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# PAIRWISE SEPARATION AXIOMS IN INTUITIONISTIC TOPOLOGICAL SPACES

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**Abstract**

The concept of intuitionistic topological space was introduced by Çoker. The aim of this paper is to show the existence of natural functors between the category of intuitionistic topological spaces and continuous mappings and the category of bitopological spaces and pairwise continuous mappings. These functors are then used to generalize bitopological notions of separation to intuitionistic topological spaces, and some relations between these and previously defined notions are then obtained.

**Keywords:** Intuitionistic fuzzy set, Intuitionistic set, Intuitionistic topology, Bitopology, Category, Construct, Adjoint functor, Pairwise separation axioms.

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## 1. Introduction

After the introduction of fuzzy sets by Zadeh [22], various mathematicians introduced generalizations of the notion of fuzzy set. Among others, Atanassov [2, 3] introduced the notion of intuitionistic fuzzy set. Chang [7] used fuzzy sets to introduce the concept of a fuzzy topology. Later this concept was extended to intuitionistic fuzzy topological spaces by Çoker in [9]. The concept of intuitionistic set, which is a classical version of an intuitionistic fuzzy set, was first given by Çoker in [8]. He studied topology on intuitionistic sets in [10]. In this context, Çoker *et. al.* [4, 6, 12, 18] studied continuity, connectedness, compactness and separation axioms in intuitionistic fuzzy topological spaces.

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In this paper we follow the suggestion of J. G. Garcia and S. E. Rodabaugh [15] that "double fuzzy set" is a more appropriate name than "intuitionistic fuzzy set", and therefore adopt the term "double-set" for the intuitionistic set, and "double-topology" for the intuitionistic topology of Doğan Çoker. As in [13] (this issue) we denote by **Dbl-Top** the construct (concrete texture over **Set**) whose objects are pairs $(X, \tau)$ where $\tau$ is a double-topology on $X$, and whose morphisms are continuous mappings between such spaces.

Following a short section on preliminaries, in Section 3 we show that there exist functors $\mathfrak{B}$ : **Dbl-Top** → **Bitop** and $\mathfrak{D}$ : **Bitop** → **Dbl-Top**, where as usual **Bitop** denotes the construct of bitopological spaces and pairwise continuous mappings, and establish that these form an adjoint pair. In terms of the representation of double-sets as **3**-sets [13, 15], it is pointed out that $\mathfrak{D}$ extends to **Bitop** the Lowen functor $\omega : \mathbf{Top} \to \mathbf{3}^X$.

In Section 4, making use of this relation between bitopological spaces and double-topological spaces, we generalize several bitopological separation axioms to the case of double-topologies, and also give a notion of pairwise continuity for double-topological spaces. Finally, in Section 5, we briefly discuss the relations between these bitopologically based concepts and some previously defined separation axioms for double-topological spaces.

For concepts, notation and results relating to category theory we follow [1]. If particular, if **A** is a category then Ob**A** denote the class of objects and Mor**A** the class of morphisms of **A**.

## 2. Preliminaries

Throughout the paper by $X$ we denote a non-empty set. In this section we shall present various fundamental definitions and propositions. The following definition is obviously inspired by Atanassov [3].

**2.1. Definition.** [8] A **double-set** (DS for short) $A$ is an object having the form

$$A = \langle x, A_1, A_2 \rangle,$$

where $A_1$ and $A_2$ are subsets of $X$ satisfying $A_1 \cap A_2 = \emptyset$. The set $A_1$ is called the *set of members of* $A$, while $A_2$ is called the *set of non-members of* $A$.

Throughout the remainder of this paper we use the simpler notation $A = (A_1, A_2)$ for a double-set.

**2.2. Remark.** Every subset $A$ of $X$ is may obviously be regarded as a double-set having the form

$$A' = (A, A^c),$$

where $A^c = X \setminus A$ is the complement of $A$ in $X$.

We recall several relations and operations between DS's as follows:

**2.3. Definition.** [8] Let the DS's $A$ and $B$ on $X$ be of the form $A = (A_1, A_2)$, $B = (B_1, B_2)$, respectively. Furthermore, let $\{A_j : j \in J\}$ be an arbitrary family of DS's in $X$, where $A_j = (A_j^{(1)}, A_j^{(2)})$. Then

(a) $A \subseteq B$ if and only if $A_1 \subseteq B_1$ and $A_2 \supseteq B_2$;

(b) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$;

(c) $\overline{A} = (A_2, A_1)$ denotes the complement of $A$;

(d) $\bigcap A_j = (\bigcap A_j^{(1)},\ \bigcup A_j^{(2)})$;

(e) $\bigcup A_j = (\bigcup A_j^{(1)},\ \bigcap A_j^{(2)})$;
(f) $[\,]\, A = (A_1,\ A_1^c)$;
(g) $\langle\,\rangle\, A = (A_2^c,\ A_2)$;
(h) $\underset{\sim}{\emptyset} = (\emptyset, X)$ and $\underset{\sim}{X} = (X, \emptyset)$.

In addition we will require the following in this paper:

(i) $(\,)\, A = (A_1, \emptyset)$ and $)(\, A = (\emptyset, A_2)$.

Now we recall the image and preimage of DS's under a function.

**2.4. Definition.** [8] Let $X$ and $Y$ be two non-empty sets and $f : X \to Y$ a function. Then:

(a) If $B = (B_1, B_2)$ is an DS in $Y$, then the *preimage of $B$ under $f$*, denoted by $f^{-1}(B)$, is the DS in $X$ defined by

$f^{-1}(B) = (f^{-1}(B_1),\ f^{-1}(B_2))$.

(b) If $A = (A_1, A_2)$ is an DS in $X$, then the *image of $A$ under $f$*, denoted by $f(A)$, is the DS in $Y$ defined by

$f(A) = (f(A_1),\ f_{-}(A_2))$,

where $f_{-}(A_2) = (f(A_2^c))^c$.

The following includes several useful results:

**2.5. Proposition.** [8, 11] *Let $A, A_j, (j \in J)$, be DS's in $X$, $B, B_k, (k \in K)$, DS's in $Y$ and $f : X \to Y$ a function. Then:*

(a) $A_1 \subseteq A_2 \implies f(A_1) \subseteq f(A_2)$;
(b) $B_1 \subseteq B_2 \implies f^{-1}(B_1) \subseteq f^{-1}(B_2)$;
(c) $A \subseteq f^{-1}(f(A))$, *and if $f$ is injective then* $A = f^{-1}(f(A))$;
(d) $f(f^{-1}(B)) \subseteq B$, *and if $f$ is surjective then* $f(f^{-1}(B)) = B$;
(e) $f^{-1}(\bigcup B_k) = \bigcup f^{-1}(B_k)$;
(f) $f^{-1}(\bigcap B_k) = \bigcap f^{-1}(B_k)$;
(g) $f(\bigcup A_j) = \bigcup f(A_j)$;
(h) $f(\bigcap A_j) \subseteq \bigcap f(A_j)$, *and if $f$ is injective, then* $f(\bigcap A_j) = \bigcap f(A_j)$;
(i) $f^{-1}(\underset{\sim}{Y}) = \underset{\sim}{X}$;
(j) $f^{-1}(\underset{\sim}{\emptyset}) = \underset{\sim}{\emptyset}$;
(k) *If $f$ is surjective, then* $f(\underset{\sim}{X}) = \underset{\sim}{Y}$;
(l) $f(\underset{\sim}{\emptyset}) = \underset{\sim}{\emptyset}$;
(m) *If $f$ is surjective, then* $\overline{f(A)} \subseteq f(\overline{A})$. *If, furthermore, $f$ is injective, then have* $\overline{f(A)} = f(\overline{A})$;
(n) $f^{-1}(\overline{B}) = (\overline{f^{-1}(B)})$.

**2.6. Definition.** [8, 11] Let $p \in X$ be a fixed element in $X$. Then:

(a) The DS given by $\underset{\sim}{x} = (\{x\},\ \{x\}^c)$ is called a *double-point* (DP for short) in $X$;
(b) The DS $\underset{\approx}{x} = (\emptyset,\ \{x\}^c)$ is called a *vanishing double-point* (VDP for short) in $X$.

**2.7. Definition.** [8, 11]

(a) Let $\underset{\sim}{x}$ be a DP in $X$ and $A = (A_1, A_2)$ an DS in $X$. Then $\underset{\sim}{x}$ is said to *belong to* $A$ ($\underset{\sim}{x} \in A$ for short) iff $x \in A_1$.

(b) Let $\underset{\approx}{x}$ be a VDP in $X$ and $A = (A_1, A_2)$ an DS in $X$. Then $\underset{\approx}{x}$ is said to *belong to* $A$ ($\underset{\approx}{x} \in A$ for short) iff $x \notin A_2$.

It is clear that $\underset{\sim}{x} \in A \iff \underset{\sim}{x} \subseteq A$ and that $\underset{\approx}{x} \in A \iff \underset{\approx}{x} \subseteq A$.

**2.8. Definition.** [10] A double-topology (DT for short) on a set $X$ is a family $\tau$ of DS's in $X$ satisfying the following axioms:

T1: $\underset{\sim}{\emptyset}, \underset{\sim}{X} \in \tau$,
T2: $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$,
T3: $\bigcup G_j \in \tau$ for any arbitrary family $\{G_j : j \in J\} \subseteq \tau$.

In this case the pair $(X, \tau)$ is called a *double-topological space* (DTS for short), and any DS in $\tau$ is known as a *double open set* (DOS for short). The complement $\overline{A}$ of an DOS $A$ in an DTS is called a *double closed set* (DCS for short) in $X$.

**2.9. Definition.** [10] Let $(X, \tau)$ be an DTS and $A = (A_1, A_2)$ an DS in $X$. Then the interior and closure of $A$ are defined by:

$$\operatorname{int}(A) = \bigcup\{G : G \text{ is a DOS in } X \text{ and } G \subseteq A\},$$
$$\operatorname{cl}(A) = \bigcap\{H : H \text{ is a DCS in } X \text{ and } A \subseteq H\},$$

respectively.

It is clear that $\operatorname{cl}(A)$ is a DCS and $\operatorname{int}(A)$ a DOS in $X$. Moreover, $A$ is a DCS in $X$ iff $\operatorname{cl}(A) = A$, and $A$ is a DOS in $X$ iff $\operatorname{int}(A) = A$.

**2.10. Example.** [10] Any topological space $(X, \tau_0)$ gives rise to a DT in the form $\tau = \{A' : A \in \tau_0)\}$ by identifying a subset $A$ in $X$ with its counterpart $A' = (A, A^c)$, as in Remark 2.2.

**2.11. Definition.** [10] Let $(X, \tau)$ and $(Y, \phi)$ be two DTS's and let $f : X \to Y$ be a function. Then $f$ said to be *continuous* iff the preimage of each DS in $\phi$ is an DS in $\tau$.

**2.12. Definition.** [10] Let $(X, \tau)$ and $(Y, \phi)$ be two DTS's, and let $f : X \to Y$ be a function. Then $f$ said to be *open* iff the image of each DS in $\tau$ is an DS in $\phi$.

**2.13. Example.** [10] Let $(X, \tau_0)$ and $(Y, \phi_0)$ be two topological spaces. Then

(a) If $f : X \to Y$ is $\tau_0$–$\phi_0$ continuous in the usual sense, then $f$ is also $\tau$–$\phi$ continuous for the corresponding DTS in the sense of Definition 2.11.
(b) Let $f : X \to Y$ is a $\tau_0$–$\phi_0$ open function in the usual sense. Then $f$ is also $\tau$–$\phi$ open for the corresponding DTS in the sense of Definition 2.12.

## 3. The Constructs Dbl-Top and Bitop

We begin by recalling the following result which associates a bitopology with a double topology.

**3.1. Proposition.** [10] *Let* $(X, \tau)$ *be a* DTS.

(a) $\tau_1 = \{A_1 : \exists\, A_2 \subseteq X$ *with* $A = (A_1, A_2) \in \tau\}$ *is a topology on* $X$.
(b) $\tau_2^* = \{A_2 : \exists\, A_1 \subseteq X$ *with* $A = (A_1, A_2) \in \tau\}$ *is the family of closed sets of the topology* $\tau_2 = \{A_2^c : \exists\, A_1 \subseteq X$ *with* $A = (A_1, A_2) \in \tau\}$ *on* $X$.
(c) *Using* (a) *and* (b) *we may conclude that* $(X, \tau_1, \tau_2)$ *is a bitopological space.*

Now we note:

**3.2. Proposition.** *Let $(X, \tau)$, $(Y, \phi)$ be* DTS*'s and $f : X \to Y$ a function. Then if $f$ is $\tau$–$\phi$ continuous it is $(\tau_1, \tau_2)$–$(\phi_1, \phi_2)$ pairwise continuous.*

*Proof.* By the definition of pairwise continuity we must show that under the given hypothesis $f$ is $\tau_1$–$\phi_1$ continuous and $\tau_2$–$\phi_2$ continuous [16].

Take $U \in \phi_1$. Then we have $B \subseteq X$ with $(U, B) \in \tau$, so by the continuity of $f$, $f^{-1}(U, B) = (f^{-1}(U), f^{-1}(B)) \in \tau$ and hence $f^{-1}(U) \in \tau_1$. This shows that $f$ is $\tau_1$–$\phi_1$ continuous.

On the other hand take $V \in \phi_2$. Then for some $A \subseteq X$ we have $(A, V^c) \in \tau$, and as above the continuity of $f$ leads to $(f^{-1}(A), f^{-1}(V^c)) = (f^{-1}(A), (f^{-1}(V))^c) \in \tau$, whence $f^{-1}(V) \in \tau_2$ and we have established that $f$ is $\tau_2$–$\phi_2$ continuous. □

If we define $\mathfrak{B}$ : **Dbl-Top** → **Bitop** by $\mathfrak{B}(X, \tau) = (X, \tau_1, \tau_2)$, and $\mathfrak{B}(f) = f$ for $f : X \to Y$, then it is clear from Proposition 3.1 that:

**3.3. Lemma.** $\mathfrak{B}$ : **Dbl-Top** → **Bitop** *is a functor.*

We now wish to obtain a functor in the opposite direction, and to do this we must associate a double topology on $X$ with a given bitopology on $X$.

**3.4. Proposition.** *Let $(X, u, v)$ be a bitopological space. Then the family*

$$\{(U, V^c) : U \in u, V \in v, U \subseteq V\}$$

*is a double topology on $X$.*

*Proof.* The condition $U \subseteq V$ ensures that $U \cap V^c = \emptyset$, while the given family contains $\underset{\sim}{\emptyset}$ because $\emptyset \in u, v$, and it contains $\underset{\sim}{X}$ because $X \in u, v$. Finally this family is closed under finite intersections and arbitrary unions by Definition 2.3 (d,e) and the corresponding properties of the topologies $u$ and $v$. □

**3.5. Definition.** Let $(X, u, v)$ be a bitopological space. Then we set

$$\tau_{uv} = \{(U, V^c) : U \in u, V \in v, U \subseteq V\}$$

and call this the *double topology on $X$ associated with* $(X, u, v)$.

**3.6. Proposition.** *If $(X, u, v)$ is a bitopological space and $\tau_{uv}$ the corresponding* DT *on $X$, then*

$$(\tau_{uv})_1 = u \text{ and } (\tau_{uv})_2 = v.$$

*Proof.* $U \in u$ implies $(U, \emptyset) \in \tau_{uv}$ since $U \subseteq X \in v$, so $u \subseteq (\tau_{uv})_1$. Conversely, take $U \in (\tau_{uv})_1$. Then $(U, B) \in \tau_{uv}$ for some $B \subseteq X$, and now $U \in u$ by Definition 3.5. Hence $(\tau_{uv})_1 \subseteq u$, and the first equality is proved.

The proof of the second equality may be obtained in a similar way, and we omit the details. □

**3.7. Proposition.** *Let $(X, u, v)$, $(Y, s, t)$ be a bitopological spaces and $\tau_{uv}$, $\tau_{st}$ the corresponding* DT*'s. Then if $f : X \to Y$ is $(u, v)$–$(s, t)$ pairwise continuous it is $\tau_{uv}$–$\tau_{st}$ continuous in the sense of Definition 2.11.*

*Proof.* For $S \in s$, $T \in t$ with $S \subseteq T$ we have $(S, T^c) \in \tau_{st} \implies f^{-1}(S, T^c) = (f^{-1}(S), f^{-1}(T^c)) = (f^{-1}(S), (f^{-1}(T))^c)$ by Definition 2.4 (a). Certainly $f^{-1}(S) \subseteq f^{-1}(T)$, while $f^{-1}(S) \in u$, $f^{-1}(T) \in v$, so $f^{-1}(S, T^c) \in \tau_{uv}$. Hence, $f$ is $\tau_{uv}$–$\tau_{st}$ continuous. □

If we now define $\mathfrak{D}$ : **Bitop** → **Dbl-Top** by $\mathfrak{D}(X, u, v) = (X, \tau_{uv})$ and $\mathfrak{D}(f) = f$ for $f : X \to Y$, then it is clear from Proposition 3.7 that:

**3.8. Lemma.** *$\mathfrak{D}$ : **Bitop** → **Dbl-Top** is a functor.*

We see from Proposition 3.6 that $\mathfrak{B} \circ \mathfrak{D} = id_{\mathbf{Bitop}}$. If it were also the case that $\mathfrak{D} \circ \mathfrak{B} = id_{\mathbf{Dbl\text{-}Top}}$, then $\mathfrak{B}$ and $\mathfrak{D}$ would be isomorphisms. That this is not the case is shown by the following example.

**3.9. Example.** Let $X = \{a, b\}$ and $\tau = \{(X, \emptyset), (\{a\}, \{b\}), (\emptyset, X)\}$. Then $(X, \tau)$ is a DT, $\tau_1 = \tau_2 = \{X, \{a\}, \emptyset\}$, and

$$\tau_{\tau_1\tau_2} = \{(X, \emptyset), (\{a\}, \emptyset), (\{a\}, \{b\}), (\emptyset, \{b\}), (\emptyset, X)\} \neq \tau.$$

Hence, $(\mathfrak{D} \circ \mathfrak{B})(X, \tau) \neq (X, \tau)$ so $\mathfrak{D} \circ \mathfrak{B} \neq id_{\mathbf{Dbl\text{-}Top}}$

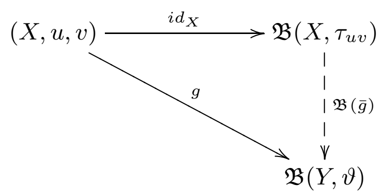
Even though $\mathfrak{B}$ and $\mathfrak{D}$ are not isomorphisms, we do have the following important result.

**3.10. Theorem.** *The functor $\mathfrak{B}$ : **Dbl-Top** → **Bitop** is an adjoint of the functor $\mathfrak{D}$ : **Bitop** → **Dbl-Top**.*

*Proof.* To prove that $\mathfrak{B}$ is an adjoint we must show that for any $(X, u, v) \in$ Ob**Bitop** there exists a $\mathfrak{B}$-universal arrow with domain $(X, u, v)$ [1, Definition 18.1]. We claim that $(id_X, (X, \tau_{uv}))$ is such an arrow, so by [1, Definition 8.30] we must show that this is a $\mathfrak{B}$-structured arrow with domain $(X, u, v)$ which is $\mathfrak{B}$-universal for **Bitop**. Certainly $(X, \tau_{uv}) \in$ Ob**Dbl-Top**, and by Proposition 3.6 we have $\mathfrak{B}(X, \tau_{uv}) = (X, u, v)$, so

$$id_X : (X, u, v) \to \mathfrak{B}(X, \tau_{uv})$$

is bicontinuous and hence a **Bitop** morphism. Hence $(id_X, (X, \tau_{uv}))$ is indeed a $\mathfrak{B}$-structured arrow with domain $(X, u, v)$. To prove it is $\mathfrak{B}$-universal for **Bitop**, let $(g, (Y, \vartheta))$ be any $\mathfrak{B}$-structured arrow with domain $(X, u, v)$. We must prove the existence of a unique **Dbl-Top** morphism $\bar{g} : (X, \tau_{uv}) \to (Y, \vartheta)$ that makes the following diagram commutative:




It is clear that the only possible choice for $\bar{g}$ is $g$, so we must show that

$$\mathfrak{B}(g) : \mathfrak{B}(X, \tau_{uv}) \to \mathfrak{B}(Y, \vartheta) = g : (X, u, v) \to (Y, \vartheta_1, \vartheta_2)$$

is pairwise continuous. However this is immediate since $g$ is given to be a **Bitop** morphism. This completes the proof that $\mathfrak{B}$ is an adjoint, and since $(X, \tau_{uv}) = \mathfrak{D}(X, u, v)$ we see that $\mathfrak{D}$ is the corresponding co-adjoint functor (see [1], Section 19). □

In view of the identification of double topologies with **3**-topologies ($\mathbf{3} = \{0, \frac{1}{2}, 1\}$) [13, 15], it is interesting to compare the above functors with the Lowen functors $\omega$ and $\iota$. Naturally, these functors effectively map between **Top** and **Dbl-Top**, rather than between **Bitop** and **Dbl-Top**, but certainly $\mathfrak{D}$ may be restricted to bitopologies of the form $(X, u, u)$, that is to topologies, and hence compared with the functor $\omega$.

**3.11. Theorem.** *If $\zeta : \mathbf{3}^X \to \mathbb{D}_X$ is the isomorphism between the Hutton algebra of **3**-sets of $X$ and the Hutton algebra of double-sets of $X$ described in [13], and $\omega$ : **Top** → **3-Top** the Lowen functor, then $\zeta \circ \omega(X, u) = \mathfrak{D}(X, u, u)$.*

*Proof.* Since **3** is a chain, $\{\chi_U \mid U \in u\} \cup \{\mathbf{k} \mid k \in \mathbf{3}\}$ is a subbase of $\omega(X,u)$ by [20, Theorem 3.3]. Applying $\zeta(\mu) = (A_\mu, B_\mu)$, $A_\mu = \{x \in X \mid \mu(x) = 1\}$, $B_\mu = \{x \in X \mid \mu(x) = 0\}$ shows that $\mathcal{S} = \{(U, U^c) \mid U \in u\} \cup \{(X,\emptyset), (\emptyset,\emptyset), (\emptyset, X)\}$ is a subbase for $\zeta \circ \omega(X,u)$. On the other hand we clearly have $\mathcal{S} \subseteq \tau_{uu}$, while if $U_1, U_2 \in u$, $U_1 \subseteq U_2$, then

$$(U_1, U_2^c) = (U_1, U_1^c) \cup ((\emptyset,\emptyset) \cap (U_2, U_2^c)),$$

so $\mathcal{S}$ is a subbase for $\tau_{uu}$ also. This completes the proof of the stated equality. □

We see from this theorem that $\mathfrak{D}$ extends the Lowen functor $\omega$ to **Bitop** in such a way that there exists an adjoint $\mathfrak{B}$ : **Dbl-Top** → **Bitop**. It would seem that this situation is of sufficient interest to warrant further study in a wider context.

## 4. Pairwise Separation Axioms in Double Topological Spaces

In this section we use the link between bitopological spaces and double topological spaces established above to generalize several bitopological separation properties to the case of double topological spaces. We limit ourselves to the strong forms of the pairwise $T_0$, $T_1$ and $T_2$ properties, and will refer to these as pairwise $T_i$, $i = 0,1,2$ throughout this paper. We will also consider the pairwise $R_0$ and pairwise $R_1$ axioms, which are related to the pairwise $T_i$ axioms by having pairwise $T_1$ = pairwise $T_0$ + pairwise $R_0$ and pairwise $T_2$ = pairwise $T_0$ + pairwise $R_1$.

The (strong) $T_0$ and $T_1$ axioms are due to Fletcher, Hoyle and Patty [14]:

**4.1. Definition.** Let $(X,u,v)$ be a bitopological space. Then $(X,u,v)$ is called

(1) *pairwise* $T_0$, if given $x \neq y$ in $X$ there exists $U \in u$ such that $x \in U,\ y \notin U$ or there exists $V \in v$ with $y \in V,\ x \notin V$.
(2) *pairwise* $T_1$, if given $x \neq y$ in $X$ there exists $U \in u$ such that $x \in U,\ y \notin U$ and there exists $V \in v$ with $y \in V,\ x \notin V$.

**4.2. Proposition.** *If $(X,u,v)$ is pairwise $T_0$ then $(X,\tau_{uv})$ satisfies the following condition:*

*Given $x \neq y$ in $X$ there exists $G \in \tau_{uv}$ with $\underset{\sim}{x} \in G$, $\underset{\sim}{y} \notin G$, or there exists $H \in \tau_{uv}$ with $\underset{\approx}{y} \in H$, $\underset{\approx}{x} \notin H$.*

*Proof.* Let $(X,u,v)$ be pairwise $T_0$ and $x \neq y$. Then there exists $U \in u$ such that $x \in U,\ y \notin U$, or there exists $V \in v$ such that $y \in V,\ x \notin V$. In the first case we may take $G = (U,\emptyset) \in \tau_{uv}$. Then $x \in U$ means that $\underset{\sim}{x} \in G$, and $y \notin U$ means that $\underset{\sim}{y} \notin G$. In the second case we may take $H = (\emptyset, V^c) \in \tau_{uv}$, and now $y \in V$ means that $\underset{\approx}{y} \in H$ and $x \notin V$ means that $\underset{\approx}{x} \notin H$. This establishes the required condition. □

This suggests the following definition for general double topologies.

**4.3. Definition.** The DTS $(X,\tau)$ is called *pairwise* $T_0$ if given $x \neq y$ in $X$ there exists $G \in \tau$ with $\underset{\sim}{x} \in G$, $\underset{\sim}{y} \notin G$, or there exists $H \in \tau$ with $\underset{\approx}{y} \in H$, $\underset{\approx}{x} \notin H$.

**4.4. Proposition.** *If $(X,\tau)$ is pairwise $T_0$ then $(X,\tau_1,\tau_2)$ is pairwise $T_0$.*

*Proof.* Take $x \neq y$ in $X$. Then there exists $G = (A,B) \in \tau$ with $\underset{\sim}{x} \in G$, $\underset{\sim}{y} \notin G$, or there exists $H = (C,D) \in \tau$ with $\underset{\approx}{y} \in H$, $\underset{\approx}{x} \notin H$. In the first case $x \in A \in \tau_1$, $y \notin A$, and in the second $y \in D^c \in \tau_2$, $x \notin D^c$, so $(X,\tau_1,\tau_2)$ is pairwise $T_0$. □

**4.5. Corollary.** *$(X, u, v)$ is pairwise $T_0$ iff $(X, \tau_{uv})$ is pairwise $T_0$.*

*Proof.* Necessity follows from Proposition 4.2, and sufficiency from Proposition 4.4 and Proposition 3.6. □

**4.6. Corollary.** *The functors $\mathfrak{B}$ and $\mathfrak{D}$ preserve the pairwise $T_0$ property.*

Now we turn our attention to the pairwise $T_1$ property.

**4.7. Proposition.** *If $(X, u, v)$ is pairwise $T_1$ then $(X, \tau_{uv})$ satisfies the following condition:*

*Given $x \neq y$ in $X$ there exists $G \in \tau_{uv}$ with $\underset{\sim}{x} \in G$, $\underset{\sim}{y} \notin G$, and there exists $H \in \tau_{uv}$ with $\underset{\approx}{y} \in H$, $\underset{\approx}{x} \notin H$.*

*Proof.* As for the proof of Proposition 4.2 with "or" replaced by "and". □

This suggests the following definition for general double topologies.

**4.8. Definition.** The DTS $(X, \tau)$ is called *pairwise $T_1$* if given $x \neq y$ in $X$ there exists $G \in \tau$ with $\underset{\sim}{x} \in G$, $\underset{\sim}{y} \notin G$, and there exists $H \in \tau$ with $\underset{\approx}{y} \in H$, $\underset{\approx}{x} \notin H$.

**4.9. Proposition.** *If $(X, \tau)$ is pairwise $T_1$ then $(X, \tau_1, \tau_2)$ is pairwise $T_1$.*

*Proof.* As for the proof of Proposition 4.4 with "or" replaced by "and". □

**4.10. Corollary.** *$(X, u, v)$ is pairwise $T_1$ iff $(X, \tau_{uv})$ is pairwise $T_1$.*

*Proof.* Necessity follows from Proposition 4.7, and sufficiency from Proposition 4.9 and Proposition 3.6. □

**4.11. Corollary.** *The functors $\mathfrak{B}$ and $\mathfrak{D}$ preserve the pairwise $T_1$ property.*

The pairwise $R_0$ axiom for bitopological spaces was introduced by Murdeshwar and Naimpally [17]:

**4.12. Definition.** $(X, u, v)$ is called *pairwise $R_0$* if $x \in U \in u$ implies $\mathrm{cl}_v(\{x\}) \subseteq U$ and $x \in V \in v$ implies $\mathrm{cl}_u(\{x\}) \subseteq V$.

**4.13. Proposition.** *If $(X, u, v)$ is pairwise $R_0$ then $(X, \tau_{uv})$ satisfies the conditions:*

*$\underset{\sim}{x} \in G \in \tau_{uv}$ implies there exists $H \in \tau_{uv}$ with $\underset{\approx}{x} \notin H$ and $\overline{H} \subseteq (\,)\, G$, and*

*$\underset{\approx}{x} \in G \in \tau_{uv}$ implies there exists $H \in \tau_{uv}$ with $\underset{\sim}{x} \notin H$ and $)(\, \overline{H} \subseteq G$.*

*Proof.* Suppose that $(X, u, v)$ is pairwise $R_0$. Then

$$\underset{\sim}{x} \in G \in \tau_{uv} \implies G = (U, V^c) \text{ with, in particular, } x \in U \in u \implies \mathrm{cl}_v(\{x\}) \subseteq U.$$

If we define $H = (\emptyset, \mathrm{cl}_v(\{x\}))$ then clearly $H \in \tau_{uv}$, $\underset{\approx}{x} \notin H$ and $\overline{H} = (\mathrm{cl}_v(\{x\}), \emptyset) \subseteq (A, \emptyset) = (\,)\, G$ (see Definition 2.3 (h)). On the other hand,

$$\underset{\approx}{x} \in G \in \tau_{uv} \implies G = (U, V^c) \text{ with, in particular, } x \in V \in v \implies \mathrm{cl}_u(\{x\}) \subseteq V.$$

If now we define $H = \big((\mathrm{cl}_u(\{x\}))^c, \emptyset\big)$, then clearly $H \in \tau_{uv}$, $\underset{\sim}{x} \notin H$ and

$$\overline{H} = (\emptyset, (\mathrm{cl}_u(\{x\}))^c) = )(\, \overline{H} \subseteq G,$$

see Definition 2.3 (h). □

This leads to the following definition for double topological spaces:

**4.14. Definition.** The DTS $(X,\tau)$ is called *pairwise* $R_0$ if it satisfies the conditions:

$\underset{\sim}{x} \in G \in \tau$ implies there exists $H \in \tau$ with $\underset{\approx}{x} \notin H$ and $\overline{H} \subseteq ()\, G$, and

$\underset{\approx}{x} \in G \in \tau$ implies there exists $H \in \tau$ with $\underset{\sim}{x} \notin H$ and $)(\, \overline{H} \subseteq G$.

**4.15. Proposition.** *If $(X,\tau)$ is pairwise $R_0$, then $(X,\tau_1,\tau_2)$ is pairwise $R_0$.*

*Proof.* Let $(X,\tau)$ be pairwise $R_0$ and take $x \in U \in \tau_1$. Then for some $B \subseteq X$ we have $G = (U,B) \in \tau$, and since $\underset{\sim}{x} \in G$ there exists $H = (C,D) \in \tau$ with $\underset{\approx}{x} \notin H$ and $\overline{H} = (D,C) \subseteq ()\, G = (U,\emptyset)$. Now $x \in D \subseteq U$, and $D$ is $\tau_2$–closed, so $\mathrm{cl}_{\tau_2}(\{x\}) \subseteq U$.

On the other hand, $x \in V \in \tau_2$ leads to $\mathrm{cl}_{\tau_1}(\{x\}) \subseteq V$ by a similar argument, so $(X,\tau_1,\tau_2)$ is pairwise $R_0$. □

**4.16. Corollary.** *$(X,u,v)$ is pairwise $R_0$ iff $(X,\tau_{uv})$ is pairwise $R_0$.*

*Proof.* Necessity follows from Proposition 4.13, and sufficiency from Proposition 4.15 and Proposition 3.6. □

**4.17. Corollary.** *The functors $\mathfrak{B}$ and $\mathfrak{D}$ preserve the pairwise $R_0$ property.*

The following theorem shows that the same relations hold between the pairwise $T_0$, $T_1$ and $R_0$ axioms for double topological spaces as hold for bitopological spaces.

**4.18. Theorem.** *The* DTS *$(X,\tau)$ is pairwise $T_1$ iff $(X,\tau)$ is pairwise $T_0$ and pairwise $R_0$.*

*Proof.* Suppose first that $(X,\tau)$ is pairwise $T_1$. Then:

- $(X,\tau)$ is pairwise $T_0$. This is clear from the definitions.
- $(X,\tau)$ is pairwise $R_0$. Suppose not. Then either
  (a) There exists $\underset{\sim}{x} \in G \in \tau$ so that for all $H \in \tau$, $\underset{\approx}{x} \in H$ or $\overline{H} \nsubseteq ()\, G$, or
  (b) There exists $\underset{\approx}{x} \in G \in \tau$ so that for all $H \in \tau$, $\underset{\sim}{x} \in H$ or $)(\, \overline{H} \nsubseteq G$.

  **Case (a)**. $\underset{\sim}{x} \in G = (A,B)$ gives $x \in A$. Let $H_0 = \bigcup\{H \in \tau \mid \underset{\approx}{x} \notin H\}$. Then $\underset{\approx}{x} \notin H_0 = (C_0,D_0) \in \tau$, so $\overline{H_0} = (D_0,C_0) \nsubseteq ()\, G = (A,\emptyset)$, whence $D_0 \nsubseteq A$. Take $y \in D_0$ with $y \notin A$. Then $y \neq x$ since $x \in A$, so by the pairwise $T_1$ axiom we have $H_1 = (C_1,D_1) \in \tau$ with $\underset{\approx}{x} \notin H_1$, $\underset{\approx}{y} \in H_1$. By the definition of $H_0$ we have $H_1 \subseteq H_0$, so $y \in D_0 \subseteq D_1$ which contradicts $\underset{\approx}{y} \in H_1$.

  **Case (b)**. A contradiction may be obtained by an argument dual to that given for Case (a), and we omit the details. This completes the proof that $(X,\tau)$ is pairwise $R_0$.

Now suppose that $(X,\tau)$ is pairwise $T_0$ and pairwise $R_0$. Take $x \neq y$ in $X$. Applying the pairwise $T_0$ axiom we have two cases.

**Case (i)**. There exists $G \in \tau$ with $\underset{\sim}{x} \in G$, $\underset{\sim}{y} \notin G$. By the pairwise $R_0$ axiom we have $H \in \tau$ with $\underset{\approx}{x} \notin H$ and $\overline{H} \subseteq ()\, G$. Letting $G = (A,B)$, $H = (C,D)$ we have $(D,C) \subseteq ()(A,B) = (A,\emptyset)$, so $D \subseteq A$. Now $\underset{\sim}{y} \notin G \implies y \notin A \implies y \notin D$, so $\underset{\approx}{y} \in H$.

**Case (ii)**. There exists $H \in \tau$ with $\underset{\approx}{y} \in H$ and $\underset{\approx}{x} \notin H$. By the pairwise $R_0$ axiom there exists $G \in \tau$ with $\underset{\sim}{y} \notin G$ and $)(\overline{G} \subseteq H$. Letting $G = (A, B)$, $H = (C, D)$ we have $)(\overline{G} = (\emptyset, A) \subseteq (C, D)$, so $D \subseteq A$. Now $\underset{\approx}{x} \notin H \implies x \in D \implies x \in A$, so $\underset{\sim}{x} \in G$.

This verifies that $(X, \tau)$ is pairwise $T_1$. □

We now turn to the pairwise $T_2$ property. Weston [21] defined this property in its strong form before the term "bitopology" was introduced by Kelly [16], and he used the name "consistent". The term "pairwise $T_2$" was first used by Kelly in [16]. It is defined as follows:

**4.19. Definition.** The bitopological space $(X, u, v)$ is called *pairwise* $T_2$, if given $x \neq y$ in $X$ there exists $U \in u,\ V \in v$ satisfying $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

**4.20. Proposition.** *If $(X, u, v)$ is pairwise $T_2$ then $(X, \tau_{uv})$ has the property:*

*Given $x \neq y$ in $X$, there exists $G,\ H \in \tau_{uv}$ with $\underset{\sim}{x} \in G$, $\underset{\approx}{y} \in H$ and $G \subseteq ()\,\overline{H}$.*

*Proof.* Take $x \neq y$ in $X$. Then we have $U \in u$, $V \in v$ satisfying $x \in U$, $y \in V$ and $U \cap V = \emptyset$, that is $U \subseteq V^c$. Now defining $G = (U, \emptyset)$, $H = (\emptyset, V^c)$ gives $G,\ H \in \tau_{uv}$, $\underset{\sim}{x} \in G$, $\underset{\approx}{y} \in H$ and $G \subseteq ()\,\overline{H}$. □

This gives the following definition for double topologies:

**4.21. Definition.** The DTS $(X, \tau)$ is called *pairwise* $T_2$ if given $x \neq y$ in $X$ there exists $G,\ H \in \tau$ satisfying $\underset{\sim}{x} \in G$, $\underset{\approx}{y} \in H$ and $G \subseteq ()\,\overline{H}$.

**4.22. Proposition.** *If $(X, \tau)$ is pairwise $T_2$ then $(X, \tau_1, \tau_2)$ is pairwise $T_2$.*

*Proof.* Take $x \neq y$ in $X$. Then we have $G = (A, B)$, $H = (C, D) \in \tau$ with $\underset{\sim}{x} \in G$, $\underset{\approx}{y} \in H$ and $G \subseteq ()\,\overline{H}$. Clearly $x \in A \in \tau_1$, $y \in D^c \in \tau_2$, while $(A, B) = G \subseteq ()\,\overline{H} = ()\,(D, C) = (D, \emptyset)$, so $A \subseteq D$, that is $A \cap D^c = \emptyset$. Hence $(X, \tau_1, \tau_2)$ is pairwise $T_2$. □

**4.23. Corollary.** *The bitopological space $(X, u, v)$ is pairwise $T_2$ iff $(X, \tau_{uv})$ is pairwise $T_2$.*

*Proof.* Necessity follows from Proposition 4.20, and sufficiency from Proposition 4.22 and Proposition 3.6. □

**4.24. Corollary.** *The functors $\mathfrak{B}$ and $\mathfrak{D}$ preserve the pairwise $T_2$ property.*

Finally, we consider the pairwise $R_1$ property, first defined by Reilly [19] and known in the literature under various different names.

**4.25. Definition.** The bitopological space $(X, u, v)$ is called *pairwise* $R_1$ if $x \notin \mathrm{cl}_u(\{y\})$ $(y \notin \mathrm{cl}_v(\{x\}))$ implies there exists $U \in u$, $v \in V$ with $U \cap V = \emptyset$ and $x \in U$, $y \in V$.

**4.26. Proposition.** *If $(X, u, v)$ is pairwise $R_1$ then $(X, \tau_{uv})$ satisfies the condition:*

*Given $x, y \in X$, if there exists $M \in \tau_{uv}$ with $\underset{\sim}{x} \in M$, $\underset{\sim}{y} \notin M$ ($\underset{\approx}{y} \in M$, $\underset{\approx}{x} \notin M$) then there exist $G,\ H \in \tau_{uv}$ with $\underset{\sim}{x} \in G$, $\underset{\approx}{y} \in H$ and $G \subseteq ()\,\overline{H}$.*

*Proof.* If we have $M \in \tau_{uv}$ with $\underset{\sim}{x} \in M$, $\underset{\sim}{y} \notin M$ then $M = (A, B)$, $x \in A \in u$, $y \notin A$, so $x \notin \mathrm{cl}_u(\{y\})$. Likewise, $\underset{\approx}{y} \in M$, $\underset{\approx}{x} \notin M$ gives $y \notin \mathrm{cl}_v(\{x\})$, so in either case we have $U \in u$, $V \in v$ satisfying $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Defining $G = (U, \emptyset)$, $H = (\emptyset, V^c)$ as in the proof of Proposition 4.20 now completes the proof. □

This leads to the following definition for double topological spaces.

**4.27. Definition.** The DTS $(X, \tau)$ is called *pairwise* $R_1$ if it satisfies the condition: Given $x, y \in X$, if there exists $M \in \tau$ with $\underset{\sim}{x} \in M$, $\underset{\sim}{y} \notin M$ ($\underset{\approx}{y} \in M$, $\underset{\approx}{x} \notin M$) then there exist $G$, $H \in \tau$ with $\underset{\sim}{x} \in G$, $\underset{\approx}{y} \in H$ and $G \subseteq () \overline{H}$.

**4.28. Proposition.** *If* $(X, \tau)$ *is pairwise* $R_1$ *then* $(X, \tau_1, \tau_2)$ *is pairwise* $R_1$.

*Proof.* First take $x, y \in X$ with $x \notin \mathrm{cl}_{\tau_1}(\{y\})$. Now we have $x \in P \in \tau_1$ with $y \notin P$, so for some $Q \subseteq X$ we have $(P, Q) \in \tau$. Setting $M = (P, Q)$ gives $\underset{\sim}{x} \in M$, $\underset{\sim}{y} \notin M$, so we have $G, H \in \tau$ satisfying $\underset{\sim}{x} \in G$, $\underset{\approx}{y} \in H$ and $G \subseteq () \overline{H}$. By a similar argument $G, H$ satisfying the same conditions may also be obtained if $y \notin \mathrm{cl}_{\tau_2}(\{x\})$. The existence of $\tau_1$, $\tau_2$ open sets separating $x$ and $y$ now follows as in the proof of Proposition 4.22. □

**4.29. Corollary.** *The bitopological space* $(X, u, v)$ *is pairwise* $R_1$ *iff* $(X, \tau_{uv})$ *is pairwise* $R_1$.

*Proof.* Necessity follows from Proposition 4.26, and sufficiency from Proposition 4.28 and Proposition 3.6. □

**4.30. Corollary.** *The functors* $\mathfrak{B}$ *and* $\mathfrak{D}$ *preserve the pairwise* $R_1$ *property.*

Again, the relations between these various axioms remain the same as for bitopological spaces.

**4.31. Theorem.** *For a* DTS $(X, \tau)$ *the following statements are valid:*

(i) *pairwise* $R_1 \implies$ *pairwise* $R_0$.
(ii) $(X, \tau)$ *is pairwise* $T_2$ *iff* $(X, \tau)$ *is pairwise* $T_0$ *and pairwise* $R_1$.

*Proof.* (i). Suppose that $(X, \tau)$ is pairwise $R_1$, but that $(X, \tau)$ is not pairwise $R_0$. Then, exactly as in the proof of Theorem 4.18, we have two cases (a) and (b).

Considering Case (a), then for $G = (A, B)$ and $H_0 = (C_0, D_0)$ as defined in the proof of Theorem 4.18, we again choose $y \in D_0$, $y \notin A$. Then $\underset{\sim}{x} \in G$, $\underset{\sim}{y} \notin G$ so by pairwise $R_1$ there exists $U = (U_1, V_1), V = (V_1, V_2) \in \tau$ satisfying $\underset{\sim}{x} \in U$, $\underset{\approx}{y} \in V$ and $U \subseteq () \overline{V}$, i.e. $U_1 \subseteq V_2$. Then

$$x \in U_1 \implies x \in V_2 \implies \underset{\approx}{x} \notin V \implies V \subseteq H_0 \implies D_0 \subseteq V_2 \implies y \in V_2,$$

which contradicts $\underset{\approx}{y} \in V$. In just the same way Case (b) also leads to a contradiction, and we deduce that $(X, \tau)$ is pairwise $R_0$.

(ii). Let $(X, \tau)$ be pairwise $T_2$. Then

- $(X,\tau)$ is pairwise $T_0$. To see this take $x \neq y$ in $X$. Then we have $G=(A,B), H=(C,D) \in \tau$ with $\underset{\sim}{x} \in G$, $\underset{\approx}{y} \in H$ and $G \subseteq ()\, \overline{H}$, i.e. $A \subseteq D$. Hence,

  $$\underset{\approx}{y} \in H \implies y \notin D \implies y \notin A \implies \underset{\sim}{y} \notin G,$$

  as required.
- $(X,\tau)$ is pairwise $R_1$. Take $x, y \in X$, $M \in \tau$. Then

  $$\underset{\sim}{x} \in M,\ \underset{\sim}{y} \notin M \implies x \neq y \implies \exists G, H \in \tau \text{ with } \underset{\sim}{x} \in G, \underset{\approx}{y} \in H, G \subseteq ()\, H$$

  since $(X,\tau)$ is pairwise $T_2$, and we reach the same conclusion if $\underset{\approx}{y} \in M$, $\underset{\approx}{x} \notin M$.

  This verifies that $(X,\tau)$ is pairwise $R_1$.

Conversely, let $(X,\tau)$ be pairwise $T_0$ and pairwise $R_1$, and take $x \neq y$ in $X$. Then using pairwise $T_0$ we have $G \in \tau$ with $\underset{\sim}{x} \in G$, $\underset{\sim}{y} \notin G$, or we have $H \in \tau$ with $\underset{\approx}{y} \in H$, $\underset{\approx}{x} \notin H$. In both cases the pairwise $R_1$ axiom gives the existence of $U, V \in \tau$ satisfying $\underset{\sim}{x} \in U$, $\underset{\approx}{y} \in V$ and $U \subseteq ()\, \overline{V}$, which is just pairwise $T_2$. □

**4.32. Corollary.** *pairwise* $T_2 \implies$ *pairwise* $T_1$.

*Proof.* Clear from Theorems 4.18 and 4.31. It can also be easily proved directly by suitably augmenting the proof of pairwise $T_2$ implies pairwise $T_0$ given above. □

## 5. Comparisons with Other Separation Properties

In this final section we mention some relationships between the pairwise separation properties defined above and the separation properties given in [5].

**5.1. Definition.** Let $(X,\tau)$ be DTS. Then $(X,\tau)$ is said to be:

(a) $T_0(i) \iff \forall\, x, y \in X\ (x \neq y)\ \exists\, U \in \tau$ such that
$(\underset{\sim}{x} \in U,\ \underset{\sim}{y} \notin U)$ or $(\underset{\sim}{y} \in U,\ \underset{\sim}{x} \notin U)$;

(b) $T_0(ii) \iff \forall\, x, y \in X\ (x \neq y)\ \exists\, U \in \tau$ such that
$(\underset{\approx}{x} \in U,\ \underset{\approx}{y} \notin U)$ or $(\underset{\approx}{y} \in U,\ \underset{\approx}{x} \notin U)$;

(c) $T_0(iii) \iff \forall\, x, y \in X\ (x \neq y)\ \exists\, U \in \tau$ such that $\underset{\sim}{x} \in U \subseteq \overline{\underset{\sim}{y}}$ or $\underset{\sim}{y} \in U \subseteq \overline{\underset{\sim}{x}}$;

(d) $T_0(iv) \iff \forall\, x, y \in X\ (x \neq y)\ \exists\, U \in \tau$ such that $\underset{\approx}{x} \in U \subseteq \overline{\underset{\approx}{y}}$ or $\underset{\approx}{y} \in U \subseteq \overline{\underset{\approx}{x}}$.

**5.2. Definition.** Let $(X,\tau)$ be DTS. Then $(X,\tau)$ is said to be:

(a) $T_1(i) \iff \forall\, x, y \in X\ (x \neq y)\ \exists\, U, V \in \tau$ such that
$(\underset{\sim}{x} \in U,\ \underset{\sim}{y} \notin U)$ and $(\underset{\sim}{y} \in V,\ \underset{\sim}{x} \notin V)$;

(b) $T_1(ii) \iff \forall\, x, y \in X\ (x \neq y)\ \exists\, U, V \in \tau$ such that
$(\underset{\approx}{x} \in U,\ \underset{\approx}{y} \notin U)$ and $(\underset{\approx}{y} \in V,\ \underset{\approx}{x} \notin V)$;

(c) $T_1(iii) \iff \forall\, x, y \in X\ (x \neq y)\ \exists\, U, V \in \tau$ such that $\underset{\sim}{x} \in U \subseteq \overline{\underset{\sim}{y}}$ and $\underset{\sim}{y} \in V \subseteq \overline{\underset{\sim}{x}}$;

(d) $T_1(iv) \iff \forall\, x, y \in X\ (x \neq y)\ \exists\, U, V \in \tau$ such that $\underset{\approx}{x} \in U \subseteq \overline{\underset{\approx}{y}}$ and $\underset{\approx}{y} \in V \subseteq \overline{\underset{\approx}{x}}$.

**5.3. Definition.** Let $(X,\tau)$ be a DTS. Then $(X,\tau)$ is said to be:

(a) $T_2(i) \iff \forall\, x, y \in X\ (x \neq y)\ \exists\, U, V \in \tau$ such that $\underset{\sim}{x} \in U$, $\underset{\sim}{y} \in V$ and $U \cap V = \underset{\sim}{\emptyset}$;

(b) $T_2(ii) \iff \forall\, x, y \in X\ (x \neq y)\ \exists\, U, V \in \tau$ such that $\underset{\approx}{x} \in U$, $\underset{\approx}{y} \in V$ and $U \cap V = \underset{\sim}{\emptyset}$;

(c) $T_2(iii) \iff \forall\, x, y \in X\ (x \neq y)\ \exists\, U, V \in \tau$ such that $\underset{\sim}{x} \in U$, $\underset{\sim}{y} \in V$ and $U \subseteq \overline{V}$;

(d) $T_2(iv) \iff \forall\, x, y \in X\ (x \neq y)\ \exists\, U, V \in \tau$ such that $\underset{\approx}{x} \in U$, $\underset{\approx}{y} \in V$ and $U \subseteq \overline{V}$.

It is clear that there is considerable similarity with the pairwise separation axioms in general, but that there are small but important difference which prevent equivalences between them. We note however the following relations, and the interested reader may well discover others.

**5.4. Proposition.** *For a* DTS $(X, \tau)$ *we have:*

(1) *Pairwise* $T_0 \implies T_0\,(iv)$.
(2) *Pairwise* $T_1 \implies T_1\,(iv)$.
(3) $T_2\,(iii) \implies$ *pairwise* $T_2$.

*Proof.* (1). Let $(X, \tau)$ be pairwise $T_0$, and take $x \neq y$ in $X$. Suppose we have $U = (U_1, U_2) \in \tau$ with $\underset{\sim}{x} \in U$, $\underset{\sim}{y} \notin U$. Now $x \in U_1$ and $U_1 \cap U_2 = \emptyset$, so $x \notin U_2$ and hence $\underset{\approx}{x} \in U$. On the other hand $y \notin U_1$ so $U_1 \subseteq \{y\}^c$ and hence $U = (U_1, U_2) \subseteq (\{y\}^c, \emptyset) = \underset{\approx}{\overline{y}}$. On the other hand, suppose we have $U = (U_1, U_2) \in \tau$ with $\underset{\approx}{y} \in U$, $\underset{\approx}{x} \notin U$. Then $x \in U_2$, whence $x \notin U_1$ and just as above we obtain $U \subseteq \underset{\approx}{\overline{x}}$. Since one of these cases must hold, $(X, \tau)$ satisfies $T_0\,(iv)$.

(2). Just as for the proof of (1).

(3). Suppose now that $(X, \tau)$ satisfies $T_2\,(iii)$, and take $x \neq y$ in $X$. Then we have $U, V \in \tau$ with $\underset{\sim}{x} \in U$, $\underset{\sim}{y} \in V$ and $U \subseteq \overline{V}$. Now, as noted above, $\underset{\approx}{y} \in V$, and clearly $U \subseteq \overline{V} \implies U \subseteq ()\,\overline{V}$, so $(X, \tau)$ is pairwise $T_2$. □

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