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SUBMERSION FROM SEMI-RIEMANNIAN MANIFOLDS ONTO LIGHTLIKE MANIFOLDS

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Abstract

In this paper, we introduce the idea of a lightlike submersion from a semi-Riemannian manifold onto a lightlike manifold, and give some examples. Then we define O'Neill's tensors for such submersions and investigate their main properties. We show that the Schouten connection is not a metric connection in a lightlike submersion. We also investigate curvature properties of the manifolds and establish a relation between the null sectional curvatures of a semi-Riemannian manifold and a lightlike manifold.

Keywords: Lightlike Manifold, Submersion, Lightlike submersion.

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1. Introduction

Let M and B be Riemannian manifolds. A Riemannian submersion $\pi : M \to B$ is a mapping of M onto B satisfying the following axioms S.1 and S.2:

S.1. π has maximal rank.

Hence, for each $b \in B$, $\pi^{-1}(b)$ is a submanifold of M of dimension $\dim M - \dim B$. The submanifolds $\pi^{-1}(b)$ are called *fibers*, and a vector field on M is vertical if it is always tangent to the fibers, horizontal if always orthogonal to the fibers. The second axiom is given by

S.2. π_* preserves the lengths of horizontal vectors.

The theory of Riemannian submersion was introduced by O'Neill and Gray in [7] and [3], respectively. Since then, it has been used as an effective tool to describe the structure of a Riemannian manifold. As it is well known, when M and B are Riemannian manifolds,

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then the fibers are always Riemannian manifolds. However, when the manifolds are semi-Riemannian manifolds, the fibers of π may not be semi-Riemannian (hence Riemannian) manifolds. Recently, Şahin defined and studied a submersion from lightlike manifolds onto semi-Riemannian manifolds in [9].

In this paper, we consider a semi-Riemannian manifold M and a lightlike manifold N, and define a lightlike submersion from M to N. In particular, we focus on the existence of lightlike submersions and give several examples. Also we show that the concept of lightlike submersion is very different from that of Riemannian submersion and semi-Riemannian submersion (For semi-Riemannian submersions, see: [8]).

2. Lightlike manifolds

In this section we give some brief information on lightlike manifolds (For more details, see [2] and [5]). Let V be a vector space of dimension n. An *inner product* on V is a symmetric bilinear form g, which is called a *non-degenerate inner product* if $g(X,Y) = 0\ \forall X \in V$ implies $Y = 0$. Otherwise it is called *degenerate* (*lightlike*). Let V be a vector space and suppose that there exists a symmetric bilinear form g on V. Then there exists a basis $\{e_i\}$ on V such that

$$\begin{aligned} g(e_i,e_i) &= 0, \text{ for } 1 \leq i \leq r, \\ g(e_j,e_j) &= -1, \text{ for } 1 \leq j \leq q, \\ g(e_k,e_k) &= 1, \text{ for } 1 \leq k \leq p, \\ g(e_I,e_J) &= 0, \text{ for } I \neq J. \end{aligned}$$

Such a basis is called *orthonormal*, and the triple (r,q,p) is called the *type* of the bilinear form g ([6, P.107]). We will denote a vector space V endowed with a bilinear form g of type (r,q,p) by $V_{r,q,p}$.

Let (M,g) be a real n-dimensional smooth manifold, where g is a symmetric tensor field of type $(0,2)$. We assume that M is paracompact. The *radical* or *null space* of T_xM is a subspace, denoted by $\operatorname{Rad} T_xM$, of T_xM defined by

$$\operatorname{Rad} T_xM = \{\xi \in T_xM : g(\xi,X) = 0,\ X \in T_x M\}. \tag{2.1}$$

The dimension of $\operatorname{Rad} T_xM$ is called the *nullity degree* of g. Suppose the mapping $\operatorname{Rad} TM$ that assigns to each $x \in M$ the radical subspace $\operatorname{Rad} T_xM$ of T_xM with respect to g_x, defines a smooth distribution of rank $r > 0$ on M. Then $\operatorname{Rad} TM$ is called the *radical distribution of* M. The manifold M is called a *lightlike manifold* if $0 < r \leq n$.

2.1. Example. We denote by $\mathbb{R}^n_{r,q,p}$ the space $\mathbb{R}^n$ endowed with the bilinear form g defined by $g(e_i,e_j)_{r,q,p} = (G_{r,q,p})_{ij}$, where e_i, $i \in \{1,\dots,n\}$ is the standard basis of E^n, and $G_{r,q,p}$ is the diagonal matrix determined by g, i,e,

$$(G)_{ij} = \text{diagonal}\,(\underbrace{0,\dots,0}_{r-times},\underbrace{-1,\dots,-1}_{q-times},\underbrace{1\dots,1}_{p-times}).$$

Hence, $\mathbb{R}^n_{r,q,p}$ is an r-lightlike manifold.

Now, consider a complementary distribution $S(TM)$ to $\operatorname{Rad} TM$ in TM. From [2, Proposition 2.1], we know that $S(TM)$ is a semi-Riemannian distribution. Therefore, we have

$$TM = S(TM) \oplus \operatorname{Rad} TM. \tag{2.2}$$

The associated quadratic form h of g is of type $(r, q, p), p + q + r = n$, locally given by

$$h = -\sum_{a=1}^{q} (\omega^a)^2 + \sum_{A=q+1}^{q+p} (\omega^A)^2, \tag{2.3}$$

where $(\omega^1, \dots, \omega^{p+q})$ are linearly independent local differential forms on M. Substituting in (2.3)

$$\omega^a = \omega^a_I dx^I \, \omega^A = \omega^A_I dx^I, \; I \in \{1, \dots, n\}$$

we obtain

$$g_{IJ} = -\sum_{a=1}^{q} \omega^a_I \omega^a_J + \sum_{A=q+1}^{q+p} \omega^A_I \omega^A_J, \; J \in \{1, \dots, n\}, \tag{2.4}$$

where $\operatorname{rank}[g_{IJ}] = p + q < n$.

Suppose $\operatorname{Rad} TM$ is an integrable distribution. Then it follows from the Frobenius theorem that the leaves of $\operatorname{Rad} TM$ determine a foliation on M of dimension r, that is, M is a disjoint union of connected subsets $\{L_t\}$, and each point x of M has a coordinate system (U, x^i), where $i \in \{1, \dots, n\}$ and $L_t \cap U$ is locally given by the equations $x^a = c^a, \; a \in \{r+1, \dots, n\}$ for real constants c^a and $x^\alpha, \; \alpha \in \{1, \dots, r\}$, are local coordinates of the leaf L_t of $\operatorname{Rad} TM$ passing through x.

Consider another coordinate system $(\bar{V}, \bar{x}^\alpha)$ on M. The transformation of coordinates on M endowed with an integrable distribution has the following special form

$$0 = d\bar{x}^a = \frac{\partial \bar{x}^a}{\partial x^b} dx^b + \frac{\partial \bar{x}^a}{\partial x^\alpha} dx^\alpha = \frac{\partial \bar{x}^a}{\partial x^\alpha} dx^\alpha,$$

which implies $\frac{\partial \bar{x}^a}{\partial x^\alpha} = 0, \; \forall a \in \{r+1, \dots, n\}, \; \alpha \in \{1, \dots, r\}$. Hence the transformation of coordinates on M is given by

$$\bar{x}^\alpha = \bar{x}^\alpha(x^1, \dots, x^n), \; \bar{x}^a = \bar{x}^a(x^{r+1}, \dots, x^n). \tag{2.5}$$

As g is degenerate on TM, by using(2.1) and (2.3) we obtain $g_{\alpha\beta} = g_{\alpha a} = 0$. Thus the matrix of g with respect to the natural frames $\{\partial\}$ becomes

$$[g_{ij}] = \begin{pmatrix} 0_{r,r} & 0_{r,n-r} \\ 0_{n-r,r} & g_{ab}(x^1, \dots, x^n) \end{pmatrix}.$$

Suppose that

$$\frac{\partial g_{ab}}{\partial x^\alpha} = 0, \forall \, a, b \in \{r+1, \dots, n\}, \; \alpha \in \{1, \dots, r\} \tag{2.6}$$

holds in a fixed adapted coordinate system, then by using the first group of equations in (2.5), we obtain that it holds in any other coordinate system adapted to the foliation induced by $\operatorname{Rad} TM$.

2.2. Definition. A lightlike manifold M on which $\operatorname{Rad} TM$ is integrable, and there exists a local coordinate system such that (2.6) is satisfied, is called a *Reinhart lightlike manifold*.

For Reinhart lightlike manifolds, we have the following theorem.

2.3. Theorem. [2]. *Let (M, g) be a lightlike manifold. Then the following assertions are equivalent:*

(1) *(M, g) is a Reinhart lightlike manifold.*
(2) $\operatorname{Rad} TM$ *is a Killing distribution.*
(3) *There exists a torsion free linear connection ∇ on M such that g is a parallel tensor field with respect to ∇.*

3. Lightlike submersions

In this section, we will introduce lightlike submersions and give several examples. It will be seen from these examples that there are many lightlike submersions. We also define O'Neill's tensors for lightlike submersions, check the usual properties and observe that a lightlike submersion does not satisfy these properties in general. Moreover we show that a Schouten connection is not a metric connection, and give the explicit expression of the derivative of the metric tensor with respect to this connection.

Let (M_1, g_1) be a semi-Riemannian manifold and (M_2, g_2) an r-lightlike manifold. We consider a smooth submersion $f : M_1 \to M_2$, then, for $p \in M_2$, $f^{-1}(p)$ is a submanifold of dimension $\dim M_1 - \dim M_2$. On the other hand, the kernel of f_* at the point p, (f_* is the derivative map), is defined by

(3.1) $\quad \operatorname{Ker} f_* = \{X \in T_p(M_1) : f_*(X) = 0\}.$

Now, consider $(\operatorname{Ker} f_*)^\perp$ defined as follows

(3.2) $\quad (\operatorname{Ker} f_*)^\perp = \{Y \in T_p(M_1) : g_1(Y, X) = 0, \forall\, X \in \operatorname{Ker} f_*\}.$

Since T_pM_1 is a semi-Riemannian vector space, $(Ker f_*)^\perp$ may not be a complement to $\operatorname{Ker} f_*$. Suppose $\triangle = \operatorname{Ker} f_* \cap (\operatorname{Ker} f_*)^\perp \neq \{0\}$. Then, consider the following four cases of submersions.

Case 1. $0 < \dim \triangle < \min\{\dim(\operatorname{Ker} f_*), \dim(\operatorname{Ker} f_*)^\perp\}$: Then $\triangle$ is the radical subspace of T_pM_1. Thus, we can construct a quasi-orthonormal basis of M_1 along $\operatorname{Ker} f_*$ as in[2]. Since $\operatorname{Ker} f_*$ is a real lightlike vector space, there is a complementary non-degenerate subspace to $\triangle$ (cf. [2, Proposition2.1]). Let $S(\operatorname{Ker} f_*)$ be a complementary non-degenerate subspace to $\triangle$ in $\operatorname{Ker} f_*$. Then we have

(3.3) $\quad \operatorname{Ker} f_* = \triangle \perp S(\operatorname{Ker} f_*).$

In a similar way we have

$$(\operatorname{Ker} f_*)^\perp = \triangle \perp S(\operatorname{Ker} f_*)^\perp,$$

where $S(\operatorname{Ker} f_*)^\perp$ is a complementary subspace of $\triangle$ in $(\operatorname{Ker} f_*)^\perp$. Since $S(\operatorname{Ker} f_*)$ is non-degenerate in T_pM_1, we can consider

$$T_pM_1 = S(\operatorname{Ker} f_*) \perp (S(\operatorname{Ker} f_*))^\perp,$$

where $(S(\operatorname{Ker} f_*))^\perp$ is the complementary subspace of $S(\operatorname{Ker} f_*)$ in T_pM_1. Also since $S(\operatorname{Ker} f_*)$ and $(S(\operatorname{Ker} f_*))^\perp$ are non-degenerate, we obtain

$$(S(\operatorname{Ker} f_*))^\perp = S(\operatorname{Ker} f_*)^\perp \perp (S(\operatorname{Ker} f_*)^\perp)^\perp.$$

Then, from [2, Proposition 2.4], we know that "there exists a quasi-orthonormal basis of T_pM_1 along $\operatorname{Ker} f_*$," thus we have

$$g(\xi_i, \xi_j) = g(N_i, N_j) = 0 \qquad g(\xi_i, N_j) = \delta_{ij}$$
$$g(W_\alpha, \xi_j) = g(W_\alpha, N_j) = 0 \qquad g(W_\alpha, W_\alpha) = \epsilon_\alpha \delta_{\alpha\beta}$$

for any $i, j \in \{1, \ldots, r\}$ and $\alpha, \beta \in \{1, \ldots, t\}$, where $\{N_i\}$ are smooth null vector fields of $(S(\operatorname{Ker} f_*)^\perp)^\perp$, $\{\xi_i\}$ is basis of $\triangle$ and W_α is a basis of $S(\operatorname{Ker} f_*)^\perp$. We denote the set of vector fields $\{N_i\}$ by $\operatorname{ltr}(\operatorname{Ker} f_*)$, and consider the following subspace

$$\operatorname{tr}(\operatorname{Ker} f_*) = \operatorname{ltr}(\operatorname{Ker} f_*) \perp S(\operatorname{Ker} f_*)^\perp.$$

We notice that $\mathrm{ltr}(\mathrm{Ker}\, f_*)$ and $\mathrm{Ker}\,(f_*)$ are not orthogonal to each other. Now, we will call $\mathcal{V} = \mathrm{Ker}\, f_*$ the *vertical space* of T_pM_1, and $\mathcal{H} = \mathrm{tr}(\mathrm{Ker}\, f_*)$ the *horizontal space* as is usual in the theory of Riemannian submersions. Thus we obtain

$$T_pM_1 = \mathcal{V}_p \oplus \mathcal{H}_p.$$

It is important to emphasize again that $\mathcal{V}$ and $\mathcal{H}$ are not orthogonal to each other.

We are now ready to give the definition of a lightlike submersion.

3.1. Definition. Let (M_1, g_1) be a semi-Riemannian manifold and (M_2, g_2) an r-lightlike manifold. Suppose that $f : M_1 \to M_2$ is a submersion such that

(1) $\dim \triangle = \dim\{(\mathrm{Ker}\, f_*) \cap (\mathrm{Ker}\, f_*)^\perp\} = r$, $0 < r < \min\{\dim(\mathrm{Ker}\, f_*), \dim(\mathrm{Ker}\, f_*)^\perp\}$.
(2) f_* preserves the length of horizontal vectors, i.e., $g_1(X, Y) = g_2(f_*X, f_*Y)$ for $X, Y \in \Gamma(\mathcal{H})$.

Then, we say that f is an *r–lightlike submersion*.

The other cases arise as follows:

Case 2. $\dim \triangle = \dim(\mathrm{Ker}\, f_*) < \dim(\mathrm{Ker}\, f_*)^\perp$. Then $\mathcal{V} = \triangle$ and $\mathcal{H} = S(\mathrm{Ker}\, f_*)^\perp \perp \mathrm{ltr}(\mathrm{Ker}\, f_*)$. We call f an *isotropic submersion*.

Case 3. $\dim \triangle = \dim(\mathrm{Ker}\, f_*)^\perp < \dim(\mathrm{Ker}\, f_*)$. Then $\mathcal{V} = S(\mathrm{Ker}\, f_*) \perp \triangle$ and $\mathcal{H} = \mathrm{ltr}(\mathrm{Ker}\, f_*)$. We call f a *co-isotropic submersion*.

Case 4. $\dim \triangle = \mathit{dim}(\mathrm{Ker}\, f_*)^\perp = \dim(\mathrm{Ker}\, f_*)$. Then $\mathcal{V} = \triangle$ and $\mathcal{H} = \mathrm{ltr}(\mathrm{Ker}\, f_*)$. We call f a *totally lightlike submersion*.

We note that, from the condition of Definition 3.1.(2), it follows that the nullity degree of M_2 and the dimension of $\triangle$ are equal. Hence we have the following

3.2. Proposition. *Let $f : M_1 \to M_2$ be a lightlike submersion. Then,*

(1) *If f is an r–lightlike or isotropic submersion, then M_2 is an r–lightlike manifold.*
(2) *If f is a co-isotropic or totally lightlike submersion, then M_2 is a totally lightlike manifold.* □

A *basic vector field* on M_1 is a horizontal vector field X which is f–related to a vector field $\tilde{X}$ on M_2, that is, $f_*(X_p) = \tilde{X}_{f(p)}$ for all $p \in M_1$. Every vector field $\tilde{X}$ on M_2 has a unique horizontal lift X to M_1, and X is basic. Thus $X \longleftrightarrow \tilde{X}$ is a one to one correspondence between basic vector fields on M_1 and arbitrary vector fields on M_2.

Now, we give one example for r–lightlike, isotropic, co-isotropic and totally lightlike submersions.

3.3. Example. Let $\mathbb{R}^4_{0,1,3}$ and $\mathbb{R}^2_{1,0,1}$ be $\mathbb{R}^4$ and $\mathbb{R}^2$ endowed with the Lorentzian metric $g_1 = -(dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_4)^2$, and degenerate metric $g_2 = (dy_2)^2$, where x_1, x_2, x_3, x_4 and y_1, y_2 are the canonical coordinates on $\mathbb{R}^4$ and $\mathbb{R}^2$, respectively. We define the following map

$$f : \mathbb{R}^4_{0,1,3} \to \mathbb{R}^2_{1,0,1},\ (x_1, x_2, x_3, x_4) \mapsto \Big(x_1 + x_3, \frac{x_2 + x_4}{\sqrt{2}}\Big).$$

Then, the kernel of f_* is

$$\mathrm{Ker}\, f_* = \mathrm{Span}\Big\{W_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3},\ W_2 = -\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}\Big\}.$$

Thus, we obtain

$$(\mathrm{Ker}\, f_*)^\perp = \mathrm{Span}\Big\{T_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3},\ T_2 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}\Big\}.$$

Hence, we have $W_1 = T_1$,

$$\triangle = \mathrm{Ker}\, f_* \cap (\mathrm{Ker}\, f_*)^\perp = \mathrm{Span}\{W_1\}.$$

Then, we get $\mathrm{ltr}(\mathrm{Ker}\, f_*) = \mathrm{Span}\{N = \frac{1}{2}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3})\}$. It is easy to check that $g_1(N, W_1) = 1$ and $g_1(N, W_2) = 0$, Thus the vertical and horizontal spaces are given by

$$\mathcal{V} = \mathrm{Span}\{W_1, W_2\},\ \mathcal{H} = \mathrm{Span}\{T_2, N\}.$$

Moreover, since $f_*(T_2) = \sqrt{2}\frac{\partial}{\partial y_2},\ f_*(N) = \frac{\partial}{\partial y_1}$, we obtain that

$$g_1(N, N) = g_2(f_*N, f_*N) = 0$$
$$g_1(T_2, T_2) = g_2(f_*T_2, f_*T_2) = 2.$$

Hence, f is a 1-lightlike submersion.

3.4. Example. Let $\mathbb{R}^6_{0,2,4}$ and $\mathbb{R}^3_{2,0,1}$ be $\mathbb{R}^6$ and $\mathbb{R}^3$ endowed with the semi-Riemannian metric $g_1 = -(dx_1)^2 - (dx_2)^2 + (dx_3)^2 + (dx_4)^2 + (dx_5)^2 + (dx_6)^2$ and the degenerate metric $g_2 = (dy_2)^2$, where $x_1, x_2, x_3, x_4, x_5, x_6$ are the canonical coordinates on R^6 and y_1, y_2, y_3 are the canonical coordinates on R^3, respectively. We define the following map

$$f : \mathbb{R}^6_{0,2,4} \to \mathbb{R}^3_{2,0,1},\ (x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (x_1\, cosh\,\alpha - x_3\, sinh\,\alpha - x_4, x_2 + x_5, x_6)$$

for $\alpha \in \mathbb{R}$. Then, the kernel of f_* is

$$\mathrm{Ker}\, f_* = \mathrm{Span}\Big\{Z_1 = cosh\,\alpha\,\frac{\partial}{\partial x_1} + sinh\,\alpha\,\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4},\ Z_2 = -\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_5}\Big\}$$

and

$$(\mathrm{Ker}\, f_*)^\perp = \mathrm{Span}\Big\{Z_1, Z_2, Z_3 = \frac{\partial}{\partial x_6}\Big\}.$$

Hence, $\Delta = \mathrm{Span}\{Z_1, Z_2\} = \mathrm{Ker}\, f_* \subset (\mathrm{Ker}\, f_*)^\perp$. Moreover, we get

$$\mathrm{ltr}(\mathrm{Ker}\, f_*) = \Big\{N_1 = \frac{1}{2}\Big\{-cosh\,\alpha\,\frac{\partial}{\partial x_1} - sinh\,\alpha\,\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\Big\},\ N_2 = \frac{1}{2}\Big\{\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_5}\Big\}\Big\}.$$

Then, it is easy to see that $f_*(Z_3) = \frac{\partial}{\partial y_3}$, $f_*(N_1) = -\frac{\partial}{\partial y_1}$, $f_*(N_2) = \frac{\partial}{\partial y_2}$. Thus f is an isotropic submersion.

3.5. Example. Let $\mathbb{R}^5_{0,2,3}$ and $\mathbb{R}^2_{2,0,0}$ be $\mathbb{R}^5$ and $\mathbb{R}^2$ endowed with the semi-Riemannian metric $g_1 = -(dx_1)^2 - (dx_2)^2 + (dx_3)^2 + (dx_4)^2 + (dx_5)^2$ and degenerate metric g_2, respectively, where x_1, x_2, x_3, x_4, x_5 are the canonical coordinates on $\mathbb{R}^5$ We denote the canonical coordinates on R^2 by y_1, y_2. We define the following map

$$f : \mathbb{R}^5_{0,2,3} \to \mathbb{R}^2_{2,0,0},\ (x_1, x_2, x_3, x_4, x_5) \mapsto \Big(x_1 + \frac{x_3 + x_4}{\sqrt{2}}, x_2 + \frac{x_3 - x_4}{\sqrt{2}}\Big).$$

Then, the kernel of f_* is

$$\mathrm{Ker}\, f_* = \mathrm{Span}\Big\{Z_1 = -\sqrt{2}\,\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4},\ Z_2 = \sqrt{2}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4},\ Z_3 = \frac{\partial}{\partial x_5}\Big\}$$

and

$$(\mathrm{Ker}\, f_*)^\perp = \mathrm{Span}\{Z_1, Z_2\} = \Delta \subset \mathrm{Ker}\, f_*.$$

Hence, we get

$$\mathrm{ltr}(\mathrm{Ker}\, f_*) = \mathrm{Span}\Big\{N_1 = \frac{1}{4}\Big\{\sqrt{2}\,\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\Big\},\ N_2 = \frac{1}{4}\Big\{-\sqrt{2}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\Big\}\Big\}.$$

Moreover, we have $f_*(N_1) = \frac{1}{\sqrt{2}} \frac{\partial}{\partial y_1}$ and $f_*(N_2) = -\frac{1}{\sqrt{2}} \frac{\partial}{\partial y_2}$. Thus f is a co-isotropic submersion.

3.6. Example. Let $\mathbb{R}^4_{0,2,2}$ and $\mathbb{R}^2_{2,0,0}$ be $\mathbb{R}^4$ and $\mathbb{R}^2$ endowed with the semi-Riemannian metric $g_1 = -(dx_1)^2 - (dx_2)^2 + (dx_3)^2 + (dx_4)^2$ and the degenerate metric g_2, respectively, where x_1, x_2, x_3, x_4 are the canonical coordinates on R^4. We denote the canonical coordinates on $\mathbb{R}^2$ by y_1, y_2. We define the following map

$$f : \mathbb{R}^4_{0,2,2} \to \mathbb{R}^2_{2,0,0},\ (x_1, x_2, x_3, x_4) \mapsto (x_1 + x_3, x_2 + x_4).$$

Then, we have

$$\operatorname{Ker} f_* = \operatorname{Span}\Big\{Z_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3},\ Z_2 = \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_4}\Big\} = (\operatorname{Ker} f_*)^\perp = \Delta.$$

Moreover, we obtain

$$\operatorname{ltr}(\operatorname{Ker} f_*) = \operatorname{Span}\Big\{N_1 = \frac{1}{2}\Big\{-\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3}\Big\},\ N_2 = \frac{1}{2}\Big\{-\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_4}\Big\}\Big\}.$$

Furthermore, we derive

$$f_*(N_1) = -\frac{\partial}{\partial y_1},\ f_*(N_2) = -\frac{\partial}{\partial y_2}.$$

Hence, f is a totally lightlike submersion.

Now, we define O'Neill's tensors for a lightlike submersion:

Let $f : M_1 \to M_2$ be a lightlike submersion and X, Y arbitrary vector fields on M_1. Let $h : TM_1 \to \mathcal{H}$ and $\nu : TM_1 \to \mathcal{V}$ denote the natural projections associated with the direct sum decomposition $TM_1 = \mathcal{H} \oplus \mathcal{V}$. Let ∇ be the Levi-Civita connection of (M_1, g_1). Then, we define a tensor field T of type (1,2) by

$$T_X Y = h\nabla_{\nu X} \nu Y + \nu \nabla_{\nu X} hY. \tag{3.4}$$

It is easy check that T has the following properties as a Riemann submersion.

(1) T reverses the horizontal and vertical subspaces.
(2) T is vertical: $T_X = T_{\nu X}$
(3) For vertical vector fields, T has the symmetry property $T_X Y = T_Y X$.

The other tensor is given by

$$A_X Y = \nu\nabla_{hX} hY + h\nabla_{hX} \nu Y, \tag{3.5}$$

and it has the following properties:

(1) A reverses the horizontal and vertical subspaces.
(2) A is horizontal: $A_X = A_{hX}$.

3.7. Lemma. *Let $f : M_1 \to M_2$ be a lightlike submersion. If X and Y are basic vector fields on M_1, then*

(a) $g_1(X, Y) = g_2(\tilde{X}, \tilde{Y}) \circ f$,
(b) $h[X, Y]$ *is the basic vector field corresponding to* $[\tilde{X}, \tilde{Y}]$.

Proof. Since f is a lightlike submersion, from Definition 3.1 (2) we have (a).

(b) follows from the identity $f_*[X, Y] = [\tilde{X}, \tilde{Y}]$. □

For Riemannian submersions, it is well known that $h\nabla_X Y$ is the basic vector field corresponding to $\nabla^{M_2}_{\tilde{X}} \tilde{Y}$, where ∇^{M_2} is the linear connection of M_2. We will show that this property is here true in a particular case.

3.8. Theorem. *Let M_1 be a semi-Riemannian manifold and M_2 a Reinhart lightlike manifold. Let also $f : M_1 \to M_2$ be a lightlike submersion. Then $h\nabla_X Y$ is the basic vector field corresponding to $\nabla^{M_2}_{\tilde{X}}\tilde{Y}$, for basic vector fields X, Y.*

Proof. From the Kozsul formula we have

$$2g_1(\nabla_X Y, Z) = X(g_1(Y,Z)) + Y(g(Z,X)) - Z(g_1(X,Y)) - g_1(X,[Y,Z]) + g_1([Z,X],Y) + g_1(Z,[X,Y]).$$

Since $X(g_1(Y,Z)) = \tilde{X}g_2(\tilde{Y},\tilde{Z}) \circ f$, from Lemma 3.1 we obtain

$$2g_1(\nabla_X Y, Z) = \tilde{X}g_2(\tilde{Y},\tilde{Z}) \circ f + \tilde{Y}g_2(\tilde{Z},\tilde{X}) \circ f - \tilde{Z}g_2(\tilde{Y},\tilde{X}) \circ f - g_2(\tilde{X},[\tilde{Y},\tilde{Z}]) \circ f + g_2(\tilde{Y},[\tilde{Z},\tilde{X}]) \circ f + g_2(\tilde{Z},[\tilde{X},\tilde{Y}]) \circ f. \tag{3.6}$$

Since M_2 is a Reinhart lightlike manifold, then from Theorem 2.3, it has a Levi-Civita connection. Hence ∇^{M_2} satisfies the Kozsul identity. Thus the right side of equation (3.6) is $2g_2(\nabla^{M_2}_{\tilde{X}}\tilde{Y}, \tilde{Z})$, hence we have

$$g_1(\nabla_X Y, Z) = g_2(\nabla^{M_2}_{\tilde{X}}\tilde{Y}, \tilde{Z}) \circ f.$$

Thus we obtain that $h\nabla_X Y$ is the basic vector field corresponding to $\nabla^{M_2}_{\tilde{X}}\tilde{Y}$. □

From (3.1) and (3.5) we have the following.

3.9. Lemma. *Let $f : (M_1, g_1) \to (M_2, g_2)$ be an r–lightlike submersion. Then we have:*
(a) $\nabla_U V = T_U V + \nu\nabla_U V$,
(b) $\nabla_V X = h\nabla_V X + T_V X$,
(c) $\nabla_X V = A_X V + \nu\nabla_X V$,
(d) $\nabla_X Y = h\nabla_X Y + A_X Y$,

for any $X, Y \in \Gamma(\mathrm{ltr}(\ker f_*))$, $U, V \in \Gamma(\mathrm{Ker}\, f_*)$, where ∇ is the Levi-Civita connection on M_1. □

We note that T and A are skew-symmetric in the Riemannian submersions. But these properties are not generally valid for a lightlike submersion because the horizontal and vertical subspaces are not orthogonal to each other. However, we have these properties for some particular cases.

3.10. Lemma. *Let $f : (M_1, g_1) \to (M_2, g_2)$ be an r–lightlike submersion. Then we have:*
(a) $g_1(T_V X, Y) = -g_1(T_V Y, X)$,
(b) $g_1(A_X V, W) = -g_1(A_X W, V)$,

for any $X, Y \in \Gamma(\mathrm{ltr}(\mathrm{Ker}\, f_*))$, $V \in \Gamma(\mathrm{Ker}\, f_*)$ and $W \in \Gamma(\Delta)$.

Proof. We only prove (a), the proof of (b) being similar. Using (3.1), we obtain

$$T_V X = h\nabla_{\nu V}\nu X + \nu\nabla_{\nu V} hX = \nu\nabla_V X \tag{3.7}$$

and

$$T_V Y = h\nabla_{\nu V}\nu Y + v\nabla_{\nu V} hY = \nu\nabla_V Y. \tag{3.8}$$

On the other hand, using $\nabla g_1 = 0$ we have

$$V g_1(X,Y) = g_1(\nabla_V X, Y) + g_1(\nabla_V Y, X). \tag{3.9}$$

Then, from (3.7), (3.8) and (3.9) we have

$$g_1(T_V X, Y) + g_1(T_V Y, X) = 0.$$

□

We also note that A has the alternation property $A_X Y = -A_Y X$ for a Riemannian submersion, but this is not generally the case for a lightlike submersion. However, we have a special case where the above property is satisfied for a lightlike submersion.

3.11. Lemma. *Let $f : (M_1, g_1) \to (M_2, g_2)$ be an r–lightlike submersion and ∇ the Levi-Civita connection of g_1. Then $A_X Z = -A_Z X$ if and only if $\nabla_N X$ does not belong to $\Gamma(S(\operatorname{Ker} f_*)^\perp)$, for any $X, Z \in \Gamma(S(\operatorname{Ker} f_*)^\perp)$ and $N \in \Gamma(\operatorname{ltr}(\operatorname{Ker} f_*))$.*

Proof. We first prove that $A_X X = 0$ for any $X \in \Gamma(S(\operatorname{Ker} f_*)^\perp)$. We may assume that X is basic.

Since $A_X X \in \Gamma(\mathcal{V})$, $A_X X = 0$ if and only if $g_1(A_X X, Y) = 0$ and $g_1(A_X X, N) = 0$ for $Y \in \Gamma(S(\operatorname{Ker} f_*))$ and $N \in \Gamma(\operatorname{ltr}(\operatorname{Ker} f_*))$. For any $Y \in \Gamma(S(\operatorname{Ker} f_*))$ we have $g_1(A_X X, Y) = g_1(\nabla_X X, Y)$. Since $g_1(X, Y) = 0$, we get $g_1(A_X X, Y) = -g_1(X, \nabla_X Y)$. Hence we have $g_1(A_X X, Y) = -g_1(X, [X, Y] + \nabla_Y X)$. Then $[X, Y] \in \Gamma(\operatorname{Ker} f_*)$ implies that

$$g_1(A_X X, Y) = -g_1(X, \nabla_Y X).$$

On the other hand, since X is constant along the vertical subspace, we have $Y g_1(X, X) = 0$ which gives $g_1(\nabla_Y X, X) = 0$, $Y \in \Gamma(S(\operatorname{Ker} f_*))$ and $X \in \Gamma((\operatorname{Ker} f_*)^\perp)$. Putting this in the above equation we arrive at

$$g_1(A_X X, Y) = 0. \tag{3.10}$$

In a similar way, from (3.5) we get $g_1(A_X X, N) = g_1(\nabla_X X, N)$. Hence we obtain $g_1(A_X X, N) = -g_1(X, \nabla_X N)$. Then we derive $g_1(A_X X, N) = -g_1(X, [X, N] + \nabla_X N)$. Since $[X, N] \in \Gamma(\operatorname{Ker} f_*)$, we obtain $g_1(A_X X, N) = -g_1(X, \nabla_N X)$. Hence we conclude that

$$g_1(A_X X, N) = 0 \tag{3.11}$$

if and only if $\nabla_N X$ does not belong to $\Gamma(S(\operatorname{Ker} f_*)^\perp)$. Then, from (3.10) and (3.11) we obtain

$$A_X Z = -A_Z X,$$

if and only $\nabla_N X$ does not belong to $\Gamma(S(\operatorname{Ker} f_*)^\perp)$, which gives the statement of lemma. □

Now, we denote by $\bar{\nabla}$ the Schouten connection associated with the distributions $\mathcal{V}$ and $\mathcal{H}$. It is defined by

$$\bar{\nabla}_X Y = h\nabla_X hY + \nu\nabla_X \nu Y \tag{3.12}$$

for $X, Y \in \Gamma(TM)$ [4]. It is easy to see that $\bar{\nabla}$ is a linear connection along a fiber with respect to the induced metric. Moreover, by direct computations, using (3.1) and (3.5), we have the following:

3.12. Proposition. *Let $f : (M_1, g_1) \to (M_2, g_2)$ be an r–lightlike submersion. Then we have*

$$\bar{\nabla}_X Y = \nabla_X Y - T_X Y - A_X Y \tag{3.13}$$

for any $X, Y \in \Gamma(TM_1)$, where $\bar{\nabla}$ is the Schouten connection and ∇ is the Levi-civita connection on M_1. □

It is important to mention that the Schouten connection is a metric connection in a non-degenerate submersion [4]. But this is not true for a lightlike submersion, in general. The reason is that T and A are not anti-symmetric in a lightlike submersion. More precisely, we have the following.

3.13. Proposition. *Let $f : (M_1, g_1) \to (M_2, g_2)$ be an r–lightlike submersion. Then we have:*

$$(\bar{\nabla}_X g_1)(Y, Z) = g_1(T_X Y, Z) + g_1(Y, T_X Z) + g_1(A_X Y, Z) + g_1(Y, A_X Z) \tag{3.14}$$

for $X, Y \in \Gamma(TM_1)$.

Proof. Since ∇ is a metric connection we have

$$(\nabla_X g_1)(Y, Z) = X g_1(Y, Z) - g_1(\nabla_X Y, Z) - g_1(Y, \nabla_X Z) = 0.$$

Thus, using (3.13), we have (3.14). □

In the rest of this section, we give the covariant derivatives of the tensors A and T. First recall that the covariant derivative of a tensor field A of type $(1, 2)$ is given by

$$(\nabla_E A)_F G = \nabla_E(A_F G) - A_{\nabla_E F} G - A_F(\nabla_E G)$$

for any three vector fields $E, F, G \in \Gamma(TM_1)$. Now if we choose $E = V \in \Gamma(\mathrm{Ker}\, f_*)$, $F = W \in \Gamma(\mathrm{Ker}\, f_*)$, then $A_F = A_W = 0$ so the first and third terms on the right side vanish. In the middle term we have

$$A_{\nabla_V W} = A_{h\nabla_V W} = A_{T_V W},$$

so we get

$$(\nabla_V A)_W = -A_{T_V W}.$$

If we take $E = X \in \Gamma(\mathrm{ltr}(\mathrm{Ker}\, f_*))$ and $F = W \in \Gamma(\mathrm{Ker}\, f_*)$, then we have

$$(\nabla_X A)_W = -A_{A_X W}.$$

All other terms are zero. In a similar way, we get,

$$(\nabla_X T)_Y = -T_{A_X Y},\ (\nabla_V T)_Y = -T_{T_V Y},\ X, Y \in \Gamma(\mathrm{ltr}(\mathrm{Ker}\, f_*)),\ V \in \Gamma(\mathrm{Ker}\, f_*).$$

3.14. Lemma. *Let $f : (M_1, g_1) \to (M_2, g_2)$ be a totally lightlike submersion. Then we have:*

(a) $g_1((\nabla_U A)_X V, Y) = g_1(T_U A_X V, Y) - g_1(A_X T_U V, Y)$,
(b) $g_1((\nabla_X T)_U Y, V) = g_1(A_X T_U Y, V) - g_1(T_{A_X Y} U, V)$,
(c) $g_1((\nabla_U T)_V X, W) = g_1(T_{T_V X} U, W) - g_1(T_{T_U X} V, W)$,
(d) $g_1((\nabla_X A)_Y U, Z) = g_1(A_X A_Y U, Z) - g_1(A_Y A_X U, Z)$,

for any $X, Y, Z \in \Gamma(\mathrm{ltr}(\mathrm{Ker}\, f_*))$, $V, U, W \in \Gamma(\mathrm{Ker}\, f_*)$, where ∇ is the Levi-Civita connection on M_1.

Proof. We only prove (a), the other assertions can be obtained in a similar way.

From the definition of covariant derivative of the tensor field A, we have

$$g_1((\nabla_U A)_X V, Y) = g_1(\nabla_U(A_X V), Y) - g_1(A_{\nabla_U X}(V), Y) - g_1(A_X(\nabla_U V), Y),$$

for any $X, Y \in \Gamma(\mathrm{ltr}(\mathrm{Ker}\, f_*))$, $V, U \in \Gamma(\mathrm{Ker}\, f_*)$. On the other hand, using Lemma 3.9 we obtain

$$\begin{aligned} g_1(\nabla_U(A_X V), Y) &= g_1(h\nabla_U A_X V, Y) + g_1(T_U A_X V, Y) \\ &= g_1(T_U A_X V, Y), \end{aligned} \tag{3.15}$$

$$\begin{aligned} g_1(A_{\nabla_U X}(V), Y) &= g_1(A_{h\nabla_U X} V, Y) + g_1(A_{T_U X} V, Y) \\ &= 0, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} g_1(A_X(\nabla_U V), Y) &= g_1(A_X T_U V, Y) + g_1(A_X v\nabla_U V, Y) \\ &= g_1(A_X T_U V, Y). \end{aligned} \tag{3.17}$$

Then, from (3.15), (3.16) and (3.17) we have

$$g_1((\nabla_U A)_X V, Y) = g_1(T_U A_X V, Y) - g_1(A_X T_U V, Y). \qquad \square$$

4. Curvature relations for lightlike submersions

For an r–lightlike submersion $f : (M_1, g_1) \to (M_2, g_2)$, since the fibers are submanifolds of M_1, we can derive equations analogous to the Gauss and Codazzi equations of a lightlike immersion. First note that geometrical features of the fibers will be distinguished by a caret $\hat{}$. For example, we write $\hat{\nabla}_V W = \nu \nabla_V W$ for the covariant derivative.

4.1. Theorem. *Let M_1 be a semi-Riemannian manifold and M_2 a Reinhart lightlike manifold. Suppose that $f : (M_1, g_1) \to (M_2, g_2)$ is an r–lightlike submersion or an isotropic submersion. Then we have:*

$$g_1(R(U,V)W, X) = g_1(\hat{R}(U,V)W, X) + g_1(T_U T_V W, X) - g_1(T_V T_U W, X),$$
$$g_1(R(U,V)W, F) = g_1((\nabla_U T)_V W, F) - g_1((\nabla_V T)_U W, F),$$

for any $X \in \Gamma(\mathrm{ltr}(\mathrm{Ker}\, f_*))$ and $U, V, W, F \in \Gamma(\Delta)$, where ∇, R and $\hat{R}$ are the Levi-Civita connection on M_1, the Riemannian curvature tensor field of M_1, and the Riemannian curvature tensor field of the fibers, respectively.

Proof. From Lemma 3.9 we obtain

$$\begin{aligned}\nabla_U \nabla_V W &= \nabla_U T_V W + \nabla_U \hat{\nabla}_V W \\ &= h\nabla_U T_V W + T_U T_V W + \hat{\nabla}_U \hat{\nabla}_V W + T_U \hat{\nabla}_V W,\\ \nabla_V \nabla_U W &= h\nabla_V T_U W + T_V T_U W + \hat{\nabla}_V \hat{\nabla}_U W + T_V \hat{\nabla}_U W,\end{aligned}$$

and

$$\begin{aligned}\nabla_{[U,V]} W &= T_{[U,V]} W + \hat{\nabla}_{[U,V]} W \\ &= h\nabla_{[U,V]} W + \hat{\nabla}_{[U,V]} W.\end{aligned}$$

Therefore we have

$$\begin{aligned}R(U,V)W = \hat{R}(U,V)W &+ h\nabla_U T_V W + T_U T_V W + T_U \hat{\nabla}_V W \\ &- h\nabla_V T_U W - T_V T_U W - T_V \hat{\nabla}_U W - h\nabla_{[U,V]} W.\end{aligned}$$

Taking the inner product of both sides of the above equation with X gives us the first equation. Taking the inner product with F, we obtain

$$\begin{aligned}g_1(R(U,V)W, F) = g_1(h\nabla_U T_V W, F) &+ g_1(T_U \nu\nabla_V W, F) - g_1(\nabla_V T_U W, F) \\ &- g_1(T_V \nabla_U W, F) - g_1(h\nabla_{[U,V]} W, F).\end{aligned} \tag{4.1}$$

On the other hand, by direct computations, we have

$$\begin{aligned}g_1(h\nabla_{[U,V]} W, F) =& g_1(h\nabla_{\nabla_U V} W, F) - g_1(h\nabla_{\nabla_V U} W, F) \\ =& g_1(T_{\nabla_U V} W, F) - g_1(T_{\nabla_V U} W, F).\end{aligned} \tag{4.2}$$

Thus, from (4.1) and (4.2) we have

$$\begin{aligned}g_1(R(U,V)W, F) =& g_1((\nabla_U T_V W - T_{\nabla_U V} W - T_V \nabla_U W), F) \\ &- [g_1((\nabla_V T_U W - T_{\nabla_V U} W - T_U \nabla_V W), F)].\end{aligned}$$

Hence,

$$g_1(R(U,V)W, F) = g_1((\nabla_U T)_V W, F) - g_1((\nabla_V T)_U W, F),$$

which is the second equation. $\square$

We recall that the null sectional curvature [1] of M at $p \in M$ with respect to U_p is defined by

$$K_M(U_p, X_p) = \frac{g(R(X_p, U_p)U_p, X_p)}{g(X_p, X_p)}, \tag{4.3}$$

where X_p is a non-null vector and U_p is a null vector in $T_p(M)$.

We denote the horizontal lift of the curvature tensor R^{M_2} of M_2 by R^*, that is, if X_1, X_2, X_3 and X_4 are basic vector fields of M_1, we write

$$g_1(R^*(X_1, X_2)X_3, X_4) = g_2(R^{M_2}(\tilde{X}_1, \tilde{X}_2)\tilde{X}_3, \tilde{X}_4).$$

Also, if X_i and X_j are basic vector fields, we will denote the horizontal lift of $\nabla^{M_2}_{\tilde{X}_i} \tilde{X}_j$ by $\nabla^*_{X_i} X_j$.

4.2. Theorem. *Let M_1 be a semi-Riemannian manifold and M_2 a Reinhart lightlike manifold. Suppose that $f : (M_1, g_1) \to (M_2, g_2)$ is an r–lightlike submersion or an isotropic submersion. Then we have:*

$$K_{M_2}(\tilde{Z}, \tilde{U}) = K_{M_1}(Z, U) - g_1(A_Z A_U U, Z) + g_1(A_U A_Z U, Z) - g_1(U, T_{\nu[Z,U]} Z),$$

where K_{M_2} is the null sectional curvature of M_2, K_{M_1} is the null sectional curvature of M_1, $Z \in \Gamma(S(\mathrm{Ker}\, f_*)^{\perp})$ and $U \in \Gamma(\mathrm{ltr}(\mathrm{Ker}\, f_*))$.

Proof. For $Z \in \Gamma(S(\mathrm{Ker}\, f_*)^{\perp})$ and $U \in \Gamma(\mathrm{ltr}(\mathrm{Ker}\, f_*))$, from (3.5), we can write

$$\nabla_Z U = h\nabla_Z U + \nu\nabla_Z U = h\nabla_Z U + A_Z U.$$

Since M_2 is a Reinhart lightlike manifold, from Theorem 3.8 we have that $h\nabla_Z U$ is the basic vector field corresponding to $\nabla^{M_2}_{\tilde{Z}} \tilde{U}$, where ∇^{M_2} is the Levi-civita connection on M_2, Z and U are the horizontal lifts of $\tilde{Z}$ and $\tilde{U}$. Then, we write the basic vector field $h\nabla_Z U$ as $\nabla^*_Z U$. Thus we have

$$\nabla_Z U = \nabla^*_Z U + A_Z U.$$

Then, by direct computations, using (3.5), we get

$$\nabla_Z \nabla_U U = \nabla^*_Z \nabla^*_U U + A_Z \nabla^*_U U + A_Z A_U U + \nu\nabla_Z A_U U.$$

Since A reverses the horizontal and vertical subspaces, we obtain

$$g_1(\nabla_Z \nabla_U U, Z) = g_1(\nabla^*_Z \nabla^*_U U, Z) + g_1(A_Z A_U U, Z). \tag{4.4}$$

In a similar way, we get

$$g_1(\nabla_U \nabla_Z U, Z) = g_1(\nabla^*_U \nabla^*_Z U, Z) + g_1(A_U A_Z U, Z). \tag{4.5}$$

On the other hand, by direct computations, we have

$$\begin{aligned} \nabla_{[Z,U]} U &= \nabla_{h[Z,U]} U + \nabla_{\nu[Z,U]} U \\ &= h\nabla_{h[Z,U]} U + \nu\nabla_{h[Z,U]} U + h\nabla_{\nu[Z,U]} U + \nu\nabla_{\nu[Z,U]} U. \end{aligned}$$

Hence, we get

$$\begin{aligned} g_1(\nabla_{[Z,U]} U, Z) &= g_1(h\nabla_{h[Z,U]} U, Z) + g_1(h\nabla_{\nu[Z,U]} U, Z) \\ &= g_1(\nabla^*_{h[Z,U]} U, Z) + g_1(\nabla_{\nu[Z,U]} U, Z). \end{aligned}$$

Then, since ∇ is a metric connection and U and Z are orthogonal, we arrive at

$$\begin{aligned} g_1(\nabla_{[Z,U]} U, Z) &= g_1(\nabla^*_{h[Z,U]} U, Z) - g_1(U, \nabla_{\nu[Z,U]} Z) \\ &= g_1(\nabla^*_{h[Z,U]} U, Z) - g_1(U, \nu\nabla_{\nu[Z,U]} Z). \end{aligned}$$

Thus, using (3.1) we obtain

$$g_1(\nabla_{[Z,U]} U, Z) = g_1(\nabla^*_{h[Z,U]} U, Z) - g_1(U, T_{\nu[Z,U]} Z). \tag{4.6}$$

Then, from (4.4), (4.5) and (4.6) we have

$$g_1(R(Z,U)U,Z) = g_1(R^*(Z,U)U,Z) + g_1(A_Z A_U U,Z) - g_1(A_U A_Z U,Z) + g_1(U, T_{\nu[Z,U]}Z).$$

Hence,

$$g_1(R(Z,U)U,Z) = g_2(R^{M_2}(\tilde{Z},\tilde{U})\tilde{U},\tilde{Z}) \circ f + g_1(A_Z A_U U,Z) - g_1(A_U A_Z U,Z) + g_1(U, T_{\nu[Z,U]}Z).$$

Thus, the proof is complete. □

References

[1] Beem, J. K., Ehrlich, P. E. and Easley, K. L. *Global Lorentzian geometry* (Pure and Applied Mathematics **202**, Marcel Dekker, Inc., New York, 1996).

[2] Duggal, K. L. and Bejancu, A. *Lightlike Submanifolds of Semi - Riemannian Manifolds and Applications* (Kluwer Acad. Publishers, Dordrecht, 1996).

[3] Gray, A. *Pseudo-Riemannian almost product manifolds and submersions*, J. Math. Mech. **16**, 715–737, 1967.

[4] Falcitelli, M., Ianus, S. and Pastore, A. M. *Riemannian Submersions and Related Topics* (World Scientific, 2004).

[5] Kupeli, D. N. *Singular Semi-Riemannian Geometry* (Kluwer Dortrecht, 1996).

[6] Nomizu, K. *Fundamentals of Linear Algebra* (McGraw Hill, New York, 1966).

[7] O'Neill, B. *The fundamental equations of a submersion*, Michigan Math. J. **13**, 459û-469, 1966.

[8] O'Neill, B. *Semi-Riemannian Geometry with Application to Relativity* (Academic Press, New York, 1983).

[9] Şahin, B. *On a submersion between Reinhart lightlike manifolds and semi-Riemannian manifolds*, Mediterranean J. Math. **5**, 273–284, 2008.