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Araştırma Makalesi / Research Article

Matrix Method Development for Structural Analysis of Euler Bernoulli Beams with Finite Difference Method

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Finite difference Method, Matrix Methods, Structural Analysis, Euler Bernoulli Beams, Matlab

Abstract

The behaviors of structural systems are generally described with ordinary or partial differential equations. Finite Difference Method (FDM) mainly replaces the derivatives in the differential equations by finite difference approximations. It can be said that finite difference formulation offers a more direct approach to the numerical solution of partial differential equations. In this study, matrix approach is proposed for structural analysis with FDM. The system analysis procedure including stiffness matrix development, applying boundary and loading conditions on a structural element is proposed. The interacting points group is determined depending on the differential equations of the structural element and system rigidity matrix is generated by using this dynamic points group. The proposed algorithms are developed for Euler Bernoulli beams in this study because of its simplicity and may be enhanced for any other structural system in future studies by using same steps.

Euler Bernoulli Kirişlerinin Sonlu Farklar Yöntemi ile Yapısal Analizi için Matris Yöntemi Geliştirilmesi

Anahtar kelimeler
Sonlu Farklar Yöntemi; Matris Yöntemler, Yapısal Analiz, Euler Bernoulli Kirişleri, Matlab.

Özet

Yapı sistemlerinin davranışı genellikle adi ya da kısmi diferansiyel denklemler ile tarif edilmektedir. Sonlu Farklar Yöntemi (SFY), diferansiyel denklemlerde yer alan türev ifadelerinin sonlu farklar yaklaşımları ile değiştirilmesi esasına dayanır. Sonlu fark formülasyonlarının sayısal çözümlere veya adi diferansiyel denklemlere göre daha doğrudan bir yaklaşım sunduğu söylenebilir. Bu çalışmada, yapıların SFY ile analizi için bir matris yaklaşımı önerilmektedir. Sistem rijitlik matrisinin geliştirilmesi, sınır koşullarının uygulanması, yapısal eleman üzerine yükleme koşullarını içeren sistem analiz prosedürü önerilmektedir. Yapı elemanın diferansiyel denklemlerine bağlı olarak etkileşimli noktalar grubu tanımlanmıştır ve bu dinamik noktalar grubu kullanılarak sistem rijitlik matrisi üretilmiştir. Bu çalışmada önerilen algoritmalar kolaylığından dolayı Euler Bernoulli kirişleri için geliştirilmiş olup gelecek çalışmalarda aynı adımlar kullanılarak herhangi bir yapısal sistem için geliştirilebilir.

1. Introduction

Finite difference method (FDM) is a technique for the numerical solution of ordinary and partial differential equations. It provides a mathematically simple and easy way to implement computationally method to solve higher order ordinary and partially differential equations. FDM schemes were first used by Euler to find approximate solutions of differential equations. After 1945, systematic research activity of FDM has been strongly developed when high-speed computers began to be available. Today, FDM provides powerful approach to solve ordinary or partial differential equations and is widely used in any field of applied sciences. By using FDM, equations with variable coefficients and even nonlinear problem can be easily solved. There are many studies about FDM applications for different engineering problems in literature (Forsythe and Wasow, 1960; Chapel and Smith, 1968; Chawla and Katti, 1982; Strikwerda 1990; Cocchi and Cappello, 1990; Thomee, 1990;

Cocchi, 2000; Jovanovic and Popovic, 2001; Jovanovic and Vulkov, 2001; Jovanovic et al., 2006; Jones et al. 2009; Liu and Yin, 2014; Kalyani et al., 2014; Moreno-García et al. 2015; D'Amico et al., 2016).

In FDM approach, derivatives in the partial differential equation are approximated by linear combinations of function values at the grid points and are expressed as difference functions. The general difference representations of differential equations are as follows:

$$\left(\frac{\partial^n}{\partial x^n}\right)_{ijk} \cong \left(\frac{\Delta^n}{\Delta x^n}\right)_{ijk}$$

$$\left(\frac{\partial^n}{\partial x^a \partial y^{n-a}}\right)_{ijk} \cong \left(\frac{\Delta^n}{\Delta x^a \Delta y^{n-a}}\right)_{ijk}$$

$$\left(\frac{\partial^n}{\partial y^n}\right)_{ijk} \cong \left(\frac{\Delta^n}{\Delta y^n}\right)_{ijk}$$

$$\left(\frac{\partial^n}{\partial x^a \partial z^{n-a}}\right)_{ijk} \cong \left(\frac{\Delta^n}{\Delta x^a \Delta z^{n-a}}\right)_{ijk}$$

$$\left(\frac{\partial^n}{\partial z^n}\right)_{ijk} \cong \left(\frac{\Delta^n}{\Delta z^n}\right)_{ijk}$$

$$\left(\frac{\partial^n}{\partial y^a \partial z^{n-a}}\right)_{ijk} \cong \left(\frac{\Delta^n}{\Delta y^a \Delta z^{n-a}}\right)_{ijk}$$

The FDM approach can be explained on a function shown in Fig. 1. The function is discretized the length by using five (equi-spaced) points. The first, second, third and fourth order differential equation of this function at i'th point is expressed as discrete difference functions, respectively.

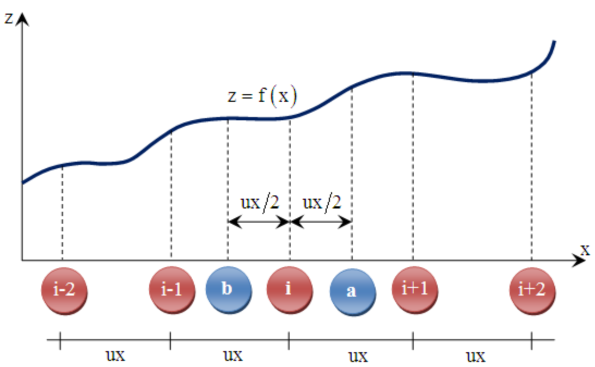

Figure 1. Finite difference representation of a one dimensional function

First-order derivative of i'th point is expressed with difference functions as follows:

$$\left(\frac{\partial z}{\partial x}\right)_i = \left(\frac{\partial f}{\partial x}\right)_i \cong \left(\frac{\Delta f}{\Delta x}\right)_i = \frac{f(i+1)-f(i-1)}{2ux}$$

Second-order derivative of i'th point is expressed with difference functions as follows:

$$\left(\frac{\Delta^2 f}{\Delta x^2}\right)_i = \frac{\Delta}{\Delta x}\left(\frac{\Delta f}{\Delta x}\right)_i = \frac{\left(\frac{\Delta f}{\Delta x}\right)_a - \left(\frac{\Delta f}{\Delta x}\right)_b}{ux}$$

$$\left(\frac{\Delta f}{\Delta x}\right)_a = \frac{f(i+1)-f(i)}{ux}$$

$$\left(\frac{\Delta f}{\Delta x}\right)_b = \frac{f(i)-f(i-1)}{ux}$$

$$\left(\frac{\Delta^2 f}{\Delta x^2}\right)_i = \frac{1}{ux}\left(\frac{f(i+1)-f(i)}{ux} - \frac{f(i)-f(i-1)}{ux}\right)$$
$$= \frac{f(i+1)-2f(i)+f(i-1)}{ux^2}$$

Third-order derivative of i'th point is expressed with difference functions as follows:

$$\left(\frac{\Delta^3 f}{\Delta x^3}\right)_i = \frac{\Delta}{\Delta x}\left(\frac{\Delta^2 f}{\Delta x^2}\right)_i = \frac{\left(\frac{\Delta^2 f}{\Delta x^2}\right)_{i+1} - \left(\frac{\Delta^2 f}{\Delta x^2}\right)_{i-1}}{2ux}$$

$$\left(\frac{\Delta^2 f}{\Delta x^2}\right)_{i+1} = \frac{f(i+2)-2f(i+1)+f(i)}{ux^2}$$

$$\left(\frac{\Delta^2 f}{\Delta x^2}\right)_{i-1} = \frac{f(i)-2f(i-1)+f(i-2)}{ux^2}$$

$$\left(\frac{\Delta^3 f}{\Delta x^3}\right)_i = \frac{1}{2ux}\times$$
$$\left(\frac{f(i+2)-2f(i+1)+f(i)}{ux^2} - \frac{f(i)-2f(i-1)+f(i-2)}{ux^2}\right)$$

$$\left(\frac{\Delta^3 f}{\Delta x^3}\right)_i = \frac{f(i+2)-2f(i+1)+2f(i-1)-f(i-2)}{2ux^3}$$

Fourth-order derivative of i'th point is expressed with difference functions as follows:

$$\left(\frac{\Delta^4 f}{\Delta x^4}\right)_i = \frac{\Delta^2}{\Delta x^2}\left(\frac{\Delta^2 f}{\Delta x^2}\right)_i = \frac{\left(\frac{\Delta^2 f}{\Delta x^2}\right)_{i+1} - 2\left(\frac{\Delta^2 f}{\Delta x^2}\right)_i + \left(\frac{\Delta^2 f}{\Delta x^2}\right)_{i-1}}{ux^2}$$

$$\left(\frac{\Delta^2 f}{\Delta x^2}\right)_{i+1} = \frac{f(i+2)-2f(i+1)+f(i)}{ux^2}$$

$$\left(\frac{\Delta^2 f}{\Delta x^2}\right)_i = \frac{f(i+1)-2f(i)+f(i-1)}{ux^2}$$

$$\left(\frac{\Delta^2 f}{\Delta x^2}\right)_{i-1} = \frac{f(i)-2f(i-1)+f(i-2)}{ux^2}$$

$$\left(\frac{\Delta^4 f}{\Delta x^4}\right)_i = \frac{1}{ux^2} \times \left(\begin{array}{l} \frac{f(i+2)-2f(i+1)+f(i)}{ux^2} \\ -2\frac{f(i+1)-2f(i)+f(i-1)}{ux^2} \\ +\frac{f(i)-2f(i-1)+f(i-2)}{ux^2} \end{array} \right)$$

$$\left(\frac{\Delta^4 f}{\Delta x^4}\right)_i = \frac{f(i+2)-4f(i+1)+6f(i)-4f(i-1)+f(i-2)}{ux^4}$$

2. Euler-Bernoulli Beam Theory

The relationship between the beam's deflection and the applied load is described in The Euler-Bernoulli equation as follows:

$$E.I(x).z^{IV}(x) = -q(x) \tag{1}$$

In this equation, E is the elastic modulus and $I(x)$ describes the moment of inertia of the beam at some position x. The curve $z(x)$ describes the deflection of the beam at some position x. $q(x)$ is a distributed load along the beam. The fourth order derivative of the deflection is obtained from eq. (1) as follows:

$$z^{IV}(x) = -\frac{1}{E}\frac{q(x)}{I(x)} \tag{2}$$

Successive derivatives of $z(x)$ have the following meanings:

$z(x)$ is the deflection.

$z'(x)$ is the slope of the beam.

$EIz''(x)$ is the bending moment in the beam.

$EIz'''(x)$ is the shear force in the beam.

3. Interacting Points of Euler Bernoulli Beams

The fourth order derivative of the deflection is presented in eq. (2). This equation is expressed with FDM as follows:

$$z^{IV}(x) = \frac{\partial^4 z}{\partial x^4} = \frac{\Delta^4 z}{\Delta x^4}$$

For i'th point, this difference equation may be expressed with discrete points as follows:

$$\left(\frac{\Delta z^4}{\Delta x^4}\right)_i = \frac{z_{i+2}-4z_{i+1}+6z_i-4z_{i-1}+z_{i-2}}{ux^4}$$

The equation may be expressed as follows:

$$\left(\frac{\Delta z^4}{\Delta x^4}\right)_i = z_{i-2}\left(\frac{1}{ux^4}\right) + z_{i-1}\left(-\frac{4}{ux^4}\right) + z_i\left(\frac{6}{ux^4}\right) + z_{i+1}\left(-\frac{4}{ux^4}\right) + z_{i+2}\left(\frac{1}{ux^4}\right)$$

If following statements are substituted in this equation,

$$k_1 = \left(\frac{6}{ux^4}\right) \quad k_2 = \left(-\frac{4}{ux^4}\right) \quad k_3 = \left(\frac{1}{ux^4}\right)$$

The fourth derivative of displacement function is obtained with interacting points as follows:

$$z^{IV}(i) = \left(\frac{\Delta^4 z}{\Delta x^4}\right)_i = \left(\begin{array}{l} k_3 z(i-2) + k_2 z(i-1) \\ +k_1 z(i) + k_2 z(i+1) \\ +k_3 z(i+2) \end{array} \right) = -\frac{1}{E}\frac{q(i)}{I(i)} \tag{3}$$

As a result, the elastic function of Euler Bernoulli beam is expressed with interacting points. The interacting points group of i'th joint on the Euler Beam is shown in Fig. 2. The coefficients of each interacting point are also shown in this figure. These coefficients are used in rigidity matrix calculations.

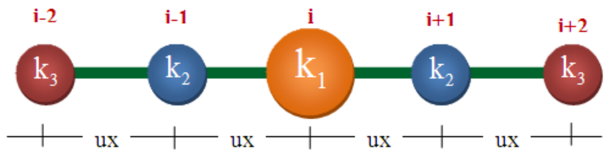

Figure 2. Interacting points group and rigidity coefficients for Euler Bernoulli Beams

3.1. Determining System Rigidity Matrix

If the beam is divided into n joints with equal distance from each other, eq. (3) is written for each of these joints, respectively. Then, the equation system is written in matrix format and the system matrices are obtained. In Fig. 3, the interacting points group which is taken into account for each point of the beam is presented. These interacting points group is acted along the beam and system equations are easily calculated.

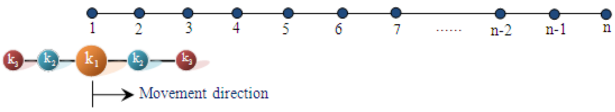

Figure 3. Euler beam divided into n equi-spaced joints and interacting points group movement

The equations which obtained from movement of interacting points group on all joints are as follows:

The displacement equation when the center of the interacting points group is on 1'th joint:

$$k_1 z(1)+k_2 z(2)+k_3 z(3)=-\frac{1}{E}\frac{q(1)}{I(1)}$$

The displacement equation when the center of the interacting points group is on 2'th joint:

$$k_2 z(1)+k_1 z(2)+k_2 z(3)+k_3 z(4)=-\frac{1}{E}\frac{q(2)}{I(2)}$$

The displacement equation when the center of the interacting points group is on 3'th joint:

$$k_3 z(1)+k_2 z(2)+k_1 z(3)+k_2 z(4)+k_3 z(5)=-\frac{1}{E}\frac{q(3)}{I(3)}$$

The displacement equation when the center of the interacting points group is on 4'th joint:

$$k_3 z(2)+k_2 z(3)+k_1 z(4)+k_2 z(5)+k_3 z(6)=-\frac{1}{E}\frac{q(4)}{I(4)}$$

The displacement equation when the center of the interacting points group is on 5'th joint:

$$k_3 z(3)+k_2 z(4)+k_1 z(5)+k_2 z(6)+k_3 z(7)=-\frac{1}{E}\frac{q(5)}{I(5)}$$

$$\vdots$$

The displacement equation when the center of the interacting points group is on n-2'th joint:

$$\begin{pmatrix} k_3 z(n-4)+k_2 z(n-3)+k_1 z(n-2) \\ +k_2 z(n-1)+k_3 z(n) \end{pmatrix}=-\frac{1}{E}\frac{q(n-2)}{I(n-2)}$$

The displacement equation when the center of the interacting points group is on n-1'th joint:

$$\begin{pmatrix} k_3 z(n-3)+k_2 z(n-2) \\ +k_1 z(n-1)+k_2 z(n) \end{pmatrix}=-\frac{1}{E}\frac{q(n-1)}{I(n-1)}$$

The displacement equation when the center of the interacting points group is on n'th joint:

$$k_3 z(n-2)+k_2 z(n-1)+k_1 z(n)=-\frac{1}{E}\frac{q(n)}{I(n)}$$

These equations may be written in matrix form as follows:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ \vdots \\ \vdots \\ \vdots \\ n-2 \\ n-1 \\ n \end{array}
\begin{array}{c}
\begin{array}{ccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots & \ldots & \ldots & n-2 & n-1 & n \end{array} \\
\begin{bmatrix}
k_1 & k_2 & k_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_2 & k_1 & k_2 & k_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_3 & k_2 & k_1 & k_2 & k_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & k_3 & k_2 & k_1 & k_2 & k_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & k_3 & k_2 & k_1 & k_2 & k_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & k_3 & k_2 & k_1 & k_2 & k_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & k_3 & k_2 & k_1 & k_2 & k_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & k_3 & k_2 & k_1 & k_2 & k_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & k_3 & k_2 & k_1 & k_2 & k_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & k_3 & k_2 & k_1 & k_2 & k_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_3 & k_2 & k_1 & k_2 & k_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_3 & k_2 & k_1 & k_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_3 & k_2 & k_1
\end{bmatrix}
\end{array}
\begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ \vdots \\ \vdots \\ \vdots \\ z_{n-2} \\ z_{n-1} \\ z_n \end{Bmatrix}
=-\frac{1}{E}
\begin{Bmatrix} q_1/I_1 \\ q_2/I_2 \\ q_3/I_3 \\ q_4/I_4 \\ q_5/I_5 \\ q_6/I_6 \\ q_7/I_7 \\ \vdots \\ \vdots \\ \vdots \\ q_{n-2}/I_{n-2} \\ q_{n-1}/I_{n-1} \\ q_n/I_n \end{Bmatrix}$$

According to this matrix expression, the general system equations may be written as follows:

$$[K]\{z\}=\{f\}$$

Where:

$[K]$	: System rigidity matrix
k_1, k_2, k_3	: Rigidity coefficients
$\{z\}$	: System displacement vector
$\{f\}$	: Load vector

The flowchart developed for automatically generating system rigidity matrix is shown in Fig. 4. The system rigidity matrix function is produced in MATLAB (2009) depending on this flowchart. The function name is "RigidityMatrix.m". The length of the beam and scanning length are input values of this function. The output data is system rigidity matrix. In main program, this function is used by typing "[k]= RigidityMatrix(lx, ux)". The MATLAB code of this function is presented in Appendix I. As it can be seen, the system equation of this approach is same with that of Finite Element Method (FEM) (Zienkiewicz 1971).

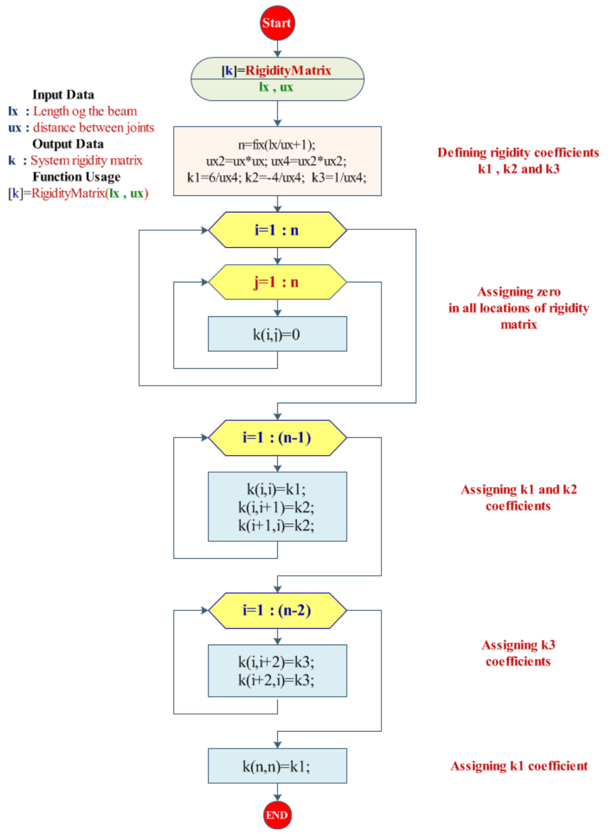

Figure 4. Flowchart developed for constructing system rigidity matrix of Euler-Bernoulli Beam with Interacting Points

While system equations are produced, the interacting points group is carried along the beam as shown in Fig. 5. However, when the point group operator is on 1., 2., (n-1). and n. joints, some points are outside the beam. Hence, the system equations seem missing. To complete equation system, some virtual joints are defined and the rigidity coefficients of these joints are determined according to the boundary conditions.

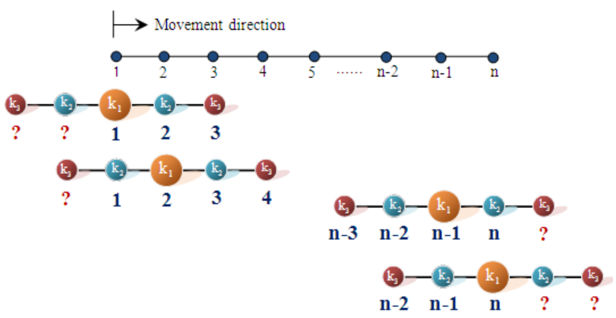

Figure 5. Some parts of interacting points group which remain outside of the beam

3.2. Boundary Conditions

Boundary conditions are determined according to supports and freedom situations. Main boundary conditions for Euler Bernoulli beam are shown in Fig. 6. In first model, a completely fixed end is given. In this boundary condition, both deflection and slope are zero. In second model, a general simple support is given. In this boundary condition, deflection and bending moment are zero. In third model, completely free end is presented. At the free end, both shear force and bending moment are zero.

The behaviors of virtual joints are determined considering these boundary conditions. Then, the system rigidity matrix is updated depending on the boundary conditions.

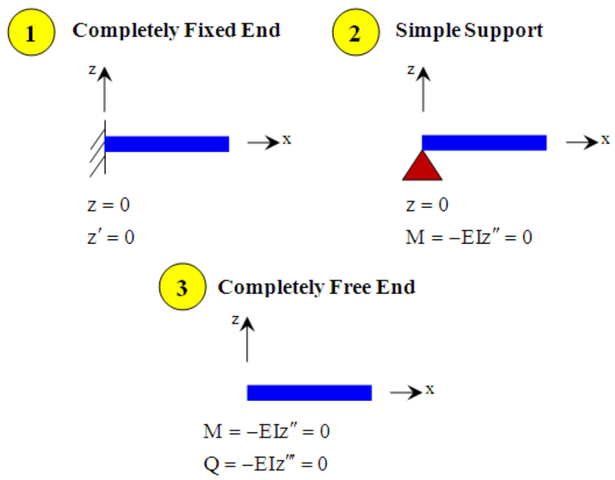

Figure 6. Main boundary conditions for Euler Bernoulli Beams

3.2.1. System Behavior at Completely Fixed End

3.2.1.1. Completely Fixed Support at Left End

Two virtual points are added on the left part of the system to consider boundary conditions as shown in Fig. 7. These virtual points are named as “a” and “b”.

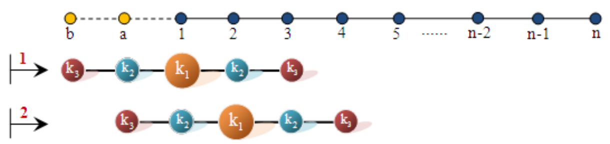

Figure 7. Virtual joints added next to the left end of the beam to consider boundary conditions

When, interacting points group operator is on first and second joints, the deflection values of “a” and “b” joints are needed. These values are calculated depending on boundary conditions. At completely fixed end, the deflection and slope which is the first order derivative of the deflection are both zero. By using this boundary condition, the deflection values of the virtual joints are determined from the following equations:

$$\left(\frac{\partial z}{\partial x}\right)_1 = \left(\frac{\Delta z}{\Delta x}\right)_1 = \frac{z(2)-z(a)}{2ux} = 0 \;\rightarrow\; z(a) = z(2)$$

$$\left(\frac{\partial z}{\partial x}\right)_1 = \left(\frac{\Delta z}{\Delta x}\right)_1 = \frac{z(3)-z(b)}{4ux} = 0 \;\rightarrow\; z(b) = z(3)$$

The contribution of the virtual joints “a” and “b” to the rigidity matrix is determined by considering the interacting point group operator shown in Fig. 7. The equations which obtained by placing the interacting points group on first and second joints are determined again as follows:

The displacement equation when the center of the interacting points group is on 1’th joint:

$$k_3 z(b) + k_2 z(a) + k_1 z(1) + k_2 z(2) + k_3 z(3) = -\frac{1}{E}\frac{q(1)}{I(1)}$$

The displacement equation when the center of the interacting points group is on 2’th joint:

$$k_3 z(a) + k_2 z(1) + k_1 z(2) + k_2 z(3) + k_3 z(4) = -\frac{1}{E}\frac{q(2)}{I(2)}$$

If $z(a) = z(2)$ and $z(b) = z(3)$ is substituted in equations above, following equations are obtained:

$$k_3 z(3) + k_2 z(2) + k_1 z(1) + k_2 z(2) + k_3 z(3) = -\frac{1}{E}\frac{q(1)}{I(1)} \qquad (4)$$

$$k_3 z(2) + k_2 z(1) + k_1 z(2) + k_2 z(3) + k_3 z(4) = -\frac{1}{E}\frac{q(2)}{I(2)} \qquad (5)$$

As it can be seen, new rigidity coefficients are added into system equations and these coefficients should be added to the rigidity matrix. Eq. (4) is the first joint equation; therefore, the coefficients of this eq. are added to the first row of the rigidity matrix. k_2 coefficient is added to the second column of the first row (it means it is multiplied with $z(2)$) and k_3 coefficient is added to the third column of the first row (it means it is multiplied with $z(3)$). Eq. (5) is the second joint equation; therefore, the coefficients of this eq. are added to the second row of the rigidity matrix. k_3 coefficient is added to the second column of the second row (it means it is multiplied with $z(2)$). As a result of these operations, system rigidity matrix is updated as follows:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \\ \vdots \end{array}
\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & \cdots & \cdots \end{array} \\
\begin{bmatrix}
k_1 & k_2+\boxed{k_2} & k_3+\boxed{k_3} & 0 & 0 & \ddots & \ddots \\
k_2 & k_1+\boxed{k_3} & k_2 & k_3 & 0 & \ddots & \ddots \\
k_3 & k_2 & k_1 & k_2 & k_3 & \ddots & \ddots \\
0 & k_3 & k_2 & k_1 & k_2 & \ddots & \ddots \\
0 & 0 & k_3 & k_2 & k_1 & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}_{n\times n} \end{array}
\times \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ \vdots \\ \vdots \end{Bmatrix} = -\frac{1}{E} \begin{Bmatrix} q_1/I_1 \\ q_2/I_2 \\ q_3/I_3 \\ q_4/I_4 \\ q_5/I_5 \\ \vdots \\ \vdots \end{Bmatrix}_{n\times 1}$$

3.2.1.2. Completely Fixed Support at Right End

After investigating left end of the beam, the same procedure is applied to the right end. Two virtual points are added next to the right end of the system to consider boundary conditions as shown in Fig. 8. These virtual points are named as “a'” and “b'”.

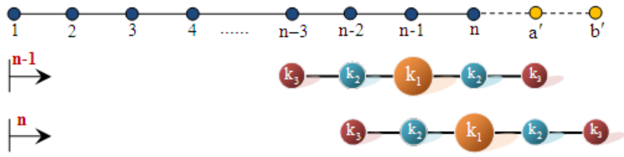

Figure 8. Virtual points added next to the right end of the beam to consider boundary conditions

When, interacting points group operator is on (n-1)’th and n’th joints, the deflection values of “a'” and “b'” joints are needed. These values are calculated depending on boundary conditions. At completely fixed end, the deflection and slope which is the first order derivative of the deflection are both zero. By using these boundary conditions, the deflection values of the virtual joints are determined from the following equations:

$$\left(\frac{\partial z}{\partial x}\right)_n=\left(\frac{\Delta z}{\Delta x}\right)_n=\frac{z(a')-z(n-1)}{2ux}=0 \;\rightarrow\; z(a')=z(n-1)$$

$$\left(\frac{\partial z}{\partial x}\right)_n=\left(\frac{\Delta z}{\Delta x}\right)_n=\frac{z(b')-z(n-2)}{4ux}=0 \;\rightarrow\; z(b')=z(n-2)$$

The contribution of the virtual joints "a'" and "b'" to the rigidity matrix is determined by considering the interacting point group operator shown in Fig. 8. The equations which obtained by placing the interacting points group on (n-1)'th and n'th joints are determined again as follows:

The displacement equation when the center of the interacting points group is on (n-1)'th joint:

$$k_3 z(n-3)+k_2 z(n-2)+k_1 z(n-1)+k_2 z(n)+k_3 z(a')=-\frac{1}{E}\frac{q(n-1)}{I(n-1)}$$

The displacement equation when the center of the interacting points group is on n'th joint:

$$k_3 z(n-2)+k_2 z(n-1)+k_1 z(n)+k_2 z(a')+k_3 z(b')=-\frac{1}{E}\frac{q(n)}{I(n)}$$

If $z(a')=z(n-1)$ and $z(b')=z(n-2)$ is substituted in equations above, following equations are obtained:

$$k_3 z(n-3)+k_2 z(n-2)+k_1 z(n-1)+k_2 z(n)+k_3 z(n-1)=-\frac{1}{E}\frac{q(n-1)}{I(n-1)} \tag{6}$$

$$k_3 z(n-2)+k_2 z(n-1)+k_1 z(n)+k_2 z(n-1)+k_3 z(n-2)=-\frac{1}{E}\frac{q(n)}{I(n)} \tag{7}$$

As it can be seen, new rigidity coefficients are added into system equations and these coefficients should be added to the rigidity matrix. Eq. (6) is the (n-1)'th joint equation; therefore, the coefficients of this eq. are added to the (n-1)'th row of the rigidity matrix. k_3 coefficient is added to the (n-1)'th column of the (n-1)'th row (it means it is multiplied with $z(n-1)$). Eq. (7) is the n'th joint equation; therefore, the coefficients of this eq. are added to the n'th row of the rigidity matrix. k_2 coefficient is added to the (n-1)'th column of the n'th row (it means it is multiplied with $z(n-1)$) and k_3 coefficient is added to the (n-2)'th column of the n'th row (it means it is multiplied with $z(n-2)$). As a result of these operations, system rigidity matrix is updated as follows:

$$\begin{array}{cc} & \begin{array}{ccccccc} \cdots & \cdots & n-4 & n-3 & n-2 & n-1 & n \end{array} \\ \begin{array}{c} \vdots \\ \vdots \\ n-4 \\ n-3 \\ n-2 \\ n-1 \\ n \end{array} & \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & k_1 & k_2 & k_3 & 0 & 0 \\ \ddots & \ddots & k_2 & k_1 & k_2 & k_3 & 0 \\ \ddots & \ddots & k_3 & k_2 & k_1 & k_2 & k_3 \\ \ddots & \ddots & 0 & k_3 & k_2 & k_1+\boxed{k_3} & k_2 \\ \ddots & \ddots & 0 & 0 & k_3+\boxed{k_3} & k_2+\boxed{k_2} & k_1 \end{bmatrix}_{n\times n} \end{array} \times \begin{Bmatrix} \vdots \\ \vdots \\ z_{n-4} \\ z_{n-3} \\ z_{n-2} \\ z_{n-1} \\ z_n \end{Bmatrix}_{n\times 1} = -\frac{1}{E}\begin{Bmatrix} \vdots \\ \vdots \\ q_{n-4}/I_{n-4} \\ q_{n-3}/I_{n-3} \\ q_{n-2}/I_{n-2} \\ q_{n-1}/I_{n-1} \\ q_n/I_n \end{Bmatrix}_{n\times 1}$$

3.2.2. System Behaviour at Simple Supported End

3.2.2.1. Simple Support at Left End

At simple supported end, the deflection and bending moment are both zero. By using these boundary conditions, the deflection values of virtual joints (a and b) shown in Fig. 7 are determined from the following equations:

$$\left(\frac{\partial^2 z}{\partial x^2}\right)_1=\left(\frac{\Delta^2 z}{\Delta x^2}\right)_1=\frac{z(2)-2z(1)+z(a)}{ux^2}=0 \;\rightarrow\; z(a)=2z(1)-z(2)$$

$$\left(\frac{\partial^2 z}{\partial x^2}\right)_1=\left(\frac{\Delta^2 z}{\Delta x^2}\right)_1=\frac{z(3)-2z(1)+z(b)}{4ux^2}=0 \;\rightarrow\; z(b)=2z(1)-z(3)$$

The displacement at the first joint is equal to zero because of the simple support. If $z(1)=0$

statement is substituted into the equations above, the displacement values of “a” and “b” virtual joints are obtained as follows:

$$z(a) = 2z(1) - z(2) = -z(2)$$

$$z(b) = 2z(1) - z(3) = -z(3)$$

The contribution of the virtual joints “a” and “b” to the rigidity matrix is determined by considering the interacting point group operator shown in Fig. 7. The equations which obtained by placing the interacting points group on first and second points are determined again as follows:

The displacement equation when the center of the interacting points group is on 1’th joint:

$$k_3 z(b) + k_2 z(a) + k_1 z(1) + k_2 z(2) + k_3 z(3) = -\frac{1}{E}\frac{q(1)}{I(1)}$$

The displacement equation when the center of the interacting points group is on 2’th joint:

$$k_3 z(a) + k_2 z(1) + k_1 z(2) + k_2 z(3) + k_3 z(4) = -\frac{1}{E}\frac{q(2)}{I(2)}$$

If $z(a) = -z(2)$ and $z(b) = -z(3)$ is substituted in equations above, following equations are obtained:

$$-k_3 z(3) - k_2 z(2) + k_1 z(1) + k_2 z(2) + k_3 z(3) = -\frac{1}{E}\frac{q(1)}{I(1)} \quad (8)$$

$$-k_3 z(2) + k_2 z(1) + k_1 z(2) + k_2 z(3) + k_3 z(4) = -\frac{1}{E}\frac{q(2)}{I(2)} \quad (9)$$

As it can be seen, new rigidity coefficients are added into system equations and these coefficients should be added to the rigidity matrix. Eq. (8) is the first joint equation; therefore, the coefficients of this eq. are added to the first row of the rigidity matrix. $-k_2$ coefficient is added to the second column of the first row (it means it is multiplied with $z(2)$) and $-k_3$ coefficient is added to the third column of the first row (it means it is multiplied with $z(3)$). Eq. (9) is the second joint equation; therefore, the coefficients of this eq. are added to the second row of the rigidity matrix. $-k_3$ coefficient is added to the second column of the second row (it means it is multiplied with $z(2)$). As a result of these operations, system rigidity matrix is updated as follows:

$$\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & \cdots & \cdots \\ \hline 1 & & & & & & & \\ 2 & & & & & & & \\ 3 & & & & & & & \\ 4 & & & & & & & \\ 5 & & & & & & & \\ \vdots & & & & & & & \\ \vdots & & & & & & & \end{array}
\begin{bmatrix} k_1 & k_2-\boxed{k_2} & k_3-\boxed{k_3} & 0 & 0 & \ddots & \ddots \\ k_2 & k_1-\boxed{k_3} & k_2 & k_3 & 0 & \ddots & \ddots \\ k_3 & k_2 & k_1 & k_2 & k_3 & \ddots & \ddots \\ 0 & k_3 & k_2 & k_1 & k_2 & \ddots & \ddots \\ 0 & 0 & k_3 & k_2 & k_1 & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}_{n\times n} \times \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ \vdots \\ \vdots \end{Bmatrix}_{n\times 1} = -\frac{1}{E}\begin{Bmatrix} q_1/I_1 \\ q_2/I_2 \\ q_3/I_3 \\ q_4/I_4 \\ q_5/I_5 \\ \vdots \\ \vdots \end{Bmatrix}_{n\times 1}$$

3.2.2.2. Simple Support at Right End

The deflection values of virtual joints (a' and b') shown in Fig. 8 are determined from the following equations considering the fact that the deflection and bending moment are both zero at simple supported end:

$$\left(\frac{\partial^2 z}{\partial x^2}\right)_n = \left(\frac{\Delta^2 z}{\Delta x^2}\right)_n = \frac{z(a') - 2z(n) + z(n-1)}{ux^2} = 0$$

$$\rightarrow z(a') = 2z(n) - z(n-1) = -z(n-1)$$

$$\left(\frac{\partial^2 z}{\partial x^2}\right)_n = \left(\frac{\Delta^2 z}{\Delta x^2}\right)_n = \frac{z(b') - 2z(n) + z(n-2)}{4ux^2} = 0 \rightarrow z(b') = -z(n-2)$$

The contribution of the virtual joints “a'” and “b'” to the rigidity matrix is determined by considering the interacting point group operator shown in Fig. 8. The equations which obtained by placing the interacting points group on (n-1)’th and n’th points are determined again as follows:

The displacement equation when the center of the interacting points group is on (n-1)’th joint:

$$k_3z(n-3)+k_2z(n-2)+k_1z(n-1)+k_2z(n)+k_3z(a')=-\frac{1}{E}\frac{q(n-1)}{I(n-1)}$$

The displacement equation when the center of the interacting points group is on n'th joint:

$$k_3z(n-2)+k_2z(n-1)+k_1z(n)+k_2z(a')+k_3z(b')=-\frac{1}{E}\frac{q(n)}{I(n)}$$

If $z(a')=-z(n-1)$ and $z(b')=-z(n-2)$ is substituted in equations above, following equations are obtained:

$$k_3z(n-3)+k_2z(n-2)+k_1z(n-1)+k_2z(n)$$
$$-k_3z(n-1)=-\frac{1}{E}\frac{q(n-1)}{I(n-1)} \qquad (10)$$

$$k_3z(n-2)+k_2z(n-1)+k_1z(n)$$
$$-k_2z(n-1)-k_3z(n-2)=-\frac{1}{E}\frac{q(n)}{I(n)} \qquad (11)$$

As it can be seen, new rigidity coefficients are added into system equations and these coefficients should be added to the rigidity matrix. Eq. (10) is the (n-1)'th joint equation; therefore, the coefficients of this eq. are added to the (n-1)'th row of the rigidity matrix. $-k_3$ coefficient is added to the (n-1)'th column of the (n-1)'th row (it means it is multiplied with $z(n-1)$). Eq. (11) is the n'th joint equation; therefore, the coefficients of this eq. are added to the n'th row of the rigidity matrix. $-k_2$ coefficient is added to the (n-1)'th column of the n'th row (it means it is multiplied with $z(n-1)$) and $-k_3$ coefficient is added to the (n-2)'th column of the n'th row (it means it is multiplied with $z(n-2)$). As a result of these operations, system rigidity matrix is updated as follows:

$$\begin{array}{cc} & \begin{array}{ccccccc} \cdots & \cdots & n-4 & n-3 & n-2 & n-1 & n \end{array} \\ \begin{array}{c} \vdots \\ \vdots \\ n-4 \\ n-3 \\ n-2 \\ n-1 \\ n \end{array} & \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & k_1 & k_2 & k_3 & 0 & 0 \\ \ddots & \ddots & k_2 & k_1 & k_2 & k_3 & 0 \\ \ddots & \ddots & k_3 & k_2 & k_1 & k_2 & k_3 \\ \ddots & \ddots & 0 & k_3 & k_2 & k_1-\boxed{k_3} & k_2 \\ \ddots & \ddots & 0 & 0 & k_3-\boxed{k_3} & k_2-\boxed{k_2} & k_1 \end{bmatrix}_{n\times n} \end{array}$$

$$\times\begin{Bmatrix} \vdots \\ \vdots \\ z_{n-4} \\ z_{n-3} \\ z_{n-2} \\ z_{n-1} \\ z_n \end{Bmatrix}_{n\times 1}=-\frac{1}{E}\begin{Bmatrix} \vdots \\ \vdots \\ q_{n-4}/I_{n-4} \\ q_{n-3}/I_{n-3} \\ q_{n-2}/I_{n-2} \\ q_{n-1}/I_{n-1} \\ q_n/I_n \end{Bmatrix}_{n\times 1}$$

3.2.3. System Behaviour at Completely Free End

3.2.3.1. Completely Free Support at Left End

At completely free end, both shear force and bending moment are zero. For free end at the left part of the beam, the deflection values of virtual joints (a and b) shown in Fig. 7 are determined from the following equations by using the boundary conditions:

$$\left(\frac{\partial^2 z}{\partial x^2}\right)_1=\left(\frac{\Delta^2 z}{\Delta x^2}\right)_1=\frac{z(2)-2z(1)+z(a)}{ux^2}=0 \;\rightarrow\; z(a)=2z(1)-z(2)$$

$$\left(\frac{\partial^2 z}{\partial x^2}\right)_1=\left(\frac{\Delta^2 z}{\Delta x^2}\right)_1=\frac{z(3)-2z(1)+z(b)}{4ux^2}=0 \;\rightarrow\; z(b)=2z(1)-z(3)$$

$$\left(\frac{\partial^3 z}{\partial x^3}\right)_1=\left(\frac{\Delta^3 z}{\Delta x^3}\right)_1=\frac{z(3)-2z(2)+2z(a)-z(b)}{2ux^3}=0$$
$$\rightarrow\; z(3)-2z(2)+2z(a)-z(b)=0$$

The displacement values of virtual joints "a" and "b" are obtained from the solution of these equations as follows:

$$z(a)=z(1)+z(2)-z(3)$$

$$z(b)=4z(1)-4z(2)+z(3)$$

The contribution of the virtual joints "a" and "b" to the rigidity matrix is determined by considering the interacting point group operator shown in Fig. 7. The equations which obtained by placing the interacting points group on first and second points are determined again as follows:

The displacement equation when the center of the interacting points group is on 1'th joint:

$$k_3 z(b)+k_2 z(a)+k_1 z(1)+k_2 z(2)+k_3 z(3)=-\frac{1}{E}\frac{q(1)}{I(1)}$$

The displacement equation when the center of the interacting points group is on 2'th joint:

$$k_3 z(a)+k_2 z(1)+k_1 z(2)+k_2 z(3)+k_3 z(4)=-\frac{1}{E}\frac{q(2)}{I(2)}$$

If $z(a)=z(1)+z(2)-z(3)$ and $z(b)=4z(1)-4z(2)+z(3)$ is substituted in equations above, following equations are obtained:

$$k_3\big(4z(1)-4z(2)+z(3)\big)+k_2\big(z(1)+z(2)-z(3)\big)$$
$$+k_1 z(1)+k_2 z(2)+k_3 z(3)=-\frac{1}{E}\frac{q(1)}{I(1)}$$

$$k_3\big(z(1)+z(2)-z(3)\big)+k_2 z(1)$$
$$+k_1 z(2)+k_2 z(3)+k_3 z(4)=-\frac{1}{E}\frac{q(2)}{I(2)}$$

As it can be seen, new rigidity coefficients are added into system equations and these coefficients should be added to the rigidity matrix. In the previous sections, the method for updating rigidity matrix explained in detail. The same procedure is applied again and as a result of the operations, system rigidity matrix is updated as follows:

$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\ \vdots\\ \vdots \end{array}\begin{array}{c}\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & \cdots & \cdots\end{array}\\ \begin{bmatrix} k_1+\boxed{k_2}+\boxed{4k_3} & k_2+\boxed{k_2}-\boxed{4k_3} & k_3-\boxed{k_2}+\boxed{k_3} & 0 & 0 & \ddots & \ddots \\ k_2+\boxed{k_3} & k_1+\boxed{k_3} & k_2-\boxed{k_3} & k_3 & 0 & \ddots & \ddots \\ k_3 & k_2 & k_1 & k_2 & k_3 & \ddots & \ddots \\ 0 & k_3 & k_2 & k_1 & k_2 & \ddots & \ddots \\ 0 & 0 & k_3 & k_2 & k_1 & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}_{n\times n}\end{array}$$

$$\times\begin{Bmatrix} z_1\\ z_2\\ z_3\\ z_4\\ z_5\\ \vdots\\ \vdots \end{Bmatrix}_{n\times1}=-\frac{1}{E}\begin{Bmatrix} q_1/I_1\\ q_2/I_2\\ q_3/I_3\\ q_4/I_4\\ q_5/I_5\\ \vdots\\ \vdots \end{Bmatrix}_{n\times1}$$

3.2.3.2. Completely Free Support at Right End

The deflection values of virtual joints (a' and b') shown in Fig. 8 are determined from the following equations considering the fact that the bending moment and shear force are both zero at free end:

$$\left(\frac{\partial^2 z}{\partial x^2}\right)_n=\left(\frac{\Delta^2 z}{\Delta x^2}\right)_n=\frac{z(a')-2z(n)+z(n-1)}{ux^2}=0$$
$$\rightarrow z(a')=2z(n)-z(n-1)$$

$$\left(\frac{\partial^2 z}{\partial x^2}\right)_n=\left(\frac{\Delta^2 z}{\Delta x^2}\right)_n=\frac{z(b')-2z(n)+z(n-2)}{4ux^2}=0$$
$$\rightarrow z(b')=2z(n)-z(n-2)$$

$$\left(\frac{\partial^3 z}{\partial x^3}\right)_n=\left(\frac{\Delta^3 z}{\Delta x^3}\right)_n=\frac{z(b')-2z(a')+2z(n-1)-z(n-2)}{2ux^3}=0$$
$$\rightarrow z(b')-2z(a')+2z(n-1)-z(n-2)=0$$

The displacement values of virtual joints "a'" and "b'" are obtained from the solution of these equations as follows:

$$z(a')=z(n)+z(n-1)-z(n-2)$$
$$z(b')=4z(n)-4z(n-1)+z(n-2)$$

The contribution of the virtual joints "a'" and "b'" to the rigidity matrix is determined by considering the interacting point group operator shown in Fig. 8. The equations which obtained by placing the interacting points group on (n-1)'th and n'th points are determined again as follows:

The displacement equation when the center of the interacting points group is on (n-1)'th joint:

$$k_3 z(n-3)+k_2 z(n-2)+k_1 z(n-1)+k_2 z(n)+k_3 z(a')=-\frac{1}{E}\frac{q(n-1)}{I(n-1)}$$

The displacement equation when the center of the interacting points group is on n'th joint:

$$k_3 z(n-2)+k_2 z(n-1)+k_1 z(n)+k_2 z(a')+k_3 z(b')=-\frac{1}{E}\frac{q(n)}{I(n)}$$

If $z(a')=z(n)+z(n-1)-z(n-2)$ and $z(b')=4z(n)-4z(n-1)+z(n-2)$ is substituted in equations above, following equations are obtained:

$$k_3 z(n-3)+k_2 z(n-2)+k_1 z(n-1)+k_2 z(n)$$
$$+k_3\big(z(n)+z(n-1)-z(n-2)\big)=-\frac{1}{E}\frac{q(n-1)}{I(n-1)}$$

$$k_3 z(n-2) + k_2 z(n-1) + k_1 z(n)$$
$$-k_2 z(n-1) - k_3 z(n-2) = -\frac{1}{E}\frac{q(n)}{I(n)}$$

As it can be seen, new rigidity coefficients are added into system equations. The system rigidity matrix is updated by adding these new rigidity coefficients as follows:

$$\begin{array}{c} \\ \vdots \\ \vdots \\ n-4 \\ n-3 \\ n-2 \\ n-1 \\ n \end{array}
\begin{array}{c} \begin{array}{ccccccc} \cdots & \cdots & n-4 & n-3 & n-2 & n-1 & n \end{array} \\
\begin{bmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & k_1 & k_2 & k_3 & 0 & 0 \\
\ddots & \ddots & k_2 & k_1 & k_2 & k_3 & 0 \\
\ddots & \ddots & k_3 & k_2 & k_1 & k_2 & k_3 \\
\ddots & \ddots & 0 & k_3 & k_2 - \boxed{k_3} & k_1 + \boxed{k_3} & k_2 + \boxed{k_3} \\
\ddots & \ddots & 0 & 0 & k_3 - \boxed{k_2} + \boxed{k_3} & k_2 + \boxed{k_2} - \boxed{4k_3} & k_1 + \boxed{k_2} + \boxed{4k_3}
\end{bmatrix} \end{array}$$

$$\times \begin{Bmatrix} \vdots \\ \vdots \\ z_{n-4} \\ z_{n-3} \\ z_{n-2} \\ z_{n-1} \\ z_n \end{Bmatrix} = -\frac{1}{E} \begin{Bmatrix} \vdots \\ \vdots \\ q_{n-4}/I_{n-4} \\ q_{n-3}/I_{n-3} \\ q_{n-2}/I_{n-2} \\ q_{n-1}/I_{n-1} \\ q_n/I_n \end{Bmatrix}$$

3.3. Applying Boundary Conditions to System Rigidity Matrix for a Computer Program

In the previous section, the additional rigidity coefficients for each boundary condition are determined in detail. In addition to this, it must be considered that if there is a support at a joint of the beam, the displacement becomes zero. In rigidity matrix, this condition is accomplished by filling the row and column of that joint with zeros. The diagonal value of the joint is also assigned as one. In computer program, a vector for representing boundary conditions is determined. The name of the vector is “bc” which is the acronym of boundary conditions. The first column element of this vector represents the boundary condition of the left end and second column element of this vector represents the boundary condition of the right end, respectively. These values are assigned according to the numbers of boundary conditions shown in Fig. 6. The flowchart developed to apply boundary conditions to the system rigidity matrix is shown in Fig. 9. The function of this flowchart is written in MATLAB and presented in Appendix II.

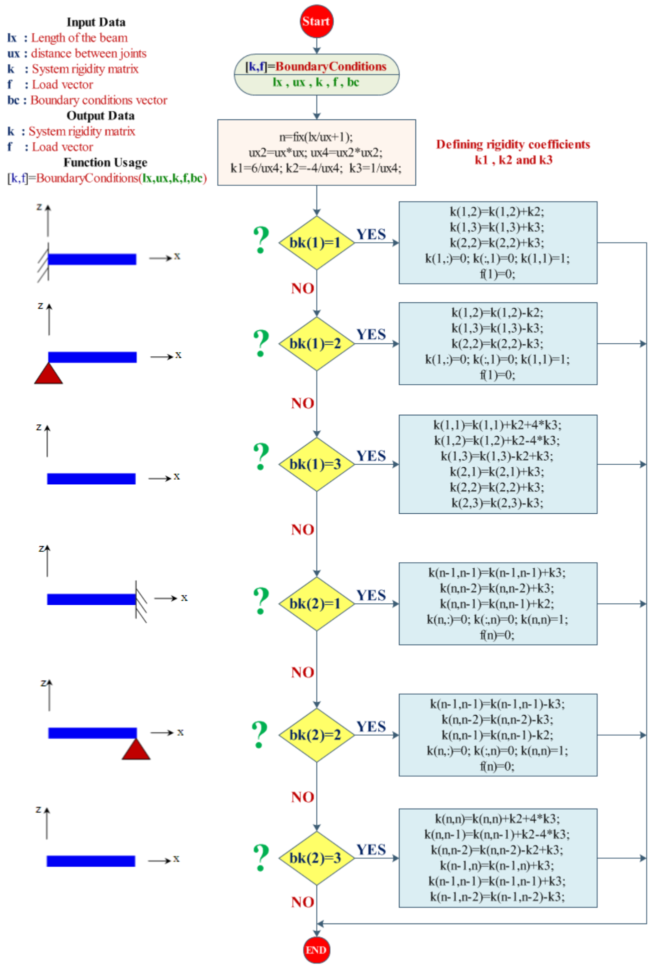

Figure 9. The flowchart developed to apply boundary conditions to the system rigidity matrix

3.4. Defining Joint Coordinates for System Analysis

The joint coordinates should be determined to analyze system with interacting points. The joint distribution along the beam is shown in Fig. 10. The coordinates of each joint are calculated as follows:

$$x(1) = 0;$$

$$x(2) = x(1) + (2-1) \times ux;$$

$$x(3) = x(1) + (3-1) \times ux;$$

$$\vdots$$

$$x(n-1)=x(1)+(n-2)\times ux;$$

$$x(n)=x(1)+(n-1)\times ux;$$

The flowchart developed to determine the coordinates of the joints is shown in Fig. 11. Interacting points group is moved along these joints after determining joint coordinates.

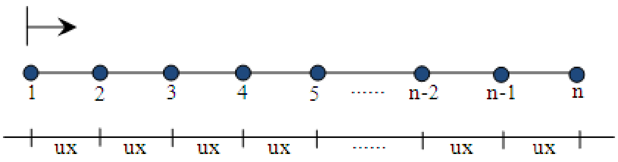

Figure 10. The joint distribution along Euler-Bernoulli Beam

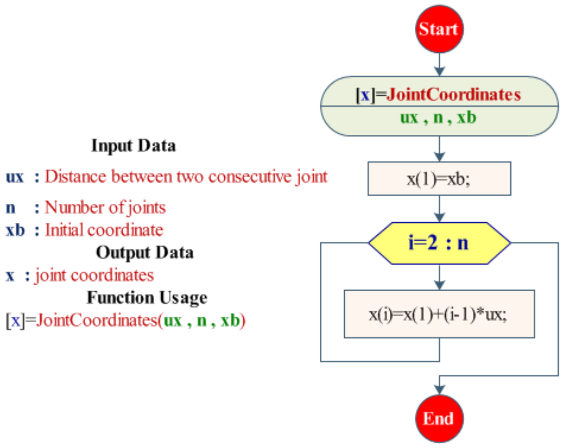

Figure 11. The flowchart developed to calculate joint coordinates

3.5. Determining Load Vector for System Analysis

The general solution equation of system is $[K]\{z\}=\{f\}$ as indicated before. In previous section, how to construct and arrange system rigidity matrix $[K]$ is explained in detail. In this section, an algorithm is presented to construct load vector automatically. The flowchart developed to construct the load vector defined in system equation is shown in Fig. 12.

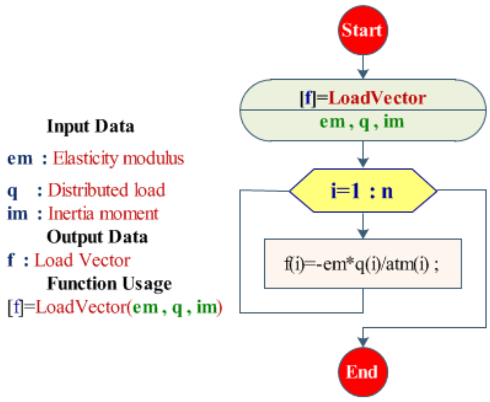

Figure 12. The flowchart developed to calculate joint loads

3.6. Determining Bending Moment Distribution

After constructing system rigidity matrix and load vector, the displacement values of each joint are calculated from the following equation:

$$\{z\}=[K]^{-1}\{f\}$$

Then, bending moment values for each joint is obtained from the following equation:

$$M(x)=EI(x)z''(x)$$

In this equation, second-order derivative of i'th joint displacement is used. This statement is expressed with difference functions as follows:

$$z''(x)=\frac{\partial^2 z}{\partial x^2}=\frac{\Delta^2 z}{\Delta x^2}=\frac{z(x+1)-2z(x)+z(x-1)}{(ux)^2}$$

$$z''(i)=\frac{z(i+1)-2z(i)+z(i-1)}{(ux)^2}$$

The moment value of i'th joint is expressed with difference functions as follows:

$$M(i)=\frac{E}{ux^2}I(i)\left[z(i+1)-2z(i)+z(i-1)\right]$$

In this statement, moment coefficient is defined as follows:

$$k(i)=\frac{E}{ux^2}I(i)$$

Then the moment value for i'th point may be expressed as follows:

$$M(i)=k(i)\left[z(i-1)-2z(i)+z(i+1)\right]$$

As a result, moment distribution of Euler Bernoulli beam is expressed with interacting points. The interacting points group which represents moment of i'th joint for the Euler Beam is shown in Fig. 13. The moment coefficients of each interacting point are also shown in this figure. These coefficients are used in moment calculations.

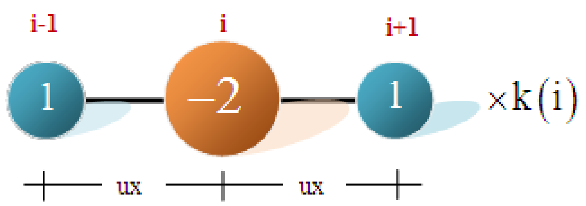

Figure 13. Interacting points group operator produced for determining moment distribution

The interacting group operator is moved from start to end of the beam to calculate moment values of each joint. When the group operator is on 1'th and n'th joints, some points of the operator are outside the beam as shown in Fig. 14. As indicated before, some virtual joints are defined and the displacements of these joints are determined according to the boundary conditions.

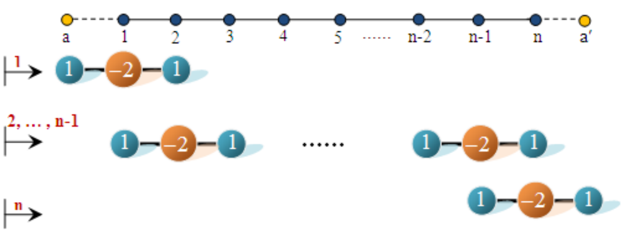

Figure 14. Moment operator moving from start to end of the beam

If there are completely fixed supports at boundary joints, the displacement values of virtual joints "a" and "b" are obtained as follows:

$$z(a) = z(2)$$

$$z(a') = z(n-1)$$

The displacement values of boundary joints are zero because of the completely fixed supports. The moment values of 1'th and n'th joints are calculated from the following equations:

$$M(1) = k(1) \times (z(2) - 2z(1) + z(2)) = 2 \times k(1) \times z(2)$$

$$M(n) = k(n) \times (z(n-1) - 2z(n) + z(n-1)) = 2 \times k(n) \times z(n-1)$$

If there are simple supports at boundary joints, the displacement values of virtual joints "a" and "b" are obtained as follows:

$$z(a) = -z(2)$$

$$z(a') = -z(n-1)$$

The displacement values of boundary joints are zero because of the simple supports. The moment values of 1'th and n'th joints are calculated from the following equations:

$$m(1) = k(1) \times (-z(2) - 2z(1) + z(2)) = 0$$
$$m(n) = k(n) \times (z(n-1) - 2z(n) - z(n-1)) = 0$$

If the boundary joints are free, the displacement values of virtual joints "a" and "b" are obtained as follows:

$$z(a) = z(1) + z(2) - z(3)$$

$$z(a') = z(n) + z(n-1) - z(n-2)$$

Then, the moment values of 1'th and n'th joints are calculated from the following equations:

$$m(1) = k(1) \times (2z(2) - z(1) - z(3))$$

$$m(n) = k(1) \times (2z(n-1) - z(n) - z(n-2))$$

3.7. Determining Shear Force Distribution

Shear force values for each joint is obtained from the following equation:

$$Q(x) = EI(x) z'''(x)$$

In this equation, third-order derivative of i'th joint displacement is used. This statement is expressed with difference functions as follows:

$$z'''(x)=\frac{\partial^3 z}{\partial x^3}=\frac{\Delta^3 z}{\Delta x^3}=\frac{z(x+2)-2z(x+1)+2z(x-1)-z(x-2)}{2(ux)^3}$$

$$z'''(i)=\frac{z(i+2)-2z(i+1)+2z(i-1)-z(i-2)}{2(ux)^3}$$

The shear force value of i'th joint is expressed with difference functions as follows:

$$Q(i)=\frac{E}{2ux^3}I(i)\left[z(i+2)-2z(i+1)+2z(i-1)-z(i-2)\right]$$

In this statement, shear coefficient is defined as follows:

$$k(i)=\frac{E}{2ux^3}I(i)$$

Then the shear force value for i'th point may be expressed as follows:

$$Q(i)=k(i)\left[z(i+2)-2z(i+1)+2z(i-1)-z(i-2)\right]$$

As a result, shear force distribution of Euler Bernoulli beam is expressed with interacting points. The interacting points group which represents shear force of i'th joint on the Euler Beam is shown in Fig. 15. The coefficients of each interacting point are also shown in this figure. These coefficients are used in shear force calculations.

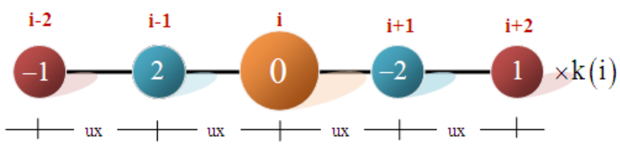

Figure 15. Interacting points group operator produced for determining shear force distribution

The shear force interacting group operator is moved from start to end of the beam to calculate shear force values of each joint. When the group operator is on 1'th, 2'th, n-1'th and n'th joints, some points of the operator are outside the beam as shown in Fig. 16. As indicated before, some virtual joints are defined and the displacements of these joints are determined according to the boundary conditions.

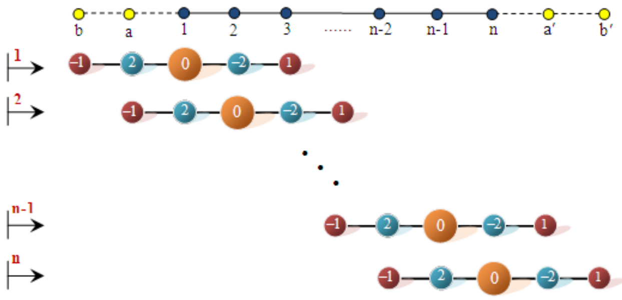

Figure 16. Shear force operator moving from start to end of the beam

If there are completely fixed supports at boundary joints, the displacement values of virtual joints (a,b,a',b') are obtained as follows:

$$z(a)=z(2)$$

$$z(b)=z(3)$$

$$z(a')=z(n-1)$$

$$z(b')=z(n-2)$$

The displacement values of boundary joints are zero because of the completely fixed supports. The shear force values of 1'th, 2'th, n-1'th and n'th joints are calculated from the following equations:

$$Q(1)=k(i)\times\left[-z(3)+2z(2)-2z(2)+z(3)\right]=0$$

$$Q(2)=k(i)\times\left[-z(2)-2z(3)+z(4)\right]$$

$$Q(n-1)=k(i)\times\left[-z(n-3)+2z(n-2)+z(n-1)\right]$$

$$Q(n)=k(i)\times\left[-z(n-2)+2z(n-1)-2z(n-1)+z(n-2)\right]=0$$

A special situation is observed from these equations. The shear force values of 1'th and n'th joints are equal to zero. This is an unexpected result. The shear force interacting point group operator doesn't give true results at these joints. Therefore, an additional solution must be suggested for calculating shear force at these joints. The shear force is obtained from the general equation of Euler-Bernoulli beam as follows:

$$\left.\begin{array}{l} EIz^{IV} = -q \\ EIz''' = Q \end{array}\right\} \rightarrow \frac{dQ}{dx} = -q$$

This equation is expressed with difference functions as follows:

$$\frac{Q_{i+1} - Q_{i-1}}{2ux} = -q_i \rightarrow Q_{i+1} - Q_{i-1} = -2 \times ux \times q_i$$

Then, shear force value of a joint is calculated in interaction with adjacent joints as follows:

$$Q_{i+1} = Q_{i-1} - 2 \times ux \times q_i$$

$$Q_{i-1} = Q_{i+1} + 2 \times ux \times q_i$$

The shear force values of 1’th and n’th joints are calculated depending on these interaction as follows:

$$Q_1 = Q_3 + 2 \times ux \times q_2$$

$$Q_n = Q_{n-2} - 2 \times ux \times q_{n-1}$$

If there are simple supports at boundary joints, the displacement values of virtual joints (a, b, a', b') are obtained as follows:

$$z(a) = -z(2)$$

$$z(b) = -z(3)$$

$$z(a') = -z(n-1)$$

$$z(b') = -z(n-2)$$

The displacement values of boundary joints are zero because of the simple supports. The shear force values of 1’th, 2’th, n-1’th and n’th joints are calculated from the following equations:

$$Q(1) = k(1) \times [-4z(2) + 2z(3)]$$
$$Q(2) = k(2) \times [z(2) - 2z(3) + z(4)]$$
$$Q(n-1) = k(n-1) \times [2z(n-2) - z(n-1) - z(n-3)]$$
$$Q(n) = k(n) \times [4z(n-1) - 2z(n-2)]$$

If the boundary joints are free, the displacement values of virtual joints (a, b, a', b') are obtained as follows:

$$z(a) = z(1) + z(2) - z(3)$$

$$z(b) = 4z(1) - 4z(2) + z(3)$$

$$z(a') = z(n) + z(n-1) - z(n-2)$$

$$z(b') = 4z(n) - 4z(n-1) + z(n-2)$$

Then, the shear force values of 1’th, 2’th, n-1’th and n’th joints are calculated from the following equations:

$$Q(1) = k(1) \times [-2z(1) + 4z(2) - 2z(3)]$$
$$Q(2) = k(2) \times [z(1) - z(2) - z(3) + z(4)]$$
$$Q(n-1) = k(n-1) \times [-z(n) + z(n-1) + z(n-2) - z(n-3)]$$
$$Q(n) = k(n) \times [2z(n) - 4z(n-1) + 2z(n-2)]$$

4. Numerical Application

As an application, a beam which has linear changeable cross section is selected as shown in Fig. 17. The boundary values of the cross section are also given in the figure. The beam is loaded with a second order polynomial distributed load. The boundary values of the load are also given in the figure. The beam has simple supports at boundary joints.

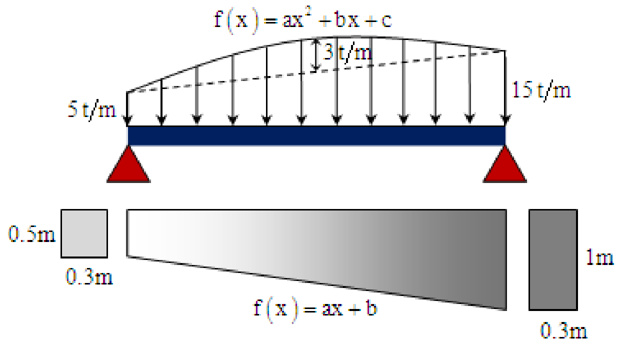

Figure 17. The beam which has linear changeable cross section with loading and boundary conditions

The length of the beam is taken as 6m and the distance between joints for analysis is taken as 0.1m. The modulus of Elasticity is 1000000 t/m². The beam is analyzed with interacting points and dispacements, bending moments and shear forces of each joint are calculated. The analysis reasults are presented in Figs. 18-20. The displacement diagram of the beam is given in Fig. 18, the bending moment distribution is shown in Fig. 19 and shear force distribution along the beam is given in Fig. 20, respectively.

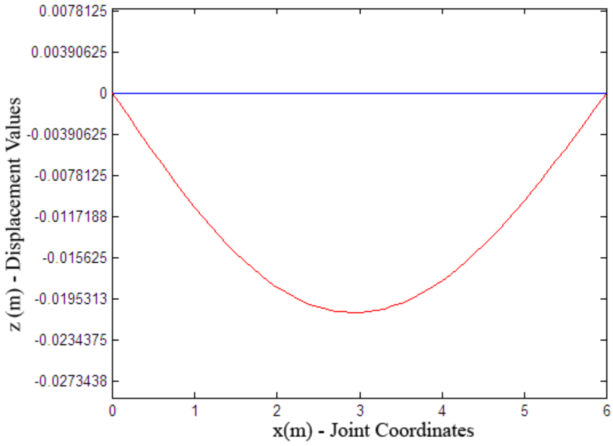

Figure 18. Displacement distribution along the beam

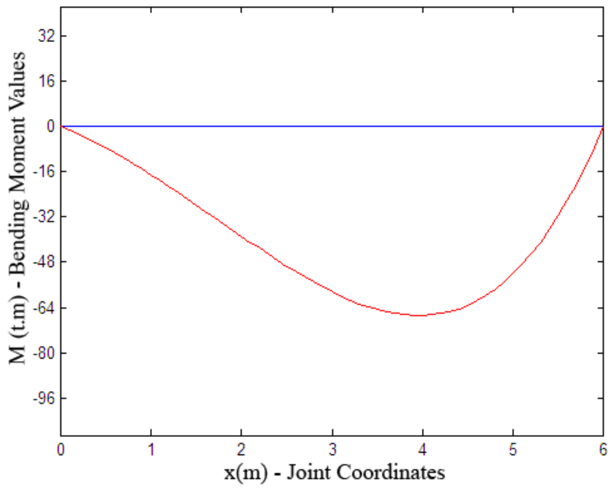

Figure 19. Bending moment distribution along the beam

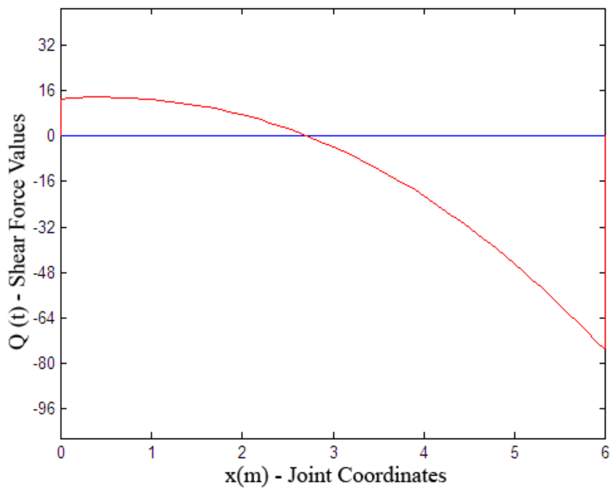

Figure 20. Shear force distribution along the beam

5. Conclusion

In this study, a novel computational structural analysis technique is presented. Proposed technique depends on matrix analysis of Finite Difference Method. In this approach, firstly interacting points group of the structural system is determined and the operator of this group is moved throughout the element. System rigidity matrix is produced and displacement vector is calculated. Then any structural effect may be calculated by using displacement values. In calculation of any structural effects, interacting points group developed for that calculation is also used. A computer program has been developed depending on the proposed algorithm. The flowcharts of the main functions are given in the paper. The functions are compiled in MATLAB which is a mathematical tool developed by The Mathworks. Some useful codes are also given in Appendix. In this paper, the proposed approach is implemented to see the practicability on Euler-Bernoulli beam which is the simplest structural element. This method may be applied to the complex systems and more accurate results may be obtained point by point. The most important advantage of this method is that any geometry and loading conditions may be taken into account and accurate results may be obtained. The proposed method transforms the differential approach of analysis into a more programmable and feasible numerical matrix formulation so that it may be used for many other structural problems with very little effort.

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Appendix I: MATLAB code written for automatically generating system rigidity matrix

```
function [k]=RigidityMatrix(lx, ux)
n=fix(lx/ux+1);
ux2=ux*ux; ux4=ux2*ux2;
k1=6/ux4;  k2=-4/ux4;  k3=1/ux4;
for i=1:n
  for j=1:n
    k(i,j)=0;
  end
end
for i=1:n-1
  k(i,i)=k1;    k(i,i+1)=k2;    k(i+1,i)=k2;
end
for i=1:n-2
  k(i,i+2)=k3;    k(i+2,i)=k3;
end
k(n,n)=k1;
```

Appendix II: MATLAB code written to apply boundary conditions to the system rigidity matrix

```
function [k, f]=BoundaryConditions(lx, ux, k , f , bc)
n=fix(lx/ux+1);
ux2=ux*ux; ux4=ux2*ux2;
k1=6/ux4;
k2=-4/ux4;
k3=1/ux4;
if bc(1)==1
  k(1,2)=k(1,2)+k2;
  k(1,3)=k(1,3)+k3;
  k(2,2)=k(2,2)+k3;
  k(1,:)=0;  k(:,1)=0;  k(1,1)=1;  f(1)=0;
elseif bc(1)==2
  k(1,2)=k(1,2)-k2;
  k(1,3)=k(1,3)-k3;
  k(2,2)=k(2,2)-k3;
  k(1,:)=0;  k(:,1)=0;  k(1,1)=1;  f(1)=0;
elseif bc(1)==3
  k(1,1)=k(1,1)+k2+4*k3;
  k(1,2)=k(1,2)+k2-4*k3;
```

```
    k(1,3)=k(1,3)-k2+k3;
    k(2,1)=k(2,1)+k3;
    k(2,2)=k(2,2)+k3;
    k(2,3)=k(2,3)-k3;
elseif bc(2)==1
    k(n-1,n-1)=k(n-1,n-1)+k3;
    k(n,n-2)=k(n,n-2)+k3;
    k(n,n-1)=k(n,n-1)+k2;
    k(n,:)=0;  k(:,n)=0;  k(n,n)=1  f(n)=0;
elseif bc(2)==2
    k(n-1,n-1)=k(n-1,n-1)-k3;
    k(n,n-2)=k(n,n-2)-k3;
    k(n,n-1)=k(n,n-1)-k2;
    k(n,:)=0;  k(:,n)=0;  k(n,n)=1  f(n)=0;
elseif bc(2)==3
    k(n,n)=k(n,n)+k2+4*k3;
    k(n,n-1)=k(n,n-1)+k2-4*k3;
    k(n,n-2)=k(n,n-2)-k2+k3;
    k(n-1,n)=k(n-1,n)+k3;
    k(n-1,n-1)=k(n-1,n-1)+k3;
    k(n-1,n-2)=k(n-1,n-2)-k3;
end
```