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ON THE COMPOSITION OF THE DISTRIBUTIONS $x^{-1}\ln^m|x|$ AND x^r

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Abstract

Let F be a distribution and f a locally summable function. The distribution $F(f)$ is defined as the neutrix limit of the sequence $\{F_n(f)\}$, where $F_n(x) = F(x) * \delta_n(x)$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. The composition of the distributions $x^{-1}\ln^m|x|$ and x^r is evaluated for $r, m = 1, 2, 3 \dots$.

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1. Introduction

Certain operations on smooth functions can be extended without difficulty to arbitrary distributions. Others (such as multiplication, convolution and change of variables) can be defined only for particular distributions.

In the theory of Schwartz distributions, no meaning can be generally given to expressions of the form $F(f(x))$, where F and f are distributions.

The technique of neglecting appropriately defined infinite quantities was devised by Hadamard and the resulting finite value extracted from a divergent integral is referred to as the Hadamard finite part. In fact his method can be regarded as a particular applications of the neutrix calculus developed by van der Corput, see [1].

Using the concepts of a neutrix and neutrix limit, the first author gave a general principle for the discarding of unwanted infinite quantities from asymptotic expansions, and this has been exploited particularly in connection with multiplication, convolution

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and the composition of distributions, see [2, 3]. Using Fisher's definition, Koh and Li gave a meaning to δ^r and $(\delta')^r$ for $r = 2, 3, \dots$, see [6], and the more general form $(\delta^{(s)}(x))^r$ was considered by Kou and Fisher in [7]. Recently the rth powers of the Dirac distribution and the Heaviside function for negative integers have been defined in [9] and [10], respectively.

In the following we let $\rho(x)$ be an infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\displaystyle\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$, see [5].

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and $\mathcal{D}'$ the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x - t)\rangle$$

for $n = 1, 2, \dots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

We now define the distribution $x^{-1} \ln^m |x|$ by

$$x^{-1} \ln^m |x| = \frac{(\ln^{m+1} |x|)'}{m + 1}$$

for $m = 1, 2, \dots$, and we define the distribution $x^{-r-1} \ln^m |x|$ inductively by

$$x^{-r-1} \ln^m |x| = \frac{m x^{-r-1} \ln^{m-1} |x| - (x^{-r} \ln^m |x|)'}{r}$$

for $r, m = 1, 2, \dots$. It follows by induction that

$$\begin{aligned} x^{-r-1} \ln |x| &= \phi(r) x^{-r-1} + \frac{(-1)^r (x^{-1} \ln |x|)^{(r)}}{r!} \\ &= \phi(r) x^{-r-1} + \frac{(-1)^r (\ln^2 |x|)^{(r+1)}}{2r!} \end{aligned}$$

where

$$\phi(r) = \begin{cases} \sum_{i=1}^{r} i^{-1}, & r = 1, 2, \dots, \\ 0, & r = 0. \end{cases}$$

The following definition was given in [3].

1.1. Definition. Let F be a distribution and f a locally summable function. We say that the distribution $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$\operatorname*{N-lim}_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x) dx = \langle h(x), \varphi(x)\rangle$$

for all test functions φ with compact support contained in (a, b), where N is the neutrix, see [1], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n,\ \ln^r n :\ \lambda > 0,\ r = 1, 2, \dots,$$

and all functions which converge to zero in the usual sense as n tends to infinity.

Note that taking a neutrix limit of a function is equivalent to picking out the Hadamard finite part from the function and taking the usual limit of that.

2. Main results

The following theorems were proved in [8] and [4] respectively.

2.1. Theorem. *The distribution* $(x^r)^{-s}$ *exists and*

$$(x^r)^{-s} = x^{-rs}$$

for $r, s = 1, 2, \dots$.

2.2. Theorem. *If* $F_s(x)$ *denotes the distribution* $x^{-s}\ln|x|$, *then the distribution* $F_s(x^r)$ *exists and*

$$F_s(x^r) = rx^{-rs}\ln|x|$$

for $r, s = 1, 2, \dots$.

The following two lemmas can be proved easily by induction.

2.3. Lemma.

$$\int_{-1}^{1} v^i \rho^{(r)}(v)\, dv = \begin{cases} \{0, & 0 \le i < r, \\ (-1)^r r!, & i = r \end{cases}$$

for $r = 0, 1, 2, \dots$.

2.4. Lemma. *If* φ *is an arbitrary function in* $\mathcal{D}$ *with support contained in the interval* $[-1, 1]$, *then*

$$\langle x^{-r}, \varphi(x)\rangle = \int_{-1}^{1} x^{-r}\Big[\varphi(x) - \sum\nolimits_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!}x^k\Big]\, dx + \\ + \sum\nolimits_{k=0}^{r-2} \frac{(-1)^{r-k-1} - 1}{(r-k-1)k!}\varphi^{(k)}(0),$$

for $r = 1, 2, \dots$, *where the second sum is empty when* $r = 1$.

The next two lemmas can be proved easily by induction and the use of Lemma 2.4.

2.5. Lemma. *If* φ *is an arbitrary function in* $\mathcal{D}$ *with support contained in the interval* $[-1, 1]$, *then*

$$\langle x^{-r}\ln^m|x|, \varphi(x)\rangle = \int_{-1}^{1} x^{-r}\ln^m|x|\Big[\varphi(x) - \sum\nolimits_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!}x^k\Big]\, dx + \\ + \sum_{k=0}^{r-2} \frac{[(-1)^{r-k-1} - 1]m!}{(r-k-1)^{m+1}k!}\varphi^{(k)}(0),$$

for $m, r = 1, 2, \dots$, *where the second sum is empty when* $r = 1$.

2.6. Lemma.

$$\int_{1}^{n} u^\alpha \ln^r u\, du = \frac{(-1)^{r+1} r!(1 - n^{\alpha+1})}{(\alpha+1)^{r+1}} + O(\ln n),$$

for $\alpha \ne -1$ *and* $r = 1, 2, \dots$, *where* $O(\ln n)$ *denotes a sum of negligible functions, each containing a positive power of* $\ln n$.

We now prove the following theorem.

2.7. Theorem. *If* $F_m(x)$ *denotes the distribution* $x^{-1}\ln^m|x|$, *then the distribution* $F_m(x^r)$ *exists and*

(1) $$F_m(x^r) = r^m x^{-r}\ln^m|x|$$

for $m, r = 1, 2, \ldots$.

Proof. We first of all put

$$[F_m(x)]_n = (x^{-1}\ln^m|x|) * \delta_n(x) = \frac{1}{m+1}\int_{-1/n}^{1/n}\ln^{m+1}|x-t|\delta_n'(t)\,dt$$

so that

$$[F_m(x^r)]_n = \frac{1}{m+1}\int_{-1/n}^{1/n}\ln^{m+1}|x^r-t|\delta_n'(t)\,dt.$$

Our problem now is to evaluate $\mathrm{N-}\lim\limits_{n\to\infty}\langle[F_m(x^r)]_n,\varphi(x)\rangle$ for all φ with support contained in the interval $[-1,1]$, where by Taylor's Theorem we have

$$\varphi(x) = \sum_{k=0}^{r-1}\frac{x^k}{k!}\varphi^{(k)}(0) + \frac{x^r}{r!}\varphi^{(r)}(\xi x)$$

where $0 < \xi < 1$, and so

$$\langle[F_m(x^r)]_n,\varphi(x)\rangle = \sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{k!}\langle[F_m(x^r)]_n, x^k\rangle + \frac{1}{r!}\langle[F_m(x^r)]_n, x^r\varphi^{(r)}(\xi x)\rangle.$$

We must therefore first of all evaluate

(2) $$\begin{aligned}\int_{-1}^{1}[F_m(x^r)]_n x^k\,dx &= \int_{-1}^{1}x^k\int_{-1/n}^{1/n}\ln^{m+1}|x^r-t|\delta_n'(t)\,dt\,dx\\ &= \begin{cases}0, & r-k \text{ odd}\\ 2\int_0^1 x^k\int_{-1/n}^{1/n}\ln^{m+1}|x^r-t|\delta_n'(t)\,dt\,dx, & r-k\text{ even}\end{cases}.\end{aligned}$$

We have

(3) $$\begin{aligned}&\int_0^1 x^k\int_{-1/n}^{1/n}\ln^{m+1}|x^r-t|\delta_n'(t)\,dt\,dx =\\ &= \int_{-1/n}^{1/n}\delta_n'(t)\int_0^{n^{-1/r}}x^k\ln^{m+1}|x^r-t|\,dx\,dt+\\ &\quad + \int_{-1/n}^{1/n}\delta_n'(t)\int_{n^{-1/r}}^{1}x^k\ln^{m+1}|x^r-t|\,dx\,dt\\ &= \frac{n^{(r-k-1)/r}}{r}\int_{-1}^{1}\rho'(v)\int_0^1 u^{-(r-k-1)/r}\ln^{m+1}|(u-v)/n|\,du\,dv+\\ &\quad + \frac{n^{(r-k-1)/r}}{r}\int_{-1}^{1}\rho'(v)\int_1^n u^{-(r-k-1)/r}\ln^{m+1}|(u-v)/n|\,du\,dv\\ &= I_1 + I_2,\end{aligned}$$

on using the substitutions $u = nx^r$ and $v = nt$. It is easily seen that

(4) $$\mathrm{N-}\lim_{n\to\infty} I_1 = 0,$$

for $k = 0, 1, \ldots, r-2$.

Now,

$$\begin{aligned} I_2 &= \frac{n^{(r-k-1)/r}}{r} \int_{-1}^{1} \rho'(v) \int_{1}^{n} u^{-(r-k-1)/r} [\ln|1-v/u| + \ln u - \ln n]^{m+1} \, du \, dv \\ &= \sum_{i=0}^{m} \binom{m+1}{i} \frac{n^{(r-k-1)/r}}{r} \int_{-1}^{1} \rho'(v) \int_{1}^{n} u^{-(r-k-1)/r} \\ &\qquad \times \ln^{m-i+1}|1-v/u| \ln^i u \, du \, dv + O(\ln n) \\ &= \sum_{i=0}^{m} J_i + O(\ln n), \end{aligned} \tag{5}$$

where $O(\ln n)$ denotes the terms containing powers of $\ln n$ which are therefore negligible, and the term containing $\ln^{m+1} u$ is zero since $\int_{-1}^{1} \rho^{(s)}(v)\, dv = 0$ for $s = 1, 2, \ldots$ by Lemma 2.3.
Now $\ln^{m-i+1}|1-v/u|$ can be expanded in the form

$$\ln^{m-i+1}|1-v/u| = \sum_{p=m-i+1}^{\infty} \frac{a_p v^p}{u^p}$$

and

$$\begin{aligned} n^{(r-k-1)/r} & \int_{-1}^{1} \rho'(v) \int_{1}^{n} u^{-(r-k-1)/r} \ln^{m-i+1}|1-v/u| \ln^i u \, du \, dv \\ &= \sum_{p=m-i+1}^{\infty} n^{(r-k-1)/r} a_p \int_{-1}^{1} v^p \rho'(v) \int_{1}^{n} u^{-p-(r-k-1)/r} \ln^i u \, du \, dv \\ &= \sum_{p=m-i+1}^{\infty} \frac{a_p (-1)^i i! [n^{-p+1} - n^{(r-k-1)/r}]}{[-p-(r-k-1)/r+1]^{i+1}} \int_{-1}^{1} v^p \rho'(v) \, dv + O(\ln n) \end{aligned}$$

on using Lemma 2.6. It follows that

$$\operatorname*{N-lim}_{n\to\infty} J_i = 0, \tag{6}$$

for $i = 0, 1, 2, \ldots, m-1$ and

$$\operatorname*{N-lim}_{n\to\infty} J_m = -\frac{(m+1)! r^m}{(r-k-1)^{m+1}}, \tag{7}$$

since $a_p = -1$ when $i = m$. It follows from equations (5) to (7) that

$$\operatorname*{N-lim}_{n\to\infty} I_2 = -\frac{(m+1)! r^m}{(r-k-1)^{m+1}}, \tag{8}$$

for $k = 0, 1, \ldots, r-2$ and it then follows from equations (2) and (8) that

$$\operatorname*{N-lim}_{n\to\infty} \int_{-1}^{1} x^k \int_{-1/n}^{1/n} \ln^m |x^r - t| \delta_n'(t) \, dt \, dx = -\frac{(m+1)! r^m [(-1)^{r-k-1} - 1]}{(r-k-1)^{m+1}} \tag{9}$$

for $k = 0, 1, \ldots, r-2$.

For the case $k = r-1$, we have from equation (2),

$$\int_{-1}^{1} x^{r-1} \int_{-1/n}^{1/n} \ln^m |x^r - t| \delta_n'(t) \, dt \, dx = 0. \tag{10}$$

When $k = r$ equation (3) still holds, but now we have

$$I_1 = \frac{n^{-1/r}}{r} \int_{-1}^{1} \rho'(v) \int_{0}^{1} u^{-1/r} \ln^{m+1} |(u-v)/n)| \, du \, dv,$$

and it follows that for any continuous function ψ,

$$\lim_{n\to\infty}\int_0^{n^{-1/r}} x^r \int_{-1/n}^{1/n} \ln^{m+1}|x^r - t|\delta_n'(t)\psi(x)\,dt\,dx = 0. \tag{11}$$

Similarly

$$\lim_{n\to\infty}\int_{-n^{-1/r}}^{0} x^r \int_{-1/n}^{1/n} \ln^{m+1}|x^r - t|\delta_n'(t)\psi(x)\,dt\,dx = 0. \tag{12}$$

Next, when $x^r \geq 1/n$, we have

$$\begin{aligned}
\int_{-1/n}^{1/n} &\ln^{m+1}(x^r - t)\delta_n'(t)\,dt \\
&= n\int_{-1}^{1} \ln^{m+1}(x^r - v/n)\rho'(v)\,dv \\
&= n\int_{-1}^{1}\Big[\ln x^r - \sum_{i=1}^{\infty}\frac{v^i}{in^ix^{ri}}\Big]^{m+1}\rho'(v)\,dv \\
&= n\ln^{m+1}x^r\int_{-1}^{1}\rho'(v)\,dv - (m+1)n\ln^m x^r\sum_{i=1}^{\infty}\int_{-1}^{1}\frac{v^i}{in^ix^{ri}}\rho'(v)\,dv \\
&\qquad + n\sum_{j=2}^{m+1}\binom{m+1}{j}\ln^{m+1-j}x^r(-1)^j\int_{-1}^{1}\Big[\sum_{i=1}^{\infty}\frac{v^i}{in^ix^{ri}}\Big]^j\rho'(v)\,dv \\
&= (m+1)r^m x^{-r}\ln^m x + O(n^{-1}).
\end{aligned}$$

It follows that

$$\left|\int_{-1/n}^{1/n}\ln^{m+1}(x^r - t)\delta_n'(t)\,dt\right| \leq (m+1)r^m x^{-r}|\ln^m x| + O(n^{-1}).$$

If now $n^{-1/r} < \eta < 1$ then

$$\begin{aligned}
\int_{n^{-1/r}}^{\eta} x^r&\left|\int_{-1/n}^{1/n}\ln^{m+1}(x^r - t)\delta_n'(t)\,dt\right|dx \\
&\leq (m+1)r^m\int_{n^{-1/r}}^{\eta}|\ln^m x|\,dx + O(n^{-1}) \\
&= (m+1)r^m(\eta\ln^m\eta - n^{-1/r}\ln^m n^{-1/r}) + \ldots + O(n^{-1}).
\end{aligned} \tag{13}$$

It follows that

$$\lim_{n\to\infty}\int_{n^{-1/r}}^{\eta} x^r\left|\int_{-1/n}^{1/n}\ln^{m+1}(x^r - t)\delta_n'(t)\,dt\right|dx = O(\eta|\ln\eta|).$$

Thus, if ψ is a continuous function then

$$\lim_{n\to\infty}\left|\int_{n^{-1/r}}^{\eta} x^r\psi(x)\int_{-1/n}^{1/n}\ln^{m+1}(x^r - t)\delta_n'(t)\,dt\,dx\right| = O(\eta|\ln\eta|) \tag{14}$$

for $r = 1, 2, \ldots$. Similarly,

$$\lim_{n\to\infty}\left|\int_{-\eta}^{-n^{-1/r}} x^r\psi(x)\int_{-1/n}^{1/n}\ln^{m+1}(x^r - t)\delta_n'(t)\,dt\,dx\right| = O(\eta|\ln\eta|) \tag{15}$$

for $r = 1, 2, \ldots$.

We now have

$$
\begin{aligned}
\Big\langle \int_{-1/n}^{1/n} & \ln^{m+1}(x^r - t)\delta_n'(t)\,dt, \varphi(x)\Big\rangle \\
&= \int_{-1}^{1} \varphi(x) \int_{-1/n}^{1/n} \ln^{m+1}(x^r - t)\delta_n'(t)\,dt\,dx \\
&= \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^{1} x^k \int_{-1/n}^{1/n} \ln^{m+1}(x^r - t)\delta_n'(t)\,dt\,dx \\
&\quad + \frac{1}{r!} \int_{-n^{-1/r}}^{n^{-1/r}} x^r \varphi^{(r)}(\xi x) \int_{-1/n}^{1/n} \ln^{m+1}(x^r - t)\delta_n'(t)\,dt\,dx \\
&\quad + \frac{1}{r!} \int_{n^{-1/r}}^{\eta} x^r \varphi^{(r)}(\xi x) \int_{-1/n}^{1/n} \ln^{m+1}(x^r - t)\delta_n'(t)\,dt\,dx \\
&\quad + \frac{1}{r!} \int_{\eta}^{1} x^r \varphi^{(r)}(\xi x) \int_{-1/n}^{1/n} \ln^{m+1}(x^r - t)\delta_n'(t)\,dt\,dx \\
&\quad + \frac{1}{r!} \int_{-\eta}^{-n^{-1/r}} x^r \varphi^{(r)}(\xi x) \int_{-1/n}^{1/n} \ln^{m+1}(x^r - t)\delta_n'(t)\,dt\,dx \\
&\quad + \frac{1}{r!} \int_{-1}^{-\eta} x^r \varphi^{(r)}(\xi x) \int_{-1/n}^{1/n} \ln^{m+1}(x^r - t)\delta_n'(t)\,dt\,dx.
\end{aligned}
$$

Using equations (9) to (14) and noting that on the intervals $[-1, -\eta]$ and $[\eta, 1]$,

$$
(16) \qquad \lim_{n\to\infty} \int_{-1/n}^{1/n} \ln^{m+1}(x^r - t)\delta_n'(t)\,dt = (m+1)r^m x^{-r} \ln^m |x|,
$$

since x^{-r} and $F_m(x)$ are continuous functions on these intervals, it follows that

$$
\begin{aligned}
\operatorname*{N-lim}_{n\to\infty} \frac{1}{m+1} & \Big\langle \int_{-1/n}^{1/n} \ln^{m+1}(x^r - t)\delta_n'(t)\,dt, \varphi(x)\Big\rangle \\
&= -\sum_{k=0}^{r-2} \frac{m!r^m[(-1)^{r-k-1} - 1]}{(r-k-1)^{m+1}k!} \varphi^{(k)}(0) \\
&\quad + O(\eta|\ln\eta|) + \int_{\eta}^{1} \frac{r^m \ln^m x\varphi^{(r)}(\xi x)}{r!} dx \\
&\quad + \int_{-1}^{-\eta} \frac{r^m \ln^m |x|\varphi^{(r)}(\xi x)}{r!}\,dx \\
&= -\sum_{k=0}^{r-2} \frac{m!r^m[(-1)^{r-k-1} - 1]}{(r-k-1)^{m+1}k!} \varphi^{(k)}(0) \\
&\quad - \int_{-1}^{1} \frac{m!r^m \ln|x|\varphi^{(r)}(\xi x)}{r!} dx,
\end{aligned}
$$

since η can be made arbitrarily small. It follows that

$$\begin{aligned}
\operatorname*{N-lim}_{n\to\infty} \frac{1}{m+1}\Big\langle \int_{-1/n}^{1/n} \ln^m(x^r - t)\delta_n'(t)\,dt, \varphi(x)\Big\rangle \\
&= -\sum_{k=0}^{r-2} \frac{m!r^m(-1)^{r-k-1}-1}{(r-k-1)^{m+1}k!}\varphi^{(k)}(0) \\
&\quad + r^m \int_{-1}^{1} x^{-r}\ln^m|x| \left[\varphi(x) - \sum_{k=0}^{r-1}\frac{x^k}{k!}\varphi^{(k)}(0)\right] dx \\
&= r^m\langle x^{-r}\ln^m|x|, \varphi(x)\rangle
\end{aligned}$$

on using Lemmas 2.4 and 2.5. This proves equation (1) on the interval $[-1, 1]$. However, equation (1) clearly holds on any closed interval not containing the origin, and the proof is complete. □

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