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Generalized closed sets and some separation axioms on weak structure

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Abstract

In this paper, we introduce and characterize the concepts of generalized closed (gw-closed, for short) sets in weak structures which introduced by Császár [3] and we give some properties of these concepts. The concept of gw-closed sets (in the sense of Al Omari and Noiri [1]) is a special case of gw-closed sets presented here. Finally, the concepts of $T_{\frac{1}{2}}$-, T_1-, normal, almost normal and weakly normal spaces are investigated by using the concepts of gw-closed, sgw-closed and mgw-closed sets in weak structures.

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1. Preliminaries

In 2002, Császár [2] introduced the concept of generalized topology and investigated some concepts such as continuity, generalized open sets. In 2005, Maki et al. [5] introduced the concept of minimal structure and investigated some of its properties. Finally, Császár [3] introduced the concept of weak structure (Let X be a non-empty set and $P(X)$ its power set. A class $w \subset P(X)$ is said to be a weak structure (WS, for short) on X if and only if $\phi \in w$). He defined a subset A is said to be w-open if $A \in w$ and its complement is called w-closed. Also, he defined two operations $i_w(A)$ and $c_w(A)$ in WS on X as the union of all w-open subsets of A and the intersection of all w-closed set containing A, respectively. Furthermore, he gave some properties of $c_w(A)$ and $i_w(A)$.

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The purpose of this paper is to introduce and study the concepts of generalized closed sets in weak structures and we give some characterizations and properties of these concepts. The concept of gw-closed sets (in the sense of Al Omari and Noiri [1]) is a special case of gw-closed sets in a weak structure. Finally, the concepts of $T_{\frac{1}{2}}$-, T_1-, normal, almost normal and weakly normal spaces are investigated by using the concepts of gw-closed, sgw-closed and mgw-closed sets in weak structures. It is shown that many results in previous papers [1, 6, 7] can be considered as special cases of our results.

1.1. Theorem. [3] *Let w be a WS on X and A, $B \subseteq X$. Then the following statements are true:*

(1) $A \subseteq c_w(A)$,
(2) *If* $A \subseteq B$, *then* $c_w(A) \subset c_w(B)$,
(3) *If A is w-closed, then* $A = c_w(A)$,
(4) $c_w(c_w(A)) = c_w(A)$,
(5) $A \supseteq i_w(A)$,
(6) *If* $A \subset B$, *then* $i_w(A) \subset i_w(B)$,
(7) $i_w(i_w(A)) = i_w(A)$,
(8) *If A is w-open, then* $A = i_w(A)$,
(9) $c_w(X - A) = X - i_w(A)$,
(10) $i_w(X - A) = X - c_w(A)$,
(11) $i_w(c_w(i_w(c_w(A)))) = i_w(c_w(A))$,
(12) $c_w(i_w(c_w(i_w(A)))) = c_w(i_w(A))$,
(13) $x \in i_w(A)$ *if only if there is a w-open set U such that* $x \in U \subset A$,
(14) $x \in c_w(A)$ *if and only if* $U \cap A \neq \phi$ *for each w-open set U containing x.*

1.2. Definition. [4] Let w be a WS on X and $A \subseteq X$. Then:

(1) $A \in r(w)$ (i.e., A is w-regular open subset) if $A = i_w(c_w(A))$,
(2) $A \in rc(w)$ (i.e., A is w-regular closed subset) if $A = c_w(i_w((A))$.

1.3. Definition. Let w be a WS on X and $A \subset X$. A point $x \in X$ is said to be w-boundary point of a subset A if and only if $x \in c_w(A) \bigcap c_w(X - A)$. By $Bd_w(A)$ we denote the set of all w-boundary points of A.

1.4. Theorem. *Let w be a WS on X and $A \subseteq X$. Then:*

(1) $Bd_w(A) = Bd_w(X - A)$,
(2) $Bd_w(A) = c_w(A) - i_w(A)$,
(3) *If A is w-open, then* $A \cap Bd_w(A) = \phi$,
(4) *If A is w-closed, then* $Bd_w(A) \subset A$.

Proof. It follows from Definition 1.3 and Theorem 1.1. □

1.5. Remark. One may notice that the converses of (3) and (4) in Theorem 1.4 are not true as shown by the following example.

1.6. Example. Let $X = \{a, b, c\}$ and $w = \{\phi, \{a\}, \{b\}, \{c\}\}$. One may notice that:

(1) The subset $A = \{a, c\}$ satisfy $A \cap Bd_w(A) = \phi$, but A is not w-open,
(2) The subset $A = \{c\}$ satisfy $Bd_w(A) \subset A$, but A is not w-closed.

2. Generalized w-Closed and Generalized w-Open Sets

2.1. Definition. Let w be a WS on X. We define the concepts of generalized closed and generalized open sets in weak structure as follows:

(1) A subset A is said to be generalized w-closed (gw-closed, for short) if $c_w(A) \subset U$, whenever $A \subset U$ and U is w-open.

(2) The complement of a generalized w-closed set is said to be generalized w-open (gw-open, for short).

The family of all gw-closed (resp. gw-open) sets in a weak structure X will be denoted by $gwC(X)$ (resp. $gwO(X)$)

2.2. Theorem. *Let w be a WS on X. A subset A is gw-open if and only if $i_w(A) \supseteq F$, whenever $A \supseteq F$ and F is w-closed.*

Proof. It follows from Theorem 1.1 and the fact the complement of w-open set is w-closed. □

2.3. Remark. By the following two examples, we show that union and intersection of two gw-closed sets is not gw-closed.

2.4. Example. Let $X = \{a, b, c\}$ and $w = \{\emptyset, \{a\}\}$. If $A = \{a, b\}$ and $B = \{a, c\}$, then A and B are gw-closed sets but $A \cap B = \{a\}$ is not gw-closed set.

2.5. Example. Let $X = \{a, b, c, d\}$ and $w = \{\emptyset, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b\}\}$. Then $A = \{a\}$ and $B = \{c, d\}$ are gw-closed sets in X, but their union $A \cup B = \{a, c, d\}$ is not gw-closed.

2.6. Theorem. *Let w be a WS on X. If $\{A_i : i \in I\}$ is a family of subsets of X, then $c_w(\bigcup A_i) \supseteq \bigcup c_w(A_i)$.*

Proof. It is clear. □

2.7. Definition. Let w be a WS on X. A family $\{A_i : i \in I\}$ is said to be w-locally finite if $c_w(\bigcup A_i) = \bigcup c_w(A_i)$.

2.8. Theorem. *Let w be a WS on X. The arbitrary union of gw-closed sets $A_i, i \in I$ in X is a gw-closed set if the family $\{A_i : i \in I\}$ is w-locally finite.*

Proof. Let w be a WS on X, let $\{A_i : i \in I\}$ be a family of gw-closed sets in X and U be a w-open set such that $\bigcup A_i \subset U$. Then $A_i \subset U$ for each $i \in I$ and hence $c_w(A_i) \subset U$ which implies $\bigcup c_w(A_i) \subseteq U$. Since the family $\{A_i : i \in I\}$ is w-locally finite, then $c_w(\bigcup A_i) = \bigcup c_w(A_i) \subseteq U$. Therefore $\bigcup A_i$ is gw-closed. □

2.9. Theorem. *Let w be a WS on X. The arbitrary intersection of gw-open sets $A_i, i \in I$ in X is a gw-open set if the family $\{A_i : i \in I\}$ is w-locally finite.*

Proof. It follows from Theorem 1.1 and Theorem 2.27 and the fact the complement of a gw-open set is a gw-closed. □

2.10. Theorem. *Let w be a WS on X. If A is a w-closed set, then A is gw-closed.*

Proof. Let A be a w-closed set and U be a w-open set in X such that $A \subset U$. Then $c_w(A) = A \subset U$ and hence A is gw-closed. □

2.11. Corollary. *Let w be a WS on X. If A is a w-open set, then A is gw-open.*

2.12. Remark. By the following example, we show that the converse of Theorem 2.10 need not be true in general.

2.13. Example. In Example 2.5, if $A = \{d\}$, then A is gw-closed and not w-closed.

2.14. Theorem. *Let w be a WS on X. If A is a gw-closed set in X, then $c_w(A) - A$ contains no non empty w-closed.*

Proof. Suppose that F is a non empty w-closed subset of $c_w(A)-A$. Now $F \subset c_w(A)-A$. Then $F \subset c_w(A) \bigcap X - A$ and hence $F \subset c_w(A)$ and $F \subset X - A$. Since $X - F$ is w-open and A is gw-closed, then $c_w(A) \subset X - F$ and hence $F \subset X - c_w(A)$. Thus $F \subset c_w(A) \cap X - c_w(A) = \phi$ and hence $F = \phi$. Therefore $c_w(A) - A$ does not contain non empty w-closed. □

2.15. Remark. In general topology, Levine [6] proved that the above theorem is true for "if and only if". But in the weak structures the converse of the above theorem need not be true in general as shown by the following example.

2.16. Example. Let $X = \{a, b, c\}$ and $w = \{\phi, \{b\}, \{c\}\}$. One may notice that if $A = \{b\}$, then $c_w(A) - A = \{a, b\} - \{b\} = \{a\}$ does not contain any non empty w-closed, but A is not a gw-closed set in X, since A is an w-open set contains itself and $c_w(A) = \{a, b\} \nsubseteq A$.

2.17. Corollary. *Let w be a WS on X and $A \subseteq X$ is a gw-closed set. If $c_w(A) - A$ is w-closed, then $c_w(A) = A$.*

Proof. Let $c_w(A)-A$ be w-closed and A be a gw-closed set in X. Then by Theorem 2.14, $c_w(A) - A$ contains no non empty w-closed set. Since $c_w(A) - A$ is a w-closed subset of itself, $c_w(A) - A = \phi$ and hence $c_w(A) = A$. □

2.18. Remark. If A is a gw-closed set in a WS on X and $c_w(A) = A$, then $c_w(A) - A$ is need not be w-closed as shown by the following example.

2.19. Example. Let $X = \{a, b, c\}$, $w = \{\phi, \{a\}, \{c\}, \{a, b\}\}$ and $A = \{b\}$. One may notice that $c_w(A) = A$ and hence $c_w(A) - A = \phi$, which is not w-closed.

2.20. Theorem. *Let w be a WS on X. Then $A \subseteq X$ is a gw-closed if $c_w(\{x\}) \bigcap A \neq \phi$ for each $x \in c_w(A)$.*

Proof. Let $c_w(\{x\}) \bigcap A \neq \phi$ for each $x \in c_w(A)$ and U be any w-open set with $A \subseteq U$. Let $x \in c_w(A)$. Then $c_w(\{x\}) \bigcap A \neq \phi$ and hence there exists $y \in c_w(\{x\}) \bigcap A$, so $y \in A \subseteq U$. Thus $\{x\} \bigcap U \neq \phi$ and hence $x \in U$. Therefore $c_w(A) \subseteq U$, which implies A is gw-closed. □

2.21. Remark. Al Omari and Noiri [1, Theorem 2.9] proved that the converse of the above theorem is true. The following example shows that the converse needn't be true generally.

2.22. Example. Let $X = \{a, b, c\}, w = \{\phi, \{a\}, \{b\}\}$. one may notice that $A = \{a, b\}$ is gw-closed and $c_w(\{c\}) = \{c\}$. So $A \bigcap c_w(\{c\}) = \phi$.

2.23. Theorem. *Let w be a WS on X. If A is a gw-closed set in X, then $c_w(A) - A$ is gw-open.*

Proof. Let A is a gw-closed set in X and F be a w-closed subset such that $F \subset c_w(A)-A$. Then by Theorem 2.14 we have $F = \phi$ and hence $F \subset i_w(c_w(A) - A))$. So by Theorem 2.2, we have $c_w(A) - A$ is gw-open. □

2.24. Remark. In topological space, Levine [6] proved that the above theorem is true for "if and only if". But in the weak structures the converse of the above theorem need not be true in general as shown by the following example.

2.25. Example. Let $X = \{a, b, c\}$, $w = \{\phi, \{a\}, \{c\}, \{a, b\}\}$ and $A = \{a\}$. One may notice that $c_w(A) - A = \{a, b\} - \{a\} = \{b\}$ which is gw-open, but A is not a gw-closed set, since A is a generalized w-open set contain itself, but $c_w(A) = \{a, b\} \not\subset A$

2.26. Theorem. *Let w be a WS on X and A be a gw-closed set with $A \subset B \subset c_w(A)$, then B is gw-closed.*

Proof. Let H be a w-open set in X such that $B \subset H$, then $A \subset H$. Since A is gw-closed, then $c_w(A) \subset H$ and hence $c_w(B) \subset c_w(A) \subset H$. Thus B is gw-closed. □

2.27. Theorem. *Let w be a WS on X and A be a gw-closed set with $A \subset B \subset c_w(A)$, then $c_w(B) - B$ contains no non empty w-closed.*

Proof. It follows from Theorems 2.14 and 2.26. □

2.28. Remark. Let w be a WS on X and A be a gw-open set with $i_w(A) \subset B \subset A$, then B is gw-open.

2.29. Remark. Let w be a WS on X. Then each subset of X is gw-closed if each w-open set is w-closed.

2.30. Remark. In topological space, Levine [6] proved that the above theorem is true for "if and only if". But in the weak structures the converse of the above theorem need not be true in general as shown by the following example.

2.31. Example. Let $X = \{a, b, c\}$, $w = \{\phi, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}, X\}$. One may notice that every subset of X is gw-closed, but $A = \{c\}$ is w-open set in X and it is not w-closed.

2.32. Remark. Let w be a WS on X. Then each subset of X is gw-closed if and only if $c_w(A) \subseteq A$ for each w-open set A in X.

2.33. Theorem. *Let w be a WS on X. If A is a gw-open set in X, then $U = X$ whenever U is w-open and $i_w(A) \cup (X - A) \subset U$.*

Proof. Let U be a w-open set in X and $i_w(A) \cup (X - A) \subset U$ for any gw-open set A. Then $X - U \subset [X - i_w(A)] \cap A$ and hence $X - U \subset (c_w(X - A)) - (X - A)$. Since $X - A$ is a gw-closed, then by Theorem 2.14, we have $X - U = \phi$ and hence $U = X$. □

3. Separation Axioms on Weak Structures

3.1. Definition. Let w be a WS on X. We define the concepts of strongly generalized closed and strongly generalized open sets in weak structure as follows:

(1) A subset A is said to be strongly generalized w-closed (sgw-closed, for short) if $c_w(A) \subseteq U$, whenever $A \subseteq U$ and U is gw-open.
(2) A subset A is said to be mildly w-closed (mgw-closed, for short) if $c_w(i_w((A)) \subseteq U$, whenever $A \subseteq U$ and U is gw-open.
(3) The complement of a sgw-closed (resp. mgw-closed) set is said to be sgw-open (resp. mgw-open).

3.2. Definition. A weak structure w on X is said to be $w - T_{\frac{1}{2}}$ if each gw-closed set A of X, $c_w(A) = A$.

3.3. Remark. (1) In a topological space X, X is $T_{\frac{1}{2}}$ [6] if and only if each singleton is either closed or open. By the following examples we show that "if X is a weak structure and each singleton is w-open or $c_w(A) = A$, then X need not be $w - T_{\frac{1}{2}}$".
(2) We think that in a weak structure X, if X is $w - T_{\frac{1}{2}}$, then there exists a singleton $x \in X$ such x is neither w-closed nor $\{x\} \neq i_w\{x\}$.

3.4. Example. Let $X = \{a, b, c\}$, $w = \{\phi, \{b\}, \{c\}, \{b, c\}\}$. One may notice that each singleton is w-open or w-closed. But there exists $A = \{a, b\}$ which is gw-closed and $c_w(A) = X \neq A$. So X is not $w - T_{\frac{1}{2}}$.

3.5. Theorem. *Let w be a WS on X. If $i_w\{x\}$ is an w-open set and each singleton is either w-closed or $\{x\} = i_w\{x\}$, then X is $w - T_{\frac{1}{2}}$.*

Proof. Let A be a gw-closed subset of X and $x \in c_w(A)$.

Case 1. If $\{x\}$ is w-closed and $x \notin A$, then $x \in (c_w(A) - A)$ and hence $\{x\} \subseteq X - A$, which implies $A \subseteq X - \{x\}$. Since A is a gw-closed set and $X - \{x\}$ is an w-open set, then $c_w(A) \subseteq X - \{x\}$ and hence $\{x\} \subseteq X - c_w(A)$. Therefore $\{x\} \subseteq c_w(A) \bigcap X - c_w(A) = \phi$, which is a contradiction. Thus $x \in A$ and hence $c_w(A) = A$.

Case 2. If $\{x\} = i_w\{x\}$ and $x \in c_w(A)$, then for each w-open set V with $x \in V$, we have $V \bigcap A \neq \phi$. Since $i_w\{x\}$ is an w-open set and $\{x\} = i_w\{x\}$, then $\{x\} \bigcap A \neq \phi$ and hence $x \in A$.Thus $c_w(A) = A$. Therefore in the two cases we have $c_w(A) = A$ and hence X is $w - T_{\frac{1}{2}}$. □

3.6. Definition. A weak structure w on X is said to be $w - T_1$ if for any points $x, y \in X$ with $x \neq y$, there exist two w-open sets U and V such that $x \in U$, $y \notin U$, $x \notin V$ and $y \in V$.

3.7. Theorem. *A weak structure w on X is $w - T_1$ if every singleton in X is w-closed.*

Proof. It is clear. □

3.8. Remark. In a topological space one may notice that:

(1) The above theorem is true for if and only if,
(2) If X is T_1, then each g-closed set in X is closed.

By the following example we show that the converse of the above theorem (the second part of item 1 above) need not be true and the item 2 above need not be true too in an WS on X in general.

3.9. Example. Let $X = \{a, b, c\}$, $w = \{\phi, \{a\}, \{b\}, \{c\}\}$. One may notice that:

(1) w is $w - T_1$, but the singleton $\{b\}$ is not w-closed.
(2) w is $w - T_1$ and the singleton $\{b\}$ is gw-closed, but is not w-closed.

3.10. Definition. A weak structure w on X is said to be:

(1) w-normal if for each two w-closed sets F and H with $F \bigcap H = \phi$, there exist two w-open sets U and V such that $F \subseteq U$, $H \subseteq V$ and $U \bigcap V = \phi$.
(2) Almost w-normal if for each w-closed set F and $H \in rc(w)$ with $F \bigcap H = \phi$, there exist two w-open sets U and V such that $F \subseteq U$, $H \subseteq V$ and $U \bigcap V = \phi$.
(3) Weakly w-normal if for each $F, H \in rc(w)$ with $F \bigcap H = \phi$, there exist two w-open sets U and V such that $F \subseteq U$, $H \subseteq V$ and $U \bigcap V = \phi$.

3.11. Theorem. *Let w be a WS on X. Consider the following statements:*

(1) *X is w-normal;*
(2) *For each w-closed set F and w-open U with $F \subseteq U$, there exist w-open sets V such that $F \subseteq V \subseteq c_w(V) \subseteq U$;*
(3) *For each w-closed set F and each gw-closed set H with $F \bigcap H = \phi$, there exist two w-open sets U and V such that $F \subseteq U, H \subseteq V$ and $U \bigcap V = \phi$;*
(4) *For each w-closed set F and gw-open U with $F \subseteq U$, there exist w-open sets V such that $F \subseteq V \subseteq c_w(V) \subseteq U$.*

Then the implications (1) ⇒ (2) *and* (3) ⇒ (4) ⇒ (2) *are hold.*

Proof. It is clear. □

3.12. Theorem. *Let w be a WS on X. If $c_w(A)$ is w-closed for each w-open or gw-closed, then the statements in Theorem 3.11 are equivalent.*

Proof. From Theorem 3.11 we need to prove (2) ⇒ (1) and (1) ⇒ (3) only.

(2) ⇒ (1): Let A and B be two disjoint w-closed subsets of X. Then $X - B$ is an w-open set containing A. Thus by (2) there exists an w-open set U such that $A \subseteq U \subseteq c_w(U) \subseteq X - B$ and hence $A \subseteq U$ and $B \subseteq X - c_w(U)$. Since $c_w(U)$ is w-closed for each w-open set U, then $X - c_w(U) = V$ is w-open and $U \bigcap V = \phi$. Hence X is w-normal.

(1) ⇒ (3). Let F be an w-closed set and H be a gw-closed set with $F \bigcap H = \phi$. Then $H \subseteq X - F$ which is w-open. Since H is gw-closed and $H \subseteq X - F$, then $c_w(H) \subseteq X - F$. Since H is gw-closed, then $c_w(H)$ is w-closed. By (1) there exist two w-open sets U and V such that $c_w(H) \subseteq U$, $F \subseteq V$ and $U \bigcap V = \phi$. Hence $H \subseteq U$, $F \subseteq V$ and $U \bigcap V = \phi$. □

3.13. Theorem. *Let w be a WS on X. Consider the following statements:*

(1) *X is almost w-normal;*
(2) *For each w-closed set F and $U \in r(w)$ with $F \subseteq U$, there exist w-open sets V such that $F \subseteq V \subseteq c_w(V) \subseteq i_w(c_w(U))$;*
(3) *For each w-closed set F and mgw-open U with $F \subseteq U$, there exist w-open sets V such that $F \subseteq V \subseteq c_w(V) \subseteq i_w(c_w(U))$;*
(4) *For each w-closed set F and sgw-open U with $F \subseteq U$, there exist w-open sets V such that $F \subseteq V \subseteq c_w(V) \subseteq i_w(c_w(U))$;*
(5) *For each w-closed set F and w-open U with $F \subseteq U$, there exist w-open sets V such that $F \subseteq V \subseteq c_w(V) \subseteq i_w(c_w(U))$.*

Then the implications (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) *are hold.*

Proof. (1) ⇒ (2): Obvious.

(2) ⇒ (3): Let F be an w-closed set and U be a mgw-open with $F \subseteq U$. Then $F \subseteq i_w(C_w(U)) \in r(w)$. By (2) there exist w-open sets V such that $F \subseteq V \subseteq c_w(V) \subseteq i_w(c_w(i_w(c_w(U)))) = i_w(c_w(U))$.

(3) ⇒ (4) ⇒ (5): Obvious. □

3.14. Theorem. *Let w be a WS on X. If $c_w(A)$ is w-closed for each w-open A, then the statements in Theorem 3.13 are equivalent.*

Proof. From Theorem 3.13 we need to prove that (5) ⇒ (1) only.

(5) ⇒ (1): Let F be an w-closed set and $H \in rc(w)$ with $F \bigcap H = \phi$. Then $F \subseteq X - H = i_w(c_w(X - H))$. By (5) there exist w-open sets V such that $F \subseteq V \subseteq c_w(V) \subseteq i_w(c_w(i_w(c_w(X-H)))) = i_w(c_w(X-H))$ and hence $F \subseteq V, H = c_w(i_w(H)) \subseteq X - c_w(V)$. Since V is an w-open, then $c_w(V)$ is w-closed and hence $X - c_w(V) = W$ which is w-open contains H. Thus $V \bigcap W = \phi$. Therefore X is almost w-normal. □

3.15. Theorem. *Let w be a WS on X. Consider the following statements:*

(1) *X is almost w-normal;*
(2) *For each w-open set U and $F \in rc(w)$ with $F \subseteq U$, there exist w-open sets V such that $F \subseteq V \subseteq c_w(V) \subseteq U$;*
(3) *For each mgw-closed set F and w-open U with $F \subseteq U$, there exist w-open sets V such that $c_w(i_w(F)) \subseteq V \subseteq c_w(V) \subseteq U$;*
(4) *For each gw-closed set F and w-open U with $F \subseteq U$, there exist w-open sets V such that $c_w(i_w(F)) \subseteq V \subseteq c_w(V) \subseteq U$.*

Then the implications $(1) \Rightarrow (2) \Rightarrow (3)$ *and* $(2) \Rightarrow (4)$ *are hold.*

Proof. $(1) \Rightarrow (2)$: Obvious.

$(2) \Rightarrow (3)$: Let F be a mgw-closed set and U be a w-open with $F \subseteq U$. Then $c_w(i_w(F)) \subseteq U$. Since $c_w(i_w(F)) \in rc(w)$, then by (2) there exists an w-open set V such that $c_w(i_w(F)) \subseteq V \subseteq c_w(V) \subseteq U$.

$(2) \Rightarrow (4)$: Let F be a gw-closed set and U be a w-open with $F \subseteq U$. Then $c_w(F) \subseteq U$ and hence $c_w(i_w(F)) \subseteq U$. Since $c_w(i_w(F)) \in rc(w)$, then by (2) there exists an w-open set V such that $c_w(i_w(F)) \subseteq V \subseteq c_w(V) \subseteq U$. □

3.16. Theorem. *Let w be a WS on X. If $c_w(A)$ is w-closed for each w-open set A or $A \in r(w)$, then the statements in Theorem 3.15 are equivalent.*

Proof. From Theorem 3.15 we need to prove that $(3) \Rightarrow (1)$ and $(4) \Rightarrow (1)$ only.

$(3) \Rightarrow (1)$: Let F be an w-closed set and $H \in rc(w)$ with $F \bigcap H = \phi$. Then $H \subseteq X - F$. Since $H \in rc(w)$, then H is mgw-closed. By (3) there exist w-open sets V such that $H = c_w(i_w(H)) \subseteq V \subseteq c_w(V) \subseteq X - F$ and hence $H \subseteq V$ and $F \subseteq X - c_w(V) = W$ which is w-open. Thus there exist two w-open sets V and W such that $H \subseteq V, F \subseteq W$ and $V \bigcap W = \phi$. Therefore X is almost w-normal.

$(4) \Rightarrow (1)$: Let F be an w-closed set and $H \in rc(w)$ with $F \bigcap H = \phi$. Then $H \subseteq X - F$. Since $H \in rc(w)$, then $c_w(i_w(H)) \subseteq X - F$. Since $H \in rc(w)$, then $i_w(H)$ is an w-open and hence $c_w(i_w(H))$ is w-closed which is gw-closed. By (4) there exist w-open sets V such that $c_w(i_w(c_w(i_w(H))) \subseteq V \subseteq c_w(V) \subseteq X - F$ and hence $H \subseteq V$ and $F \subseteq X - c_w(V) = W$ which is w-open. Thus there exist two w-open sets V and W such that $H \subseteq V, F \subseteq W$ and $V \bigcap W = \phi$. Therefore X is almost w-normal. □

3.17. Theorem. *Let w be a WS on X. Consider the following statements:*

(1) *X is weakly w-normal,*
(2) *For each $F \in rc(w)$ and $U \in r(w)$ with $F \subseteq U$, there exist w-open sets V such that $F \subseteq V \subseteq c_w(V) \subseteq i_w(c_w(U))$,*
(3) *For each $F \in rc(w)$ and mgw-open U with $F \subseteq U$, there exist w-open sets V such that $F \subseteq V \subseteq c_w(V) \subseteq i_w(c_w(U))$,*
(4) *For each $F \in rc(w)$ and sgw-open U with $F \subseteq U$, there exist w-open sets V such that $F \subseteq V \subseteq c_w(V) \subseteq i_w(c_w(U))$,*
(5) *For each $F \in rc(w)$ and w-open U with $F \subseteq U$, there exist w-open sets V such that $F \subseteq V \subseteq c_w(V) \subseteq i_w(c_w(U))$.*

Then the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$ *are hold.*

Proof. It is similar to that of Theorem 3.13. □

3.18. Theorem. *Let w be a WS on X. If $c_w(A)$ is w-closed for each w-open A, then the statements in Theorem 3.17 are equivalent.*

Proof. It is similar to that of Theorem 3.14. □

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