterbase $\mathrm{Sec}\,\mathcal{G} \cap \sigma$. Then $\mathrm{Sec}\,\mathcal{F}$ is a fuzzy grill on X and by hypothesis, it converges to some fuzzy point x_α in X. Hence for each open q-nbd U of x_α, $F \le U$ for some $F \in \mathrm{Sec}\,\mathcal{F}$. Then $U\,q\,R$ for each open q-nbd U of x_α and each $R \in \mathcal{F}$ (as $\mathrm{Sec}\,\mathcal{F} = \{F \in I^X : F\,q\,G$ for each $G \in \mathcal{F}\}$ and $F \le U$).

We claim that $U \in \mathcal{G}$ for each open q-nbd U of x_α. If not, then there is some open q-nbd U of x_α, for which $1 - U \in \mathrm{Sec}\,\mathcal{G} \cap \sigma$ (by Theorem 2.4). Then $1 - U \in \mathcal{F}$, as $\mathrm{Sec}\,\mathcal{G} \cap \sigma$ is a base for $\mathcal{F}$. Thus $U\,q\,(1 - U)$, a contradiction, proving that $\mathcal{G}$ converges to x_α (by Remark 2.6).

Conversely, let us assume that every open fuzzy grill on X converges in X, and suppose $\mathcal{G}$ is any fuzzy grill on X. Then $\mathcal{G} \cap \tau$ is an open fuzzy grill on X, by Remark 2.16. By hypothesis, $\mathcal{G} \cap \tau$ converges to some fuzzy point x_α in X, so that $U \in \mathcal{G} \cap \tau \subseteq \mathcal{G}$ for each open q-nbd U of x_α. Hence $\mathcal{G}$ converges to x_α. □

2.19. Theorem. *In an fts X, every fuzzy grill converges iff every closed prefilter adheres.*

Proof. Let $\mathcal{G}$ be any open fuzzy grill on X in which every closed prefilter adheres. By Theorem 2.17, $\mathrm{Sec}\,\mathcal{G} \cap \sigma$ is a closed prefilter on X and hence it adheres at some fuzzy point x_α in X. Then $P\,q\,U$, for each $P \in \mathrm{Sec}\,\mathcal{G} \cap \sigma$ and each open q-nbd U of x_α. Clearly, $U \in \mathcal{G}$, for each open q-nbd U of x_α; for otherwise, there is some open q-nbd U of x_α such that $(1 - U) \in \mathrm{Sec}\,\mathcal{G} \cap \sigma$, a contradiction. Hence $\mathcal{G}$ converges to x_α and consequently by Theorem 2.18, necessity follows.

Conversely, let $\mathcal{F}$ be a closed prefilter on X. Then $\mathrm{Sec}\,\mathcal{F} \cap \tau$ is an open fuzzy grill on X (by Theorem 2.17), and hence by Theorem 2.18, it converges to some fuzzy point x_α in X. Thus $U \in \mathrm{Sec}\,\mathcal{F} \cap \tau$, for each open q-nbd U of x_α. Hence for each open q-nbd U of x_α and each $F \in \mathcal{F}$, $F\,q\,U$, so that $\mathcal{F}$ adheres at x_α. □

2.20. Lemma. [13] *If U is a fuzzy open set and V any fuzzy set in an fts X, then $U\,\bar{q}\,V \implies U\,\bar{q}\,\mathrm{cl}\,V$.*

2.21. Theorem. *In an fts X, every open prefilter adheres iff every closed fuzzy grill Θ-converges.*

Proof. Let us suppose that every open prefilter on X adheres in X, and let $\mathcal{G}$ be any closed fuzzy grill on X. Then by Theorem 2.17, $\mathrm{Sec}\,\mathcal{G} \cap \tau$ is an open prefilter on X and hence adheres at some fuzzy point x_α in X. Thus,

(2) for each open q-nbd U of x_α and each $P \in \mathrm{Sec}\,\mathcal{G} \cap \tau$, $P\,q\,U$.

We claim that $\mathrm{cl}\,U \in \mathcal{G}$ for each open q-nbd U of x_α. Indeed, if $\mathrm{cl}\,U \notin \mathcal{G}$ for some open q-nbd U of x_α, then $1 - \mathrm{cl}\,U \in \mathrm{Sec}\,\mathcal{G} \cap \tau$, a contradiction to (2). By Remark 2.6, $\mathcal{G}$ Θ-converges to x_α.

Conversely, for any open prefilter $\mathcal{F}$ on X, $\mathrm{Sec}\,\mathcal{F} \cap \sigma$ is a closed fuzzy grill on X (by Theorem 2.17) and hence Θ-converges to some fuzzy point x_α in X. Then

$$\{\mathrm{cl}\,U : U \text{ is an open q-nbd of } x_\alpha\} \subseteq \mathrm{Sec}\,\mathcal{F} \cap \sigma$$

by Theorem 2.6, which implies that $\mathrm{cl}\, U\, q\, F$ for each $F \in \mathcal{F}$ and each open q-nbd U of x_α. Then $\mathcal{F}$, Θ-adheres at x_α. Since $\mathcal{F}$ is an open prefilter it follows from Lemma 2.20 that $\mathcal{F}$ adheres at x_α. □

2.22. Theorem. *In an fts X, every prefilter adheres iff every closed prefilter adheres.*

Proof. Necessity follows from the fact that each closed prefilter on X is a base for some prefilter on X.

For sufficiency, let $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a prefilter on X. It is to be shown that for some fuzzy point x_t in X, $x_t \leq \bigcap_{\alpha\in\Lambda} \mathrm{cl}\, F_\alpha$.

We claim that $\mathcal{F}^* = \{\mathrm{cl}\, F_\alpha : \alpha \in \Lambda\}$ is a closed prefilter on X. In fact, $0_X \notin \mathcal{F}^*$ is clear. Next we note that

$$\begin{aligned} \mathrm{cl}\, F_\alpha \in \mathcal{F}^* \text{ and } \mathrm{cl}\, F_\alpha \leq B \in \sigma &\implies F_\alpha \in \mathcal{F} \text{ and } F_\alpha \leq B \\ &\implies B \in \mathcal{F} \\ &\implies \mathrm{cl}\, B = B \in \mathcal{F}^*. \end{aligned} \tag{3}$$

Finally, for $\alpha, \beta \in \Lambda$ consider $\mathrm{cl}\, F_\alpha, \mathrm{cl}\, F_\beta \in \mathcal{F}^*$. Then $F_\alpha, F_\beta \in \mathcal{F} \implies F_\alpha \cap F_\beta \in \mathcal{F} \implies \mathrm{cl}\,(F_\alpha \cap F_\beta) \in \mathcal{F}^*$. But $\mathrm{cl}\,(F_\alpha \cap F_\beta) \subseteq \mathrm{cl}\, F_\alpha \cap \mathrm{cl}\, F_\beta \in \sigma$, and (3) now gives $\mathrm{cl}\, F_\alpha \cap \mathrm{cl}\, F_\beta \in \mathcal{F}^*$. Hence $\mathcal{F}^*$ a closed prefilter on X.

By hypothesis, $\mathcal{F}^*$ adheres at some fuzzy point x_t in X. Then

$$x_t \leq \bigcap \{\mathrm{cl}\, F : F \in \mathcal{F}^*\} = \bigcap_{\alpha\in\Lambda} \mathrm{cl}\,(\mathrm{cl}\, F_\alpha) = \bigcap_{\alpha\in\Lambda} \mathrm{cl}\, F_\alpha.$$

Hence $\mathcal{F}$ adheres at x_t. □

2.23. Theorem. *In an fts X, every open fuzzy grill converges iff every prefilter adheres.*

Proof. The proof follows immediately from Theorems 2.19 and 2.22. □

3. Applications

In this section, we shall apply the theory of fuzzy grills developed in the foregoing section to the study and characterizations of two celebrated fuzzy covering properties, viz. fuzzy compactness and especially fuzzy almost compactness of an fts. At the outset, we recall the definitions and a few well known characterizations of these concepts.

3.1. Definition. An fts X is said to be *fuzzy compact* [3] (*fuzzy almost compact* [6]) if every fuzzy cover $\mathcal{U}$ of X (i.e., $\sup\{U(x) : U \in \mathcal{U}\} = 1$ for each $x \in X$) by fuzzy open sets contains a finite subfamily $\mathcal{U}_0$ such that $\sup\{U(x) : U \in U_0\} = 1$ (resp. $\sup\{(\mathrm{cl}\, U)(x) : U \in \mathcal{U}_0\} = 1$) for each $x \in X$.

Although the definition of fuzzy filterbase as taken in [7] is different from that of prefilterbase being considered here, from the results obtained in [7] it is nonetheless not at all difficult to observe the following result:

3.2. Theorem. *An fts X is fuzzy compact iff every prefilterbase on X adheres in X.*

Fuzzy almost compactness has been studied in [11, 13, 16], among many others. For our purpose we recall the following characterizations of a fuzzy almost compact space.

3.3. Theorem. [11, 13] *For an fts X, the following are equivalent:*

(a) *X is fuzzy almost compact.*
(b) *Every open prefilterbase adheres in X.*
(c) *Every family $\mathcal{F}$ of fuzzy closed sets with the property that every finite intersection of members of $\mathcal{F}$ contains a non-null fuzzy open set, has a non-null intersection.*

(d) *Every prefilter* Θ*-adheres in* X*.*

From what we have developed in the previous section, we immediately have :

3.4. Theorem. *For an fts* X*, the following are equivalent:*

(a) X *is fuzzy compact.*
(b) *Every fuzzy grill converges in* X*.*
(c) *Every open fuzzy grill converges in* X*.*
(d) *Every closed prefilter adheres.*

Proof. The proof follows from Theorems 2.18, 2.19, 2.23 and 3.2. □

Again, from Theorems 2.21 and 3.2 it follows that:

3.5. Theorem. *An fts* (X, τ) *is fuzzy almost compact if and only if every closed fuzzy grill* Θ*-converges.* □

We shall use the following result in the sequel.

3.6. Theorem. *An fts* X *is fuzzy almost compact iff every open fuzzy ultrafilter adheres in* X*.*

Proof. By virtue of Theorem 3.3, (a) $\Longrightarrow$ (b), every open fuzzy ultrafilter obviously adheres in a fuzzy almost compact space.

Conversely, if X is not fuzzy almost compact, then there is a fuzzy open cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of X such that for any finite subset Λ_0 of Λ, $1 - \bigcup_{\alpha\in\Lambda_0} \operatorname{cl} U_\alpha \neq 0_X$. Then

$$\mathcal{B} = \{1 - \bigcup_{\alpha\in\Lambda_0} \operatorname{cl} U_\alpha : \Lambda_0 \text{ is a finite subset of } \Lambda\}$$

is an open prefilterbase on X. Let $\mathcal{F}$ be the open prefilter generated by $\mathcal{B}$. Thus $\mathcal{F}$ is contained in some open fuzzy ultrafilter F^* on X. By hypothesis, F^* adheres at some fuzzy point x_t in X, i.e., $x_t \leq \operatorname{cl} F$, for each $F \in \mathcal{F}^*$. As $\mathcal{U}$ is a fuzzy cover of X, there exists $U_\beta \in \mathcal{U}$ such that $x_t\, q\, U_\beta$, and also, $x_t \leq \operatorname{cl}(1 - \operatorname{cl} U_\beta)$ [as $1 - \operatorname{cl} U_\beta \in \mathcal{B} \subseteq F^*$]. Thus $U_\beta\, q\, (1 - \operatorname{cl} U_\beta)$, which is a contradiction. □

3.7. Definition. A fuzzy grill or an open fuzzy grill $\mathcal{G}$ on an fts X is said to be Θ*-conjoint* if for every finite subfamily $\{G_1, G_2, \ldots, G_n\}$ of $\mathcal{G}$, $\operatorname{int}[\bigcap_{i=1}^n \Theta\text{-cl}\, G_i] \neq 0_X$.

3.8. Lemma. *Let* $\mathcal{F}$ *be a fuzzy open ultrafilter on an fts* (X, τ)*, and let* B, C *be fuzzy open sets such that* $B \cup C \in \mathcal{F}$*. Then* $B \in \mathcal{F}$ *or* $C \in \mathcal{F}$*.*

Proof. We note first that either

(i) $B \cap F \neq 0_X$ for all $F \in \mathcal{F}$, or
(ii) $C \cap F \neq 0_X$ for all $F \in \mathcal{F}$.

Indeed, if both (i) and (ii) are false, then we have $F_1 \in \mathcal{F}$ with $B \cap F_1 = 0_X$ and $F_2 \in \mathcal{F}$ with $C \cap F_2 = 0_X$, so $F_1 \cap F_2 \cap (B \cup C) = 0_X$, which contradicts $F_1 \cap F_2 \cap (B \cup C) \in \mathcal{F}$. Suppose that (i) holds and define

$$\mathcal{G} = \{D \in \tau : B \cap F \leq D \text{ for some } F \in \mathcal{F}\}.$$

It is clear that $\mathcal{G}$ is a fuzzy open prefilter, and $\mathcal{F} \subseteq \mathcal{G}$ because $B \cap F \leq F$ for all $F \in \mathcal{F}$. Since $\mathcal{F}$ is a fuzzy open ultrafilter we obtain $\mathcal{G} = \mathcal{F}$. However $B \in \mathcal{G}$ since $1_X \in \mathcal{F}$ and $B \cap 1_X \leq B$, so $B \in \mathcal{F}$. In the same way, if (ii) holds then $C \in \mathcal{F}$. □

3.9. Theorem. *An fts* X *is fuzzy almost compact iff every open* Θ*-conjoint fuzzy grill adheres in* X*.*

Proof. Consider any open Θ-conjoint fuzzy grill $\mathcal{G}$ on a fuzzy almost compact space X. Then $\{\mathrm{cl}\, G : G \in \mathcal{G}\}$ is a collection of fuzzy closed sets in X such that for each finite subfamily $\{A_1, A_2, \ldots, A_n\}$ of $\mathcal{G}$, $\mathrm{int}\,[\bigcap_{i=1}^{n} \mathrm{cl}\, A_i] = \mathrm{int}\,[\bigcap_{i=1}^{n} \Theta\text{-cl}\, A_i]$ (as each A_i is fuzzy open) $\neq 0_X$. By the implication (a) $\Longrightarrow$ (c) of Theorem 3.3, $\bigcap\{\mathrm{cl}\, A : A \in \mathcal{G}\} \neq 0_X$. If $x_\alpha \leq \bigcap\{\mathrm{cl}\, A : A \in \mathcal{G}\}$, then $\mathcal{G}$ adheres at the fuzzy point x_α.

For the converse we are only to prove, by virtue of Theorem 3.6, that every open fuzzy ultrafilter adheres in X. Let $\mathcal{F}$ be an open fuzzy ultrafilter on X. Then $\mathcal{F}$ is an open fuzzy grill on X (by Lemma 3.8). It is obvious that for any finite subfamily $\mathcal{F}_0$ of $\mathcal{F}$, $\mathrm{int}\,(\bigcap \mathcal{F}_0) = \bigcap \mathcal{F}_0 \neq 0_X$. And, $\mathrm{int}\,[\bigcap\{\Theta\text{-cl}\, F : F \in \mathcal{F}_0\}] \geq \mathrm{int}\,[\bigcap \mathcal{F}_0] \neq 0_X$, which shows that $\mathcal{F}$ is an open Θ-conjoint fuzzy grill on X. Hence $\mathcal{F}$ adheres in X. □

We now show that in the necessity part of the above theorem open fuzzy grills could be replaced by fuzzy grills. For this we require the following result.

3.10. Lemma. *For any fuzzy set A in an fts (X, τ), $\Theta\text{-cl}\, A$ is expressible as an intersection of fuzzy regular closed sets.*

Proof. It suffices to show that

$$(4) \qquad \Theta\text{-cl}\, A = \bigcap\{\mathrm{cl}\, U : U \in \tau \text{ and } A \leq U\}.$$

Since for any fuzzy open set U, $\mathrm{cl}\, U = \Theta\text{-cl}\, U$ [13], in (4) left hand side $\leq$ right hand side. Next, let x_α be any fuzzy point such that $x_\alpha \not\leq$ left hand side, but $x_\alpha \leq$ right hand side. Then there is an open q-nbd V of x_α such that $\mathrm{cl}\, V\, \bar{q}\, A$, so that $A \leq 1 - \mathrm{cl}\, V$. As $x_\alpha \leq$ right hand side, $x_\alpha \leq \mathrm{cl}\,(1 - \mathrm{cl}\, V) = \Theta\text{-cl}\,(1 - \mathrm{cl}\, V)$. Then $\mathrm{cl}\, V\, q\, (1 - \mathrm{cl}\, V)$, which is impossible. The contradiction shows that in (4) right hand side $\leq$ left hand side, and the proof is complete. □

3.11. Theorem. *In a fuzzy almost compact space X, every Θ-conjoint fuzzy grill Θ-adheres in X.*

Proof. Consider any Θ-conjoint fuzzy grill $\mathcal{G}$ on X. Then $\{\Theta\text{-cl}\, A : A \in \mathcal{G}\}$ is a collection of fuzzy sets such that:

$$(5) \qquad \mathrm{int}\,[\bigcap_{i=1}^{n} \Theta\text{-cl}\, A_i] \neq 0_X \text{ for each finite subfamily } \{A_1, A_2, \ldots, A_n\} \text{ of } \mathcal{G}.$$

For each $A \in \mathcal{G}$, we have by Lemma 3.10, that,

$$(6) \qquad \Theta\text{-cl}\, A = \bigcap\{B_\alpha : \alpha \in \Lambda_A\},$$

where each B_α is fuzzy regular closed in X and Λ_A is an index set. Consider $\mathcal{B} = \{B_\alpha : \alpha \in \Lambda_A \text{ and } A \in \mathcal{G}\}$. Then $\mathcal{B}$ is a collection of fuzzy regular closed sets, and hence of fuzzy closed sets in X. Now, let $\{B_{\alpha_i} : i = 1, 2, \ldots, k\}$ be any finite subcollection of $\mathcal{B}$. To each B_{α_i} $(i = 1, 2, \ldots, k)$, there corresponds some $A_{\alpha_i} \in \mathcal{G}$ such that $\Theta\text{-cl}\, A_{\alpha_i} \leq B_{\alpha_i}$. Then

$$\bigcap\{\Theta\text{-cl}\, A_{\alpha_i} : i = 1, 2, \ldots, k\} \leq \bigcap\{B_{\alpha_i} : i = 1, 2, \ldots, k\}.$$

Hence, using (5), $0_X \neq \mathrm{int}\,[\bigcap\{\Theta\text{-cl}\,(A_{\alpha_i}) : i = 1, 2, \ldots, k\}] \leq \mathrm{int}\,[\bigcap\{B_{\alpha_i} : i = 1, 2, \ldots, k\}]$. As X is fuzzy almost compact, by Theorem 3.3 we have $\bigcap\{B_\alpha : \alpha \in \bigcup\{\Lambda_A : A \in \mathcal{G}\}\} \neq 0_X$. But by (6), $\bigcap\{B_\alpha : \alpha \in \bigcup\{\Lambda_A : A \in \mathcal{G}\}\} = \bigcap\{\Theta\text{-cl}\, A : A \in \mathcal{G}\}$. Hence there is a fuzzy point x_t in X such that $x_t \leq \bigcap\{\Theta\text{-cl}\, A : A \in \mathcal{G}\}$, i.e., $A \in \mathcal{G}(\Theta, x_t)$ for each $A \in \mathcal{G}$. Then by Theorem 2.12, $\mathcal{G}$, Θ-adheres in X. □

We could neither prove nor disprove the converse of the above theorem. However, we would now like to establish a weaker converse by proving that a condition, slightly stronger that that of Θ-conjointness of fuzzy grills, implies fuzzy almost compactness of an fts; the converse being also true in a fuzzy almost regular space. So let us first recall the definition of a fuzzy almost regular space as follows.

3.12. Definition. [12] An fts X is called *fuzzy almost regular* if for each fuzzy regular open set V in X and each fuzzy point x_α in X with $x_\alpha \, q \, V$, there exists a fuzzy regular open set U such that $x_\alpha \, q \, U \leq \mathrm{cl}\, U \leq V$, or equivalently, iff for any fuzzy set A in X, $\Theta\text{-cl}\,(\Theta\text{-cl}\, A) = \Theta\text{-cl}\, A$ [12].

3.13. Theorem. *A fuzzy almost regular fts X is fuzzy almost compact iff every fuzzy grill $\mathcal{G}$ on X with the property that $\bigcap_{i=1}^{n} \Theta\text{-cl}\, G_i \neq 0_X$ for every finite subfamily $\{G_1, \dots, G_n\}$ of $\mathcal{G}$, Θ-adheres in X.*

Proof. Let X be a fuzzy almost regular, almost compact space, and $\mathcal{G} = \{G_\alpha : \alpha \in \Lambda\}$ a fuzzy grill on X with the stated property. Let us consider the collection

$$\mathcal{F} = \{ \bigcap_{\alpha \in \Lambda_0} \Theta\text{-cl}\, G_\alpha : \Lambda_0 \text{ is a finite subset of } \Lambda\}.$$

Then $\mathcal{F}$ is a prefilterbase on X. By fuzzy almost compactness of X, there is a fuzzy point x_t $(0 < t \leq 1)$ such that $x_t \leq \Theta\text{-cl}\,(\Theta\text{-cl}\, G)$, for each $G \in \mathcal{G}$ (by Theorem 3.3). As X is fuzzy almost regular, it follows that $x_t \leq \Theta\text{-cl}\, G$, for each $G \in \mathcal{G}$ and hence $\mathcal{G}$, Θ-adheres in X.

Conversely, let $\mathcal{F}$ be a prefilter on an fts X (not necessarily fuzzy almost regular), in which the given condition holds. Now $\mathcal{F}$ is contained in some fuzzy ultrafilter $\mathcal{U}$ on X. Then $\mathcal{U}$ is a fuzzy grill on X. Moreover, for any finite subcollection $\{U_1, U_2, \dots, U_n\}$ of $\mathcal{U}$, $\bigcap_{i=1}^{n} \Theta\text{-cl}\, U_i \geq \bigcap_{i=1}^{n} U_i \neq 0_X$. Thus by hypothesis, $\mathcal{U}$, Θ-adheres in X and hence $\mathcal{F}$, Θ-adheres in X. Then by Theorem 3.3, X is fuzzy almost compact. $\square$

A different version of necessity in the above theorem is given as follows.

3.14. Theorem. *For a fuzzy almost regular space X, fuzzy almost compactness of X implies that every fuzzy grill $\mathcal{G}$ on X with the property that $(\Theta\text{-cl}\, A) \, q \, (\Theta\text{-cl}\, B)$ for any two members A, B of $\mathcal{G}$, Θ-adheres in X.*

Proof. Let (X, τ) be a fuzzy almost regular, almost compact space and $\mathcal{G}$ a fuzzy grill on X with the stated property such that $\mathcal{G}$ does not Θ-adhere in X. For each $x \in X$ and each $n \in \mathbb{N}$ (the set of natural numbers), consider the fuzzy point $x_{\frac{1}{n}}$. Then there exists $A_{x(n)} \in \mathcal{G}$ such that $x_{\frac{1}{n}} \not\leq \Theta\text{-cl}\,(A_{x(n)})$. By Lemma 3.10,

$$\Theta\text{-cl}\,(A_{x(n)}) = \bigcap \{K_\mu : \mu \in \Lambda_{x(n)}\},$$

where each K_μ is fuzzy regular closed and $\Lambda_{x(n)}$ is an index set corresponding to the fuzzy set $A_{x(n)}$. Then $x_{\frac{1}{n}} \not\leq K_{\mu(x(n))}$, for some $\mu(x(n)) \in \Lambda_{x(n)}$. Now, it can be shown (by using Theorem 3.9 of [13]) that in a fuzzy almost regular space, every fuzzy regular closed set is fuzzy Θ-closed. Thus there is an open q-nbd $V_{x(n)}$ of $x_{\frac{1}{n}}$ such that $\mathrm{cl}\,(V_{x(n)}) \, \bar{q} \, K_{\mu(x(n))}$. As $V_{x(n)}$ is fuzzy open, we have $\mathrm{cl}\,(V_{x(n)}) = \Theta\text{-cl}\,(V_{x(n)}) = \Theta\text{-cl}\,(\Theta\text{-cl}\,(V_{x(n)}))$ (since X is fuzzy almost regular). Thus $\Theta\text{-cl}\,(\Theta\text{-cl}\,(V_{x(n)})) \, \bar{q} \, (\Theta\text{-cl}\,(A_{x(n)}))$. As $A_{x(n)} \in \mathcal{G}$, by the given property of $\mathcal{G}$ it follows that $\Theta\text{-cl}\,(V_{x(n)}) \notin \mathcal{G}$. Now, $V_{x(n)} \, q \, x_{\frac{1}{n}} \implies V_{x(n)}(x) + \frac{1}{n} > 1 \implies V_{x(n)}(x) > 1 - \frac{1}{n}$, and so $\{V_{x(n)} : x \in X, n \in \mathbb{N}\}$ forms a fuzzy open cover of X. By fuzzy almost compactness of X, there is a finite subset $\{x_1, x_2, \dots, x_n\}$ of X and finitely many positive integers $m_1, m_2, \dots, m_n$ such that $1_X = \bigcup_{i=1}^{n} \mathrm{cl}\,(V_{x_i(m_i)}) = \bigcup_{i=1}^{n} \Theta\text{-cl}\,(V_{x_i(m_i)}) \notin \mathcal{G}$, a contradiction. Hence $\mathcal{G}$ must Θ-adhere in X. $\square$

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