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Araştırma Makalesi

Some Identities of Fibonacci and Lucas Quaternions by Quaternion Matrices

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ABSTRACT

In this paper, we consider one of the most known Fibonacci matrix $Q=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$ and the Fibonacci quaternion matrix $M_{Q_F^n}=\begin{pmatrix}Q_{n+1} & Q_n\\ Q_n & Q_{n-1}\end{pmatrix}$, where Q_n is the n-th Fibonacci quaternion. In particular we define some new quaternion matrices. Our object is to derive some identities concerning Fibonacci and Lucas quaternions by using some new quaternion matrices with terms Fibonacci and Lucas numbers.

Keywords: *Fibonacci numbers, Lucas numbers, Fibonacci quaternions, Lucas quaternions*

Kuaterniyon Matrisleri ile Bazı Fibonacci ve Lucas Kuaterniyon Özdeşlikleri

ÖZET

Bu makalede en çok bilinen Fibonacci matrislerinden biri olan $Q=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$ matrisi ve Q_n $n.$ Fibonacci kuaterniyonu olmak üzere $M_{Q_F^n}=\begin{pmatrix}Q_{n+1} & Q_n\\ Q_n & Q_{n-1}\end{pmatrix}$ Fibonacci kuaterniyon matrisi ele alınmıştır. Ayrıca bazı yeni kuaterniyon matrisleri tanımlanmıştır. Bu çalışmada terimleri Fibonacci ve Lucas kuaterniyonları olan yeni kuaterniyon matrislerini kullanarak, Fibonacci ve Lucas kuaterniyonları ile ilgili bazı özdeşlikler elde edilecektir.

Anahtar Kelimeler: *Fibonacci sayıları, Lucas sayıları, Fibonacci kuaterniyonları, Lucas kuaterniyonları*

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I. Introduction

The recurrence relation for Fibonacci sequence $(F_n)_{n\geq 1}$ is defined as

$$F_{n+1} = F_n + F_{n-1}$$

with initial conditions $F_0 = 0,\ F_1 = 1$. Moreover Lucas sequence $(L_n)_{n\geq 1}$ is recursively defined as

$$L_{n+1} = L_n + L_{n-1}$$

with initial conditions $L_0 = 2,\ L_1 = 1.$ The terms F_n and L_n are called the n -th Fibonacci and Lucas numbers, respectively [9,11,12].

Binet's formulas for Fibonacci and Lucas numbers are given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $x^2 - x - 1 = 0.$

William R. Hamilton observed quaternions firstly in 1843 [4]. Let $a, b, c, d \in \mathbb{R}$. Then the hyper complex number $q = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ is called a quaternion where

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1},\ \mathbf{ij} = \mathbf{k} = -\mathbf{ji},\ \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \mathbf{ki} = \mathbf{j} = -\mathbf{ik}.$$

The $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is called the base of the quaternion $q,$ and the conjugate of this quaternion is defined as $\overline{q} = a\mathbf{1} - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$ in [4].

The group of quaternions is denoted as **H**. Addition is closed and commutative over **H**, but quaternion multiplication is not commutative over **H**.

The n -th Fibonacci quaternion is defined as

$$Q_n = F_n + F_{n+1}\mathbf{i} + F_{n+2}\mathbf{j} + F_{n+3}\mathbf{k},\ n \geq 0$$

where F_n is the n -th Fibonacci number. The n -th Lucas quaternion is defined as

$$K_n = L_n + L_{n+1}\mathbf{i} + L_{n+2}\mathbf{j} + L_{n+3}\mathbf{k},\ n \geq 0$$

where L_n is the n -th Lucas number [5, 6].

In [2], Binet's formulas for these quaternions are given by

$$Q_n = \frac{\hat{\alpha}\alpha^n - \hat{\beta}\beta^n}{\alpha - \beta} \text{ and } K_n = \hat{\alpha}\alpha^n + \hat{\beta}\beta^n$$

where $\hat{\alpha} = \mathbf{1} + \alpha\mathbf{i} + \alpha^2\mathbf{j} + \alpha^3\mathbf{k}$ and $\hat{\beta} = \mathbf{1} + \beta\mathbf{i} + \beta^2\mathbf{j} + \beta^3\mathbf{k}$.

It can be seen that

$$Q_{n+1} = Q_n + Q_{n-1}, \tag{1.1}$$

$$K_{n+1} = K_n + K_{n-1}, \tag{1.2}$$

$$K_n = Q_{n-1} + Q_{n+1}, \tag{1.3}$$

and

$$5Q_n = K_{n-1} + K_{n+1} \tag{1.4}$$

for all $n \geq 1$, in [2]. In addition to these, we can give the following identities which were derived in [1]

$$\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\alpha} = 2(2 + 2\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}), \tag{1.5}$$

$$\hat{\alpha}\hat{\alpha} = 2\hat{\alpha} - (1+\alpha^2)(1+\alpha^4), \tag{1.6}$$

and

$$\beta\beta = 2\beta - (1+\beta^2)(1+\beta^4). \tag{1.7}$$

II. Main Theorems

Matrix methods are used to obtain many results for various identities about Fibonacci and Lucas numbers. Halıcı defined a Fibonacci quaternion matrix $\begin{pmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{pmatrix}$ and proved Cassini's identity in [2]. Moreover Halıcı introduced complex Fibonacci quaternions and obtained a new 4×4 complex quaternionic matrix by means of 8×8 matrix with Fibonacci number components, in [3].

One of the most known Fibonacci matrix is the matrix Q which is given by $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Adapting this matrix to quaternions, Patel and Ray considered the Fibonacci quaternion matrix $M_{Q_F^n} = \begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix}$ and stated the following theorem in [10, Theorem 3.12].

Theorem 1: The matrix equation

$$M_{Q_F^n} = \begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \tag{2.1}$$

holds for any integer $n \geq 1$.

In the following corollary, we can give Cassini identity by evaluating the determinant in both sides of the matrices given in (2.1). Since the proof is obvious, we omit its proof.

Corollary 1: The equation

$$Q_{n+1}Q_{n-1} - Q_n^2 = (-1)^n (2 + \mathbf{2i} + \mathbf{4j} + \mathbf{3k})$$

holds for all $n \geq 1$, [7].

In addition to Theorem 1 we can add a lemma. We will use this lemma in the proof of Corollary 2.

Lemma 1: For all $n \geq 1$, we have

$$\begin{pmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{pmatrix}.$$

Proof: If we consider the $(n-1)$-th power of the matrix $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, then we get

$$\begin{pmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} = \begin{pmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{pmatrix} \begin{pmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{pmatrix} = \begin{pmatrix} Q_2F_n + Q_1F_{n-1} & Q_2F_{n-1} + Q_1F_{n-2} \\ Q_1F_n + Q_0F_{n-1} & Q_1F_{n-1} + Q_0F_{n-2} \end{pmatrix}.$$

Since $Q_2F_{n-1} + Q_1F_{n-2} = (Q_1 + Q_0)F_{n-1} + Q_1F_{n-2} = Q_1(F_{n-1} + F_{n-2})$, it follows that

$$\begin{pmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} = \begin{pmatrix} Q_2F_n + Q_1F_{n-1} & Q_1F_n + Q_0F_{n-1} \\ Q_2F_{n-1} + Q_1F_{n-2} & Q_1F_{n-1} + Q_0F_{n-2} \end{pmatrix}$$

$$= \begin{pmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{pmatrix} \begin{pmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{pmatrix}.$$

Therefore, in the sequel we can give the following corollary.

Corollary 2: For all integers $n, m \geq 1$, we have

$$Q_{n+1}Q_{m+1} + Q_n Q_m = Q_2 Q_{n+m} + Q_1 Q_{n+m-1}.$$

Proof: Using the equality (2.1) and Lemma 1, we get

$$\begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix}\begin{pmatrix} Q_{m+1} & Q_m \\ Q_m & Q_{m-1} \end{pmatrix} = \begin{pmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1}\begin{pmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{m-1}$$

$$= \begin{pmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{pmatrix}\begin{pmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{m-1}$$

$$= \begin{pmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{pmatrix}\begin{pmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n+m-2} = \begin{pmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{pmatrix}\begin{pmatrix} Q_{n+m} & Q_{n+m-1} \\ Q_{n+m-1} & Q_{n+m-2} \end{pmatrix}$$

$$= \begin{pmatrix} Q_2 Q_{n+m} + Q_1 Q_{n+m-1} & Q_2 Q_{n+m-1} + Q_1 Q_{n+m} \\ Q_1 Q_{n+m} + Q_0 Q_{n+m-1} & Q_1 Q_{n+m-1} + Q_0 Q_{n+m-2} \end{pmatrix}.$$

On the other side we have

$$\begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix}\begin{pmatrix} Q_{m+1} & Q_m \\ Q_m & Q_{m-1} \end{pmatrix} = \begin{pmatrix} Q_{n+1}Q_{m+1} + Q_n Q_m & Q_{n+1}Q_m + Q_n Q_{m-1} \\ Q_n Q_{m+1} + Q_{n-1}Q_m & Q_n Q_m + Q_{n-1}Q_{m-1} \end{pmatrix}.$$

Thus it follows that

$$Q_{n+1}Q_{m+1} + Q_n Q_m = Q_2 Q_{n+m} + Q_1 Q_{n+m-1},$$

$$Q_{n+1}Q_m + Q_n Q_{m-1} = Q_2 Q_{n+m-1} + Q_1 Q_{n+m},$$

$$Q_n Q_{m+1} + Q_{n-1}Q_m = Q_1 Q_{n+m} + Q_0 Q_{n+m-1},$$

and

$$Q_n Q_m + Q_{n-1}Q_{m-1} = Q_1 Q_{n+m-1} + Q_0 Q_{n+m-2}.$$

We can define another quaternion matrix as $M_{Q_L^n} = \begin{pmatrix} K_{n+1} & K_n \\ K_n & K_{n-1} \end{pmatrix}$ with Lucas quaternion entries K_n.

Theorem 2: Let n is an integer with $n \geq 1$, then we have

$$M_{Q_L^n}=\begin{pmatrix}K_{n+1} & K_n\\ K_n & K_{n-1}\end{pmatrix}=\begin{pmatrix}K_2 & K_1\\ K_1 & K_0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^{n-1}=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^{n-1}\begin{pmatrix}K_2 & K_1\\ K_1 & K_0\end{pmatrix}. \tag{2.2}$$

Proof: We will use the induction method for proving left part of (2.2). It is easy to see that the equation holds for $n=1.$ Now assume that the relation holds for all positive integers m with $1<m<n.$ Thus we have

$$\begin{pmatrix}K_2 & K_1\\ K_1 & K_0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^{m-1}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}K_{m+1} & K_m\\ K_m & K_{m-1}\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}K_{m+2} & K_{m+1}\\ K_{m+1} & K_m\end{pmatrix},$$

by the identity (1.2).

Consequently it is true for $n=m+1$ and hence for all n. Then it completes the first part of our proof. Moreover considering the $(n-1)-\text{th}$ power of the matrix $Q=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix},$ we get

$$\begin{pmatrix}K_2 & K_1\\ K_1 & K_0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^{n-1}=\begin{pmatrix}K_2 & K_1\\ K_1 & K_0\end{pmatrix}\begin{pmatrix}F_n & F_{n-1}\\ F_{n-1} & F_{n-2}\end{pmatrix}=\begin{pmatrix}K_2F_n+K_1F_{n-1} & K_2F_{n-1}+K_1F_{n-2}\\ K_1F_n+K_0F_{n-1} & K_1F_{n-1}+K_0F_{n-2}\end{pmatrix}.$$

After appropriate algebraic manipulation

$$K_2F_{n-1}+K_1F_{n-2}=(K_1+K_0)F_{n-1}+K_1F_{n-2}=K_1(F_{n-1}+F_{n-2})+K_0F_{n-1}=K_1F_n+K_0F_{n-1},$$

we find

$$\begin{pmatrix}K_2 & K_1\\ K_1 & K_0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^{n-1}=\begin{pmatrix}K_2F_n+K_1F_{n-1} & K_1F_n+K_0F_{n-1}\\ K_2F_{n-1}+K_1F_{n-2} & K_1F_{n-1}+K_0F_{n-2}\end{pmatrix}$$

$$=\begin{pmatrix}F_n & F_{n-1}\\ F_{n-1} & F_{n-2}\end{pmatrix}\begin{pmatrix}K_2 & K_1\\ K_1 & K_0\end{pmatrix}=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^{n-1}\begin{pmatrix}K_2 & K_1\\ K_1 & K_0\end{pmatrix}.$$

Thus the following corollary follows from Theorem 2.

Corollary 3: For all integers $n,m\geq 1,$ we have

$$K_{n+1}K_{m+1}+K_nK_m=K_2K_{n+m}+K_1K_{n+m-1}.$$

Proof: If we use the equality (2.2), we get

$$\begin{pmatrix} K_{n+1} & K_n \\ K_n & K_{n-1} \end{pmatrix}\begin{pmatrix} K_{m+1} & K_m \\ K_m & K_{m-1} \end{pmatrix} = \begin{pmatrix} K_2 & K_1 \\ K_1 & K_0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1}\begin{pmatrix} K_2 & K_1 \\ K_1 & K_0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{m-1}$$

$$= \begin{pmatrix} K_2 & K_1 \\ K_1 & K_0 \end{pmatrix}\begin{pmatrix} K_2 & K_1 \\ K_1 & K_0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{m-1}$$

$$= \begin{pmatrix} K_2 & K_1 \\ K_1 & K_0 \end{pmatrix}\begin{pmatrix} K_2 & K_1 \\ K_1 & K_0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n+m-2}$$

$$= \begin{pmatrix} K_2 & K_1 \\ K_1 & K_0 \end{pmatrix}\begin{pmatrix} K_{n+m} & K_{n+m-1} \\ K_{n+m-1} & K_{n+m-2} \end{pmatrix}$$

$$= \begin{pmatrix} K_2K_{n+m} + K_1K_{n+m-1} & K_2K_{n+m-1} + K_1K_{n+m-2} \\ K_1K_{n+m} + K_0K_{n+m-1} & K_1K_{n+m-1} + K_0K_{n+m-2} \end{pmatrix}.$$

Considering the left hand side of the matrix multiplication, it follows that

$$K_{n+1}K_{m+1} + K_nK_m = K_2K_{n+m} + K_1K_{n+m-1},$$

$$K_{n+1}K_m + K_nK_{m-1} = K_2K_{n+m-1} + K_1K_{n+m-2},$$

$$K_nK_{m+1} + K_{n-1}K_m = K_1K_{n+m} + K_0K_{n+m-1},$$

and

$$K_nK_m + K_{n-1}K_{m-1} = K_1K_{n+m-1} + K_0K_{n+m-2}.$$

Lemma 2: For all integers $n \geq 1$, we have

$$K_{n+1}K_{n-1} - K_n^2 = -5(Q_{n+1}Q_{n-1} - Q_n^2).$$

Proof: It follows from the determinant of the following matrix equation

$$\begin{pmatrix} K_{n+1} & K_n \\ K_n & K_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix}.$$

If we consider the matrix given in [8, Corollary 4], then the following theorem can be given for quaternions.

Theorem 3: For any integer $n \geq 1,$ we have

$$\begin{pmatrix} K_n/2 & 5Q_n/2 \\ Q_n/2 & K_n/2 \end{pmatrix} = \begin{pmatrix} K_1/2 & 5Q_1/2 \\ Q_1/2 & K_1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 5/2 \\ 1/2 & 1/2 \end{pmatrix}^{n-1}. \tag{2.3}$$

Proof: We will use the induction method to prove this equation. It is clear for $n=1.$ Now assume that the relation is true for all integers m with $1<m<n.$ Then the equation

$$\begin{pmatrix} K_1/2 & 5Q_1/2 \\ Q_1/2 & K_1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 5/2 \\ 1/2 & 1/2 \end{pmatrix}^{m} = \begin{pmatrix} K_1/2 & 5Q_1/2 \\ Q_1/2 & K_1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 5/2 \\ 1/2 & 1/2 \end{pmatrix}^{m-1} \begin{pmatrix} 1/2 & 5/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} K_m/2 & 5Q_m/2 \\ Q_m/2 & K_m/2 \end{pmatrix} \begin{pmatrix} 1/2 & 5/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} K_{m+1}/2 & 5Q_{m+1}/2 \\ Q_{m+1}/2 & K_{m+1}/2 \end{pmatrix}$$

holds for $m+1,$ by the identities (1.3) and (1.4). It completes the proof.

Corollary 4: For all $n \geq 1,$ we have

$$K_n^2 - 5Q_n^2 = 4(-1)^n(2+2\mathbf{i}+6\mathbf{j}+3\mathbf{k}).$$

Proof: Calculating the determinant of both sides of the equation (2.3), we get

$$\det\begin{pmatrix} K_n/2 & 5Q_n/2 \\ Q_n/2 & K_n/2 \end{pmatrix} = \det\left(\begin{pmatrix} K_1/2 & 5Q_1/2 \\ Q_1/2 & K_1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 5/2 \\ 1/2 & 1/2 \end{pmatrix}^{n-1}\right) = \frac{1}{4}(-1)^{n-1}(K_1^2 - 5Q_1^2).$$

By Binet's formulas, the identity (1.5) and the fact $\alpha\beta = -1,$ it follows that

$$K_n^2 - 5Q_n^2 = (-1)^{n-1}\left[(\alpha\alpha + \beta\beta)^2 - 5\left(\frac{\alpha\alpha - \beta\beta}{\alpha - \beta}\right)^2\right] = 2(-1)^n(\alpha\beta + \beta\alpha) = 4(-1)^n(2+2\mathbf{i}+6\mathbf{j}+3\mathbf{k}).$$

Corollary 5: For all integers $n, m \geq 1,$ we have

$$K_n K_m + 5Q_n Q_m = K_1 K_{n+m-1} + 5Q_1 Q_{n+m-1},$$

and

$$K_nQ_m + Q_nK_m = K_1Q_{n+m-1} + Q_1K_{n+m-1}.$$

Proof: Considering the equation (2.3), we can multiply the following matrices,

$$\begin{pmatrix} K_n/2 & 5Q_n/2 \\ Q_n/2 & K_n/2 \end{pmatrix}\begin{pmatrix} K_m/2 & 5Q_m/2 \\ Q_m/2 & K_m/2 \end{pmatrix} = \begin{pmatrix} K_1/2 & 5Q_1/2 \\ Q_1/2 & K_1/2 \end{pmatrix}\begin{pmatrix} 1/2 & 5/2 \\ 1/2 & 1/2 \end{pmatrix}^{n-1}\begin{pmatrix} K_1/2 & 5Q_1/2 \\ Q_1/2 & K_1/2 \end{pmatrix}\begin{pmatrix} 1/2 & 5/2 \\ 1/2 & 1/2 \end{pmatrix}^{m-1}$$

$$= \begin{pmatrix} K_1/2 & 5Q_1/2 \\ Q_1/2 & K_1/2 \end{pmatrix}\begin{pmatrix} K_1/2 & 5Q_1/2 \\ Q_1/2 & K_1/2 \end{pmatrix}\begin{pmatrix} 1/2 & 5/2 \\ 1/2 & 1/2 \end{pmatrix}^{n-1}\begin{pmatrix} 1/2 & 5/2 \\ 1/2 & 1/2 \end{pmatrix}^{m-1}$$

$$= \begin{pmatrix} K_1/2 & 5Q_1/2 \\ Q_1/2 & K_1/2 \end{pmatrix}\begin{pmatrix} K_1/2 & 5Q_1/2 \\ Q_1/2 & K_1/2 \end{pmatrix}\begin{pmatrix} 1/2 & 5/2 \\ 1/2 & 1/2 \end{pmatrix}^{n+m-2}$$

$$= \begin{pmatrix} K_1/2 & 5Q_1/2 \\ Q_1/2 & K_1/2 \end{pmatrix}\begin{pmatrix} K_{n+m-1}/2 & 5Q_{n+m-1}/2 \\ Q_{n+m-1}/2 & K_{n+m-1}/2 \end{pmatrix}$$

$$= \begin{pmatrix} (K_1K_{n+m-1} + 5Q_1Q_{n+m-1})/4 & 5(K_1Q_{n+m-1} + Q_1K_{n+m-1})/4 \\ (Q_1K_{n+m-1} + K_1Q_{n+m-1})/4 & (5Q_1Q_{n+m-1} + K_1K_{n+m-1})/4 \end{pmatrix}.$$

Thus it follows that

$$K_nK_m + 5Q_nQ_m = K_1K_{n+m-1} + 5Q_1Q_{n+m-1},$$

and

$$K_nQ_m + Q_nK_m = K_1Q_{n+m-1} + Q_1K_{n+m-1}.$$

III. References

[1] B. Demirtürk Bitim and N. Topal, “Quaternions via generalized Fibonacci and Lucas number components,” *Mat. Rep.*, accepted 24.04.2017.

[2] S. Halıcı, “On Fibonacci quaternions”, *Adv. Appl. Clifford Algebr*., vol. 22, no. 2, pp. 321-327, 2012.

[3] S. Halıcı, “On complex Fibonacci quaternions,” *Adv. Appl. Clifford Algebr*., vol. 23, no. 1, pp.105-112, 2013.

[4] W. R. Hamilton, *Elements of Quaternions*, London, Longmans and Green, 1866.

[5] A. F. Horadam, “Complex Fibonacci numbers and Fibonacci quaternions,” *Amer. Math. Monthly*, vol. 70, pp. 289-291, 1963.

[6] A. F. Horadam, “Quaternion recurrence relations,” *Ulam Quarterly*, vol. 2, no. 2, pp. 23-33, 1993.

[7] M. R. Iyer, “Some results on Fibonacci quaternions”, *Fibonacci Quart*., vol. 7, pp. 201-210, 1969.

[8] R. Keskin, B. Demirtürk, “Some new Fibonacci and Lucas identities by matrix methods”, *Internat. J. Math. Ed. Sci. Tech*., vol. 41, no. 3, pp. 379-387, 2009.

[9] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, New York, John Wiley & Sons, 2001.

[10] B. K. Patel and P. K. Ray, “On the properties of (p,q)-Fibonacci and (p,q)-Lucas quaternions,” *Mat. Rep*., vol. 21, no. 1, pp. 1-10, 2019.

[11] P. Ribenboim, *My Numbers, My Friends*, New York, Springer Verlag, 2000.

[12] S. Vajda, *Fibonacci and Lucas Numbers and The Golden Section*, Chichester, Ellis Horwood Limited Publ., 1989.