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Research Article

A SIMPLE MATHEMATICAL MODEL THROUGH FRACTIONAL-ORDER DIFFERENTIAL EQUATION FOR PATHOGENIC INFECTION

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ABSTRACT

The model in this study, examined the time-dependent changes in the population sizes of pathogen-immune system, is presented mathematically by fractional-order differential equations (FODEs) system. Qualitative analysis of the model was examined according to the parameters used in the model. The proposed system has always namely free-infection equilibrium point and the positive equilibrium point exists when specific conditions dependent on parameters are met, According to the threshold parameter R_0, it is founded the stability conditions of these equilibrium points. Also, the qualitative analysis was supported by numerical simulations.

Keywords: *Fractional-Order Differential Equation, Numerical Simulation, Stability Analysis*

2010 AMS Subject Classification: *34D20, 92B05*

1. INTRODUCTION

Transferring process of a situation or incident by using mathematical symbols is called as mathematical modeling [1]. In these kinds of modeling, the use of fractional-order differential and integral operators has increased recently [2,3]. In this sense, fractional-order calculations are widely used especially in physics, thermodynamics, viscoelasticity, electrical circuit theory, mechatronic systems, signal process, chemical mixtures, chaos theory, engineering, biological systems, economics, and many various areas [4-8]. Stability of the equilibrium point for fractional-order differential equations (FODEs) and its systems is at least as much as their integer order. Accordingly, the behavior of the system can be estimated by the stability analysis of equilibrium points of the suggested system via

mathematical modeling. The biological population modeling formed by using FODE is an ample source in terms of mathematical ideas [9-11].

The diseases induced by organisms called as pathogens such as tumor, bacteria, fungus or viruses have been considered as the main cause of fatal diseases through the human history. Basically, it is a quite complex process for both the pathogen and host. In spite of developing different treatment strategies against to diseases caused by these pathogens, main and first process of fight against these is played by the individuals immune system. Immune system is called as the system of biological structures and processes in an organism protecting host against the possible harmful organisms by recognizing and responding to the antigens of pathogens [12]. In this context, dynamics between immune system cells of host and pathogen that causes disease are important to understand the nature of disease and are tried to be explained in [13-19] in different ways.

The proposed mathematical model in form, nonlinear autonomous two-dimensional fractional-order differential equation system considered the main mechanisms of pathogen and immune system cells in host, was presented in this study.

2. PRELIMINARIES AND DEFINITIONS

Definition 2.1. *(Fractional Derivative and Integral in Caputo Sense) For* $t > 0, \beta \in R^+$*, fractional-order integral of the function* $f(t)$ *is defined as*

$$I^{\beta} f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s) ds \quad (2.1)$$

and fractional-order derivative the function $f(t)$ *is defined as*

$$D^{\alpha} f(t) = I^{n-\alpha} D^n f(t), \qquad D = \frac{d}{dt} \quad (2.2)$$

for $\alpha \in (n-1, n]$ *[3,20-23].*

Lemma 2.1. *Stability Analysis of Equilibrium Point of Nonlinear Autonomous Two-Dimensional Fractional-Order Differential Equations System is as the followings:*

Proof. With initial conditions

$$x(0) = x_0 \text{ and } y(0) = y_0, \quad (2.3)$$

let us consider the following system

$$\begin{aligned} D^{\alpha} x(t) &= f_1(x, y) \\ D^{\alpha} y(t) &= f_2(x, y) \end{aligned} \quad (2.4)$$

for $\alpha \in (0,1]$. Also, we have supposed that equilibrium point obtained from equation system $f_1(\bar{x}, \bar{y}) = f_2(\bar{x}, \bar{y}) = 0$ is shown by $(\bar{x}, \bar{y})$. The Jacobian Matrix of system (2.4) is founded from $J = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix}$. If the eigenvalues λ_1 and λ_2 obtained from the equation $\text{Det}\left(J_{(x,y)=(\bar{x},\bar{y})} - \lambda I_2\right) = 0$ provide the conditions

$$\left(|\arg(\lambda_1)| > \frac{\alpha\pi}{2}, \quad |\arg(\lambda_2)| > \frac{\alpha\pi}{2}\right), \quad (2.5)$$

then, the equilibrium point $(\bar{x}, \bar{y})$ is locally asymptotically stable (LAS) for system (2.4). Stability region of equilibrium point for FODE systems is larger than its integer order [24].

Conditions expressed in (2.5) can be detailed as the following. The characteristic polynomial belonging to the eigenvalues λ_1 and λ_2 obtained from Det $\left(J_{(x,y)=(\bar{x},\bar{y})} - \lambda I_2\right) = 0$ is

$$p(\lambda) = \lambda^2 + a_1\lambda + a_2 = 0. \quad (2.6)$$

When both the conditions (2.5) and the polynomial (2.6) are taken into account together; LAS conditions of the equilibrium point $(\bar{x}, \bar{y})$ are that coefficients of the polynomial (2.6) provide either Routh–Hurwitz conditions $(a_1, a_2 > 0)$ [1] or

$$a_1 < 0, 4a_2 > (a_1)^2, \left|\tan^{-1}\left(\frac{\sqrt{4a_2 - (a_1)^2}}{a_1}\right)\right| > \frac{\alpha\pi}{2}. \quad (2.7)$$

[25].

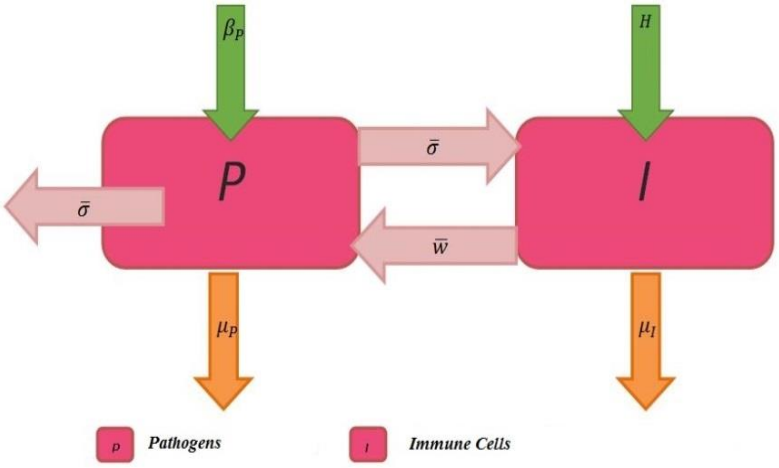
3. MATHEMATICAL MODEL

It has been identified pathogen load and level of immunity in a diseased individual. Therefore, it is presumed that the population sizes of pathogen load and immune system cells at time t denote by $P(t)$ and $I(t)$, respectively. The variable I can be denoted some accurate amount, like the density of specific B-cells or antibodies. In this context, the model in the form of FODEs system with the initial conditions $P(t_0) = P_0$ and $I(t_0) = I_0$ is

$$\begin{aligned} D^{\alpha}P(t) &= \beta_P P(t)\left(1 - \frac{P(t)}{C_p}\right) - \mu_P P(t) - \bar{\sigma}P(t)I(t) \\ D^{\alpha}I(t) &= \bar{\omega}I(t)P(t) - \mu_I I(t) + H \end{aligned} \quad (3.1)$$

where α shows the orders of derivative and the parameters $\beta_P, C_p, \mu_P, \bar{\sigma}, \bar{\omega}, \mu_I$ and H are positive constants. In the model (3.1), the growth rate of pathogen is β_P and the carrying capacity of pathogen is C_p In this sense, it is supposed that pathogen multiply according to logistic rule. Also, the natural death rate of pathogen is μ_P, the rate at which the immune system cells destroy pathogens is $\bar{\sigma}$, the multiplying rate of immune system cells in the existence of pathogen is $\bar{\omega}$ (specific immune system cells), the natural death rate of immune system cells is μ_I and the base production density of immune system cells in fixed quantity is H (aspecific immune system cells).

Figure 2.1. Schematic demonstration of interaction between immune system cells and pathogen in the model (3.1).



Let us consider as $P(t) = C_p p(t)$ and $I(t) = C_p i(t)$. When the parameter transformations $\bar{\sigma}C_p = \sigma, \bar{\omega}C_p = \omega$ and $h = \frac{H}{C_p}$ are applied to the system (3.1), it is obtained that

$$\begin{aligned} D_t^\alpha p &= \beta_P p(1-p) - \mu_P p - \sigma p i \\ D_t^\alpha i &= \omega i p - \mu_I i + h \\ 0 &< \alpha \leq 1 \end{aligned} \qquad (3.2)$$

Therefore, stability analysis of the system (3.1) can be sustained through the system (3.2).

3.1. Matrix Form of the System (3.2). FODEs system in (3.2) can be rewritten in matrix form as follows,

$$\begin{aligned} D^\alpha X(t) &= AX(t) + x_1(t)BX(t) + R \\ X(0) &= X_0 \end{aligned} \qquad (3.3)$$

where $0 < \alpha \leq 1, t \in (0,1], n \in \mathbb{N}^+$, $p(t) = x_1(t)$, $i(t) = x_2(t)$, $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, $X_0 = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$, $R = \begin{pmatrix} 0 \\ h \end{pmatrix}$, $A = \begin{pmatrix} \beta_P - \mu_P & 0 \\ 0 & -\mu_I \end{pmatrix}$ and $B = \begin{pmatrix} -\beta_P & -\sigma \\ 0 & \omega \end{pmatrix}$.

Definition 3.1. *For $X(t) = \left(x_1(t)\ x_2(t)\right)^T$, $C^*[0,T]$ set is a continuous set of the vector $X(t)$ at interval $[0,T]$. Norm of the vector $X(t) \in C^*[0,T]$ in (3.3) is $\|X(t)\| = \sum_{i=1}^{5} sup_t |x_i(t)|$.*

Proposition 3.1. *Let us consider Definition 3.1. and $X(t) = (x_1(t)\ x_2(t)\ )^T$ in $\mathbb{R}_+^2 = \{X \in R^2 : X \geq 0\}$ and $D^\alpha f(x) \in C[a,b]$ for $f(X) \in C[a,b]$, $0 < \alpha \leq 1$. According to generalized mean value theorem, it is $f(x) = f(a) + \frac{1}{\Gamma(\alpha)} D^\alpha f(\xi)(x-a)^\alpha$ for $\forall\ x \in [a,b]$ and $0 \leq \xi \leq x$. Also,*

- *When $D^\alpha f(x) > 0$ for $\forall\ x \in [a,b]$, the function $f(x)$ increases for each $x \in [a,b]$.*
- *When $D^\alpha f(x) < 0$ for $\forall\ x \in [a,b]$, the function $f(x)$ decreases for each $x \in [a,b]$.*

Also, the vector field is the points in $\mathbb{R}_+^2$, since $D^\alpha x_1(t)|_{x_1=x_2=0} = 0$ and $D^\alpha x_2(t)|_{x_1=x_2=0} = h$.

Proposition 3.2. *If $X(t) \in C^*[0,T]$, the system (3.2) has a single solution.*

Proof. We have $D^{\alpha}X(t) = AX(t) + x_1(t)BX(t) + R$ in (3.3). In this situation, it is $F(X(t)) \in C^*[0,T]$ for $X(t) \in C^*[0,T]$. Also, for vectors that would be like $X(t), Y(t) \in C^*[0,T]$ and $X(t) \neq Y(t)$, we have the followings

$$\|F(X(t)) - F(Y(t))\| =$$

$$\|(AX(t) + x_1(t)BX(t) + R) - (AY(t) + y_1(t)BY(t) + R)\|$$

$$\|AX(t) + x_1(t)BX(t) - AY(t) - y_1(t)BY(t)\|$$

$$\left\|A(X(t) - Y(t)) + x_1(t)BX(t) - y_1(t)BY(t) - \left(\underbrace{x_1(t)BY(t) - x_1(t)BY(t)}_{0}\right)\right\|$$

$$\|A(X(t) - Y(t)) + x_1(t)B(X(t) - Y(t)) + (x_1(t) - y_1(t))BY(t)\|$$

$$\leq \left(\|A(X(t) - Y(t))\| + \|x_1(t)B(X(t) - Y(t))\| + \|(x_1(t) - y_1(t))BY(t)\|\right)$$

$$\leq \left(\|A\|\|(X(t) - Y(t))\| + \|B\||x_1(t)|\|(X(t) - Y(t))\| + \|B\|\underbrace{|(x_1(t) - y_1(t))|}_{\leq\|(X(t)-Y(t))\|}\|Y(t)\|\right)$$

$$\leq \left(\|A\| + \|B\|\left(\underbrace{|x_1(t)|}_{\leq\|X(t)\|} + \|Y(t)\|\right)\right)\|(X(t) - Y(t))\|$$

and thus it is

$$\|F(X(t)) - F(Y(t))\| \leq L\|(X(t) - Y(t))\|$$

where $L = \|A\| + \|B\|(M_1 + M_2) > 0$ and M_1 and M_2 are positive constants, such that $X(t), Y(t) \in C^*[0,T]$, $\|X(t)\| \leq M_1$, $\|Y(t)\| \leq M_2$. Therefore the system (3.2) has a single solution.

4. QUALITATIVE ANALYSIS OF PROPOSED MODEL

In this part, equilibrium points of model, expressed in (3.2), are found and the stability analysis of these points is made.

Definition 4.1. *In system (3.2), the parameters are redefined as*

$$A_1 = \frac{\beta_P - \mu_P}{\beta_P}, A_2 = \frac{\mu_I}{\omega}, A_3 = \frac{\sigma}{\beta_P}, A_4 = \frac{h}{\omega} \quad (4.1)$$

for ease of stability analysis. Because the parameters used in system (3.1) are positive, it is

$$A_2, A_3, A_4 > 0 \quad (4.2)$$

Therefore, the system (3.2) can be rewritten as,

$$\begin{aligned} D_t^{\alpha} p &= f(p,i) = p\frac{1}{\beta_P}(A_1 - p - A_3 i) \\ D_t^{\alpha} i &= g(p,i) = \omega(ip - A_2 i + A_4) \\ 0 &< \alpha \leq 1 \end{aligned} \quad (4.3)$$

Proposition 4.1. *The system (4.3) has always the equilibrium point $E_0\left(0, \frac{A_4}{A_2}\right)$ namely free-infection equilibrium point. Also, when $A_1A_2 - A_3A_4 > 0$, the positive equilibrium point*

$E_1\left(\frac{(A_1+A_2)-\sqrt{(A_1-A_2)^2+4\,A_3A_4}}{2}, \frac{(A_1-A_2)+\sqrt{(A_1-A_2)^2+4\,A_3A_4}}{2A_3}\right)$ *exists.*

Proof. Let us consider that the general expression of equilibrium points of the system (4.3) is $E_j = (\bar{p}, \bar{\imath})$ for j = 1,2. They can be found by solving the equation system $f(\bar{p}, \bar{\imath}) = g(\bar{p}, \bar{\imath}) = 0$ in (4.3). Thus, we have

$$\begin{matrix} \bar{p}(A_1 - \bar{p} - A_3\bar{\imath}) & = 0 \\ (\bar{\imath}\bar{p} - A_2\bar{\imath} + A_4) & = 0 \end{matrix} \quad (4.4)$$

By the first equation in (4.4), it is either $\bar{p} = 0$ or $A_1 - \bar{p} - A_3\bar{\imath} = 0$ is obtained.

- Let $\bar{p} = 0$. In this situation, $\bar{\imath} = \frac{A_4}{A_2}$ is obtained from the second equation of (4.4). Therefore the free-infection equilibrium point is $E_0\left(0, \frac{A_4}{A_2}\right)$.
- Consider the other situation being $A_1 - \bar{p} - A_3\bar{\imath} = 0$ that is

$$\bar{\imath} = \frac{A_1 - \bar{p}}{A_3}. \quad (4.5)$$

When this value in (4.5) is written in the second equation in (4.4), the second degree polynomial

$$\bar{p}^2 - (A_1 + A_2)\bar{p} + (A_1A_2 - A_3A_4) = 0 \quad (4.6)$$

related to $\bar{p}$ is obtained. Discriminant of (4.6) is $\Delta = (A_1 - A_2)^2 + 4\,A_3A_4$. In this sense, it is clear that $\Delta > 0$ due to (4.2). The roots $\bar{p}$ are found as,

$$\begin{matrix} \overline{p_1} = \frac{(A_1 + A_2) - \sqrt{(A_1 - A_2)^2 + 4\,A_3A_4}}{2} \\ \text{and} \\ \overline{p_2} = \frac{(A_1 + A_2) + \sqrt{(A_1 - A_2)^2 + 4\,A_3A_4}}{2}. \end{matrix} \quad (4.7)$$

For the roots $\overline{p_1}$ and $\overline{p_2}$ are positive real number, it must be $(A_1 + A_2) \mp \sqrt{(A_1 - A_2)^2 + 4\,A_3A_4} > 0$ and so,

$$A_1A_2 - A_3A_4 > 0 \quad (4.8)$$

Thus, the values $\bar{\imath}_j$ for j = 1,2 that correspond to $\overline{p_1}$ and $\overline{p_2}$ can be found from (4.5) as

$$\begin{matrix} \overline{\imath_1} = \frac{(A_1 - A_2) + \sqrt{(A_1 - A_2)^2 + 4\,A_3A_4}}{2A_3} \\ \text{and} \\ \overline{\imath_2} = \frac{(A_1 - A_2) - \sqrt{(A_1 - A_2)^2 + 4\,A_3A_4}}{2A_3}, \end{matrix} \quad (4.9)$$

respectively. It is clear from (4.2) that $\overline{\imath_1}$ is always positive and $\overline{\imath_2}$ is always negative. Accordingly when (4.8) is provided, the positive equilibrium point $E_1(\overline{p_1}, \overline{\imath_1})$ is as follows; $E_1\left(\frac{(A_1+A_2)-\sqrt{(A_1-A_2)^2+4\,A_3A_4}}{2}, \frac{(A_1-A_2)+\sqrt{(A_1-A_2)^2+4\,A_3A_4}}{2A_3}\right)$.

Thus, as a result, the following table can be given.

Table 4.1. The biological existence conditions of equilibrium points of system (4.3)

Equilibrium Points	**Condition of biological existence**
$\mathrm{E}_0\left(0,\frac{\mathrm{A}_4}{\mathrm{A}_2}\right)$,	Always exists
$E_1\left(\frac{(A_1+A_2)-\sqrt{(A_1-A_2)^2+4\,A_3A_4}}{2},\frac{(\mathrm{A}_1-\mathrm{A}_2)+\sqrt{(\mathrm{A}_1-\mathrm{A}_2)^2+4\,\mathrm{A}_3\mathrm{A}_4}}{2\mathrm{A}_3}\mathrm{A}_3\mathrm{A}_4\right)$,	$\mathrm{A}_1\mathrm{A}_2-\ \mathrm{A}_3\mathrm{A}_4>0$

Proposition 4.2. *For the system (4.3), when equilibrium points shown in Table 4.1 are considered, the stability of these points are obtained as follows:*

(i) *If $A_1A_2-A_3A_4<0$, then the equilibrium point $E_0\left(0,\frac{A_4}{A_2}\right)$ is locally asymptotically stable (LAS).*

(ii) *We have assumed that the condition (4.8) be provided. In this situation, the positive equilibrium point E_1 is locally asymptotically stable (LAS).*

Proof.

For the stability analysis, Jacobian matrix obtained from the right side of the system (4.3) is assigned as:

$$\mathrm{J}=\begin{pmatrix}\frac{1}{\beta_\mathrm{P}}(\mathrm{A}_1-2\mathrm{p}-\mathrm{A}_3\mathrm{i}) & -\mathrm{A}_3\mathrm{p}\frac{1}{\beta_\mathrm{P}}\\ \omega\mathrm{i} & \omega(\mathrm{p}-\mathrm{A}_2)\end{pmatrix} \quad (4.10)$$

(i) Jacobian Matrix calculated in $\mathrm{E}_0\left(0,\frac{\mathrm{A}_4}{\mathrm{A}_2}\right)$ is as following

$$\mathrm{J}\left(\mathrm{E}_0\left(0,\frac{\mathrm{A}_4}{\mathrm{A}_2}\right)\right)=\begin{pmatrix}\beta_\mathrm{P}\frac{\mathrm{A}_1\mathrm{A}_2-\mathrm{A}_3\mathrm{A}_4}{\mathrm{A}_2} & 0\\ \omega\frac{\mathrm{A}_4}{\mathrm{A}_2} & -\omega\mathrm{A}_2\end{pmatrix} \quad (4.11)$$

Here, it is clear that eigenvalues are real number. Accordingly, for the stability of this equilibrium point, it is enough to look at Routh-Hurwitz criteria. Also, the characteristic equation belonging to this matrix is of second degree, since Jacobian Matrix in (4.11) is 2x2. As a result of this criteria, for eigenvalues to be negative, and so LAS of this point, the condition $\mathrm{TrJ}(\mathrm{E}_0)=\frac{\beta_\mathrm{B}\,(\mathrm{A}_1\mathrm{A}_2-\mathrm{A}_3\mathrm{A}_4)}{\mathrm{A}_2}-\mathrm{A}_2\,\omega<0$ and $\mathrm{DetJ}(\mathrm{E}_0)=-\omega\,\beta_\mathrm{B}\,(\mathrm{A}_1\mathrm{A}_2-\mathrm{A}_3\mathrm{A}_4)>0$ must be provided. Thus, if

$$\mathrm{A}_1\mathrm{A}_2-\mathrm{A}_3\mathrm{A}_4<0, \quad (4.12)$$

from aforementioned inequality with respect to $\mathrm{TrJ}(\mathrm{E}_0)$ and $\mathrm{DetJ}(\mathrm{E}_0)$,then $\mathrm{E}_0\left(0,\frac{\mathrm{A}_4}{\mathrm{A}_2}\right)$ is LAS.

(ii) We have the positive equilibrium point E_1 under condition $\mathrm{A}_1\mathrm{A}_2-\mathrm{A}_3\mathrm{A}_4>0$. Jacobian matrix calculated in E_1 is written as

$$J(E_1) = \begin{pmatrix} -\beta_S \bar{p} & -\sigma \bar{p} \\ \omega \bar{i} & -\frac{h}{\bar{i}} \end{pmatrix} \quad (4.13)$$

Characteristic equation obtained from (4.13) is as follows:

$$\lambda^2 + \left(\beta_P \bar{p} + \frac{h}{\bar{i}}\right)\lambda + \bar{p}\left(\omega\sigma\bar{i} + \beta_P \frac{h}{\bar{i}}\right) = 0 \quad (4.14)$$

When Lemma 2.1. is considered, as coefficients of (4.14) are positive real number. According to Routh-Hurwitz criteria, the eigenvalues λ are either negative real number or complex numbers having negative real parts. In this respect, when the positive equilibrium point E_1 biologically exists, it is also the stable equilibrium point.

Stability conditions of equilibrium points expressed in Table 4.1. are summarized in the following table.

Table 4.2. Stability conditions of equilibrium points of (4.3).

Equilibrium Point	**Stability Condition**
$E_0\left(0, \frac{A_4}{A_2}\right)$	$A_1A_2 - A_3A_4 < 0$
$E_1\left(\frac{(A_1+A_2)-\sqrt{(A_1-A_2)^2+4\,A_3A_4}}{2}, \frac{(A_1-A_2)+\sqrt{(A_1-A_2)^2+4\,A_3A_4}}{2A_3}A_3A_4\right)$.	When It exists as biological (that is, $A_1A_2 - A_3A_4 > 0$)

Definition 4.2. *For (4.3) system, the threshold parameter R_0, minimum infection free parameter, has the following property:*

If $R_0 < 1$, then equilibrium point E_0 is LAS, and if $R_0 > 1$, then positive equilibrium point E_1 is LAS. Here R_0 parameter is defined as following

$$R_0 = \frac{A_1A_2}{A_3A_4} \quad (4.15)$$

Table 4.3. According to the parameter R_0, stability conditions of equilibrium points of (4.3).

Equilibrium Point	**Stability Condition**
$E_0\left(0, \frac{A_4}{A_2}\right)$	$R_0 < 1$
$E_1\left(\frac{(A_1+A_2)-\sqrt{(A_1-A_2)^2+4\,A_3A_4}}{2}, \frac{(A_1-A_2)+\sqrt{(A_1-A_2)^2+4\,A_3A_4}}{2A_3}A_3A_4\right)$.	$R_0 > 1$

5. RESULTS

In this part, qualitative analysis of system is supported via numerical simulations by giving values to parameters used in system (3.2).

Table 5.1. For system (3.2), values and expressions of parameter values.

Parameters	Descriptions	Values
β_P	Growth rate of pathogen,	0.8 day^{-1}
C_p	Carrying capacity of pathogen,	10^9 cells
μ_P	Natural death rate of pathogen,	0.312 day^{-1}
μ_I	Natural death rate of immune cells,	0.1512 day^{-1}
$\bar{\sigma}$	The rate at which the immune system cells destroy pathogens,	$3*10^{-6}$ $cells^{-1}$ $days^{-1}$
$\bar{\omega}$	Proliferation rate of immune system cells in pathogenesis,	10-9 cells−1 days−1
H	Intensity of base production of immune system cells,	106 cells
α	The orders of the derivative in the system,	0.25, 0.50, 0.75, 0.9, 0.99
$[P_0\ I_0]$	Initial condition	[10000 1000]

In the light of data obtained from Table 5.1., it is founded as following

$$A_1 = 0.61,\ A_2 = 1.512,\ A_3 = 3750,\ A_4 = 0.001$$

and the equilibrium points

$$E_0(0{,}661375.7) \text{ and } E_1(-927316121.8{,}409951).$$

Because it is

$$A_1A_2 - A_3A_4 = -2.82768,$$

the positive equilibrium point E_1 is biologically meaningless and E_0 is locally asymptotically stable. This situation with different derivative orders is clearly seen in Figure 5.1 and Figure 5.2.

Figure 5.1. In case of $\alpha = 0{,}25, 0.50, 0.75, 0.90$ and 0.99 in system (3.2), respectively, temporal courses of pathogen obtained by using datas in the Table 5.1.

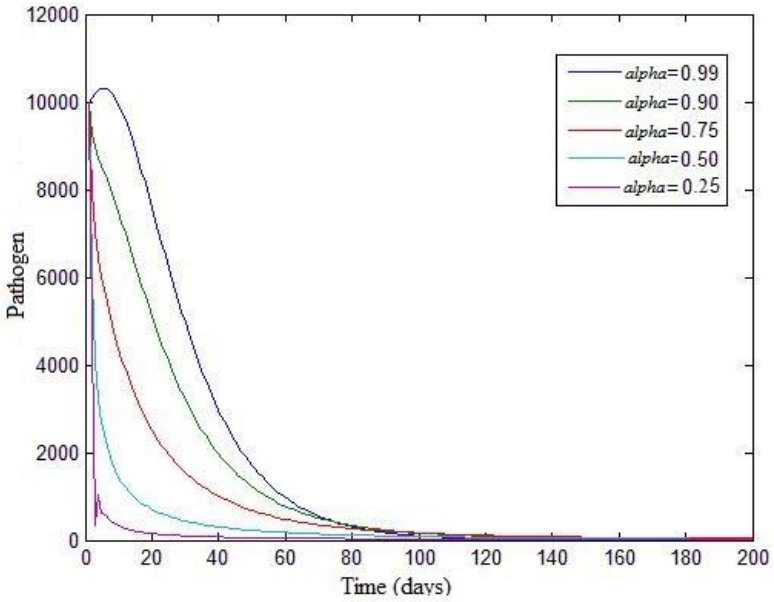
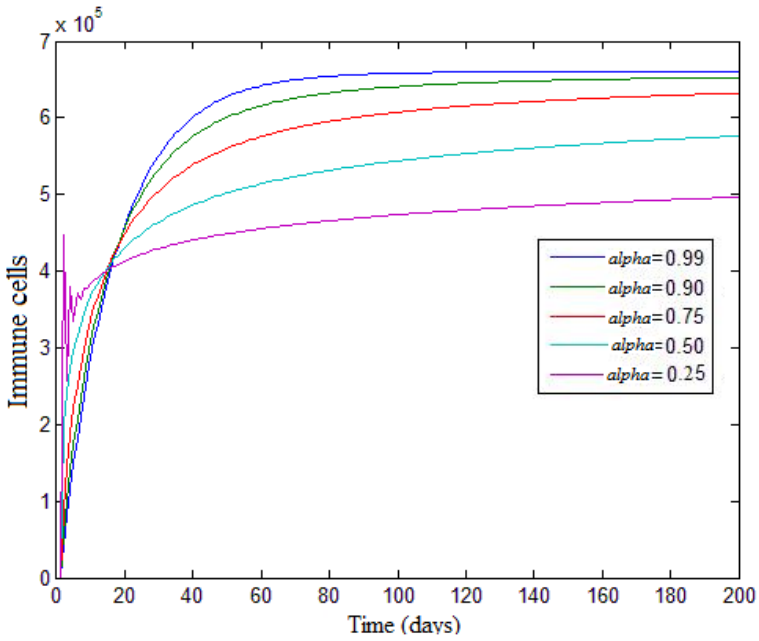


Figure 5.2. In case of $\alpha = 0{,}25, 0.50, 0.75, 0.90$ and 0.99 in system (3.2), respectively, temporal courses of Immune system cells obtained by using datas in the Table 5.1.



In the numerical studies in this section, the parameters obtained from the literature for mycobacterium tuberculosis were used. Within about 100 days, as can be seen from the figures above, the pathogen population disappears and the immune system cells increase.

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