CONSTRUCTIVE MATHEMATICAL ANALYSIS
2 (2019), No. 2, pp. 49-56
http://dergipark.gov.tr/cma
ISSN 2651-2939

On the Geometric Series of Linear Positive Operators

RADU PĂLTĂNEA

ABSTRACT. We study the existence and the norm of operators obtained as power series of linear positive operators with particularization to Bernstein operators. We also obtain a Voronovskaja-kind theorem.

Keywords: Positive linear operators, Geometric series of operators, Bernstein operators, Voronovskaja theorem

2010 Mathematics Subject Classification: 41A36, 41A25.

1. INTRODUCTION.

Let $L : C[0,1] \to C[0,1]$ be a positive linear operator. Denote by L^k, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the iterates of L, defined by $L^0 = I$, where I is the identical operator and $L^k = L \circ \ldots \circ L$, where L appears k times.
By geometric series of operator L we understand the series

$$G_L = \sum_{k=0}^{\infty} L^k. \tag{1.1}$$

The geometric series of operators were studied in [11], [1], [2], [3], [12]. The existence of these operators needs some restrictions of the domain of definition. It is necessary to consider some special subspaces of functions. Let $\psi : [0,1] \to \mathbb{R}$, $\psi(x) = x(1-x)$. The more simple is the space

$$\psi C[0,1] = \{f \in C[0,1] : \ \exists g \in C[0,1], \ f = \psi g\},$$

which is a Banach space if it is endowed with the norm

$$\|f\|_\psi := \sup_{x\in(0,1)} \frac{|f(x)|}{\psi(x)}, \tag{1.2}$$

where $f \in \psi C[0,1]$.
Denote by B_n the Bernstein operators. In [11] there is proved that operators $A_n : \psi C[0,1] \to \psi C[0,1]$, given by

$$A_n = \frac{1}{n}\sum_{k=0}^{\infty}(B_n)^k$$

are well defined and the following result is true:
Theorem A *For any $g \in C[0,1]$ we have*

$$\lim_{n\to\infty} \|A_n(\psi g) - 2F(g)\|_\psi = 0, \tag{1.3}$$

where

$$F(g)(x) = (1-x)\int_0^x tg(t)dt + x\int_x^1 (1-t)g(t)dt, \ x \in [0,1]. \tag{1.4}$$

Received: 31 December 2018; Accepted: 1 March 2019; Published Online: 8 March 2019
*Corresponding author: R. Păltănea; radupaltanea@yahoo.com
DOI: 10.33205/cma.506015

Note that $(F(g)(x))'' = -g(x)$, for $x \in [0,1]$. Because the convergence in norm $\|\cdot\|_\psi$ implies the convergence in sup-norm $\|\cdot\|$, ($\|f\| = \max_{x\in[0,1]}|f(x)|$), we have
Corollary A *For any $g \in C[0,1]$ we have*

$$\lim_{n\to\infty} \|A_n(\psi g) - 2F(g)\| = 0. \tag{1.5}$$

In [3], the geometric series are consider for a large class of operators, defined on an more extended space $C_\psi[0,1]$ given by

$$C_\psi[0,1] := \{f : [0,1] \to \mathbb{R} \mid \exists g \in B[0,1] \cap C(0,1) :\ f = \psi g\},$$

or equivalently by:

$$C_\psi[0,1] := \{f \in C[0,1] \mid \exists M > 0 :\ |f(x)| \le M\psi(x),\ x \in [0,1]\}.$$

Space $C_\psi[0,1]$ is also a Banach space with regard the norm $\|\cdot\|_\psi$, defined in (1.2), but is not a Banach space with respect the sup-norm, $\|\cdot\|$. Theorem A is generalized in this extended context and also an inverse Voronovskaja theorem is obtained.
A more general space is

$$C_0[0,1] = \{f \in C[0,1],\ f(0) = 0 = f(1)\},$$

endowed by the usual sup-norm $\|\cdot\|$. Clearly, $\Psi C[0,1] \subset C_\psi[0,1] \subset C_0[0,1]$, but the topologies are different.
In paper [12] the geometric series are considered for multidimensional Bernstein operators for a simplex, on the space of continuous functions which vanish at the vertexes. In the unidimensional case we obtain the space $C_0[0,1]$. The definition of geometric series of Bernstein operators is possible because the norms of operators B_n on space $C_0[0,1]$ are strictly less than 1.
The first aim of the present paper is to study the norm of the operators defined by geometric series and in the particular case the norm of the series of powers of Bernstein operators. This allow us to extend Theorem A on space $C_0[0,1]$. In the final section, we derive a Voronovskaja type theorem for the geometric series of Bernstein operators.
For a general theory on Bernstein operators see the papers [9], [6], [5]. For specific problems regarding Voronovskaja theorem we indicate the papers [4], [7]. For general methods of estimating the degree of approximation we mention [10] and [8].

2. Preliminaries

Lemma 2.1. *We have:*

i) *$C_0[0,1]$ is a Banach space with regard the norm $\|\cdot\|$.*
ii) *With regard to the norm $\|\cdot\|$ we have:*

$$\overline{\psi C[0,1]} = C_0[0,1].$$

Proof. i) It is immediate.
ii) If $f \in C_0[0,1]$, then $B_n(f) \in \psi C[0,1]$, for $n \in \mathbb{N}$, where B_n are the Bernstein operators. Since $\lim_{n\to\infty}\|f - B_n(f)\| = 0$, it follows $f \in \overline{\psi C[0,1]}$. The inverse inclusion follows since, obviously $\psi C[0,1] \subset C_0[0,1]$ and $C_0[0,1]$ is closed. □

Lemma 2.2. *If $L : C[0,1] \to C[0,1]$ satisfies condition $L(e_j) = e_j$, $j = 0,1$, then*

$$L(C_0[0,1]) \subset C_0[0,1].$$

Proof. It is well known that an operator $L : C[0,1] \to C[0,1]$ which satisfies the given condition has the property $L(f)(0) = f(0)$ and $L(f)(1) = f(1)$, for any $f \in C[0,1]$. □

Definition 2.1. *Denote by $\Lambda_0[0,1]$, the class of positive linear operators $L : C[0,1] \to C[0,1]$ which satisfy the following conditions:*

a) $L(e_j) = e_j$, *for* $j = 0, 1$;
b) $\|L\|_{\mathcal{L}(C_0[0,1],C_0[0,1])} < 1$.

Lemma 2.3. *For any $L \in \Lambda_0[0,1]$ we have:*

i) *operator $G_L : C_0[0,1] \to C_0[0,1]$, given in (1.1) is well defined if we consider the convergence with regard to the sup-norm $\|\cdot\|$;*
ii) *operator G_L is positive and linear;*
iii) $(I - L) \circ G_L = I$, *in the Banach space* $(C_0[0,1], \|\cdot\|)$;
iv) $G_L \circ (I - L) = I$, *in the Banach space* $(C_0[0,1], \|\cdot\|)$.

Proof. i) Because the series $\sum_{k=0}^{\infty} \|L^k\|_{\mathcal{L}(C_0[0,1],C_0[0,1])}$ is convergent it follows that for each $f \in C_0[0,1]$, series $\sum_{k=0}^{\infty} L^k(f)$ is convergent in space $(C_0[0,1], \|\cdot\|)$.
Point ii) is obvious. The proof of points iii) and iv) is standard. □

3. THE NORM OF OPERATORS G_L

In this section we give estimates of the norm $\|G_L\|_{\mathcal{L}(C_0[0,1],C_0[0,1])}$ for operators $L \in \Lambda_0[0,1]$.
In the next lemma, for $x \in (0,1)$ we make the following conventions. If $t = 1$, then $\int_x^t \frac{t-u}{u(1-u)}\,du = \int_x^1 \frac{du}{u}$ and if $t = 0$, then $\int_x^t \frac{t-u}{u(1-u)}\,du = \int_0^x \frac{du}{1-u}$.

Lemma 3.4. *For all $x \in (0,1)$ and $t \in [0,1]$ we have*

$$0 \le \int_x^t \frac{t-u}{u(1-u)}\,du \le \frac{(t-x)^2}{x(1-x)}. \tag{3.6}$$

Proof. The left inequality is clear. For the second one first we consider that $0 < x \le t \le 1$. For a fixed $t \in [0,1]$ we have

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{t-u}{u(1-u)}\right) = \frac{-u^2 + 2ut - t}{u^2(1-u)^2} \le -\frac{(t-u)^2}{u^2(1-u)^2} \le 0.$$

From this it follows relation (3.6). The case $0 \le t \le x < 1$ can be reduced to the case above. Indeed if we made the chang of variable $u_1 = 1 - u$ and denote $x_1 = 1 - x$, $t_1 = 1 - t$ then we obtain

$$\int_x^t \frac{t-u}{u(1-u)}\,du = \int_{x_1}^{t_1} \frac{t_1 - u_1}{u_1(1-u_1)}\,du_1 \le \frac{(t_1-x_1)^2}{x_1(1-x_1)} = \frac{(t-x)^2}{x(1-x)}.$$

□

Consider function $\Phi \in C_0[0,1]$, defined by

$$\Phi(x) = x\ln x + (1-x)\ln(1-x),\ x \in (0,1),\ \Phi(0) = 0,\ \Phi(1) = 0. \tag{3.7}$$

Theorem 3.1. *If $L \in \Lambda_0[0,1]$, then*

$$\|G_L\|_{\mathcal{L}(C_0[0,1],C_0[0,1])} \ge \frac{\|\Phi\|}{\alpha_L} = \frac{\ln 2}{\alpha_L}, \tag{3.8}$$

where

$$\alpha_L = \sup_{x\in(0,1)} \frac{L((e_1 - x)^2)(x)}{\psi(x)}. \tag{3.9}$$

Proof. For $x \in (0,1)$ and $t \in [0,1]$, the Taylor formula yields

$$\Phi(t) = \Phi(x) + \Phi'(x)(t-x) + \int_x^t (t-u)\Phi''(u)du.$$

Since $\Phi''(u) = \frac{1}{u(1-u)}$, $u \in (0,1)$, by taking into account Lemma 3.4 we obtain

$$\Phi(t) \quad \leq \quad \Phi(x) + \Phi'(x)(t-x) + \frac{(t-x)^2}{x(1-x)}.$$

Applying operator L we obtain

$$L(\Phi)(x) \leq \Phi(x) + \alpha_L.$$

We use the immediate equality $L((e_1 - x)^2)(x) = L(e_2)(x) - e_2(x)$ and the equalities $L(\Phi)(0) - \Phi(0) = 0$ and $L(\Phi)(1) - \Phi(1) = 0$. Since function Φ is convex and L preserves linear functions we have $L(\Phi) - \Phi \geq 0$. From these we deduce that $\frac{1}{\alpha_L}(L(\Phi) - \Phi) \in C_0[0,1]$ and $\left\|(\alpha_L)^{-1}(L(\Phi) - \Phi)\right\| \leq 1$. Therefore

$$\|G_L\|_{\mathcal{L}(C_0[0,1],C_0[0,1])} \geq \|G_L((\alpha_L)^{-1}(L(\Phi) - \Phi))\|.$$

But using Lemma 2.3 - iv) we obtain

$$G_L(L(\Phi) - \Phi) = -\Phi.$$

Consequently we obtain relation (3.8). □

4. Convergence of geometric series of Bernstein operators in the space $C_0[0,1]$

Let B_n, $n \in \mathbb{N}$ be the Bernstein operators. It is clear that $B_n \in \Lambda_0[0,1]$, for any $n \in \mathbb{N}$, see [13]. From Lemma 2.3, G_{B_n} is well defined on space $C_0[0,1]$.

Theorem 4.2. *For $n \in \mathbb{N}$, $n \geq 2$ we have*

$$n \ln 2 \leq \|G_{B_n}\|_{\mathcal{L}(C_0[0,1],C_0[0,1])} \leq 1 + 3n\ln 2. \tag{4.10}$$

Proof. For simplicity let denote $G_n = G_{B_n}$. The left inequality follows from Theorem 3.1, by taking into account that $\alpha_{B_n} = \frac{1}{n}$, for $n \in \mathbb{N}$.
We pass to the right inequality. Let $x \in (0,1)$ we have

$$\Phi''(x) = \frac{1}{x(1-x)}, \quad \Phi^{(3)}(x) = \frac{2x-1}{x^2(1-x)^2}, \quad \Phi^{(4)}(x) = \frac{2(1-3\Psi(x))}{\Psi^3(x)}.$$

Since $\Phi^{(4)} \geq 0$, using the Taylor formula for $x \in (0,1)$, $t \in [0,1]$:

$$\begin{aligned} \Phi(t) &= \sum_{k=0}^{3} \frac{\Phi^{(k)}(x)(t-x)^k}{k!} + \int_x^t \frac{(t-u)^3}{3!}\Phi^{(4)}(u)du \\ &\geq \sum_{k=0}^{3} \frac{\Phi^{(k)}(x)(t-x)^k}{k!} \end{aligned}$$

We have $B_n((e_1 - xe_0)^3)(x) = \frac{1}{n^2}(1-2x)x(1-x)$. Applying operator B_n we obtain:

$$B_n(\Phi)(x) \geq \Phi(x) + \frac{1}{2n} - \frac{1}{6n^2} \cdot \frac{(1-2x)^2}{x(1-x)}.$$

Take here $x = \frac{k}{n}$, $1 \leq k \leq n-1$. We obtain, for $n \geq 2$:

$$\max_{1\leq k\leq n-1} \frac{1}{6n^2} \cdot \frac{\left(1 - 2\frac{k}{n}\right)^2}{\frac{k}{n}\left(1 - \frac{k}{n}\right)} = \frac{1}{6n^2} \cdot \frac{\left(1 - \frac{2}{n}\right)^2}{\frac{1}{n}\left(1 - \frac{1}{n}\right)} \leq \frac{1}{6n}.$$

Hence

$$B_n(\Phi)\Big(\frac{k}{n}\Big) - \Phi\Big(\frac{k}{n}\Big) \geq \frac{1}{3n},\ 1 \leq k \leq n-1. \tag{4.11}$$

Since $G_n = I + G_n \circ B_n$ we obtain

$$\|G_n\|_{\mathcal{L}(C_0[0,1],C_0[0,1])} \leq 1 + \|G_n \circ B_n\|_{\mathcal{L}(C_0[0,1],C_0[0,1])}. \tag{4.12}$$

Fix $f_0 \in C_0[0,1]$ arbitrary such that, $f_0 \geq 0$ and $f_0\left(\frac{k}{n}\right) = 1$, $1 \leq k \leq n-1$. It is easy to see that

$$\|G_n \circ B_n\|_{\mathcal{L}(C_0[0,1],C_0[0,1])} = \|G_n(B_n(f_0))\|. \tag{4.13}$$

From relation (4.11) and since $f_0(0) = 0 = f_0(1)$ and $(B_n(\Phi) - \Phi)(0) = 0 = (B_n(\Phi) - \Phi)(1)$ it follows that

$$f_0\Big(\frac{k}{n}\Big) \leq 3n\left[B_n(\Phi)\Big(\frac{k}{n}\Big) - \Phi\Big(\frac{k}{n}\Big)\right],\ 0 \leq k \leq n$$

and from this we obtain

$$B_n(f_0) \leq 3nB_n(B_n(\Phi) - \Phi).$$

Applying operator G_n to this inequality we arrive to

$$G_n(B_n(f_0)) \leq 3nG_n \circ (B_n - I)(B_n(\Phi)).$$

By tacking into account Lemma 2.3 - iv) we get

$$G_n(B_n(f_0)) \leq -3nB_n(\Phi).$$

Now, since $f_0 \geq 0$ it follows $G_n(B_n(f_0)) \geq 0$ and from the inequality above we obtain

$$\|G_n(B_n(f_0))\| \leq 3n\|B_n(\Phi)\| \tag{4.14}$$

From relations (4.12), (4.13), (4.14) and inequality $\|B_n(\Phi)\| \leq \|\Phi\|$ we deduce relation (4.15). □

Lemma 4.5. *Let F be the operator defined in relation (1.4). We have:*

i) $F\left(\psi^{-1}\right) = -\Phi$.
ii) *If $f \in C[0,1]$, then $F(\psi^{-1}f)$ is well defined and $F(\psi^{-1}f) \in C_0[0,1]$.*

Proof. i) It follows by a simple direct calculus.
ii) Let $x \in (0,1)$. Then $0 \leq F(\psi^{-1}|f|)(x) \leq \|f\|F(\psi^{-1})(x) = -\|f\|\Phi(x) < \infty$. Since $F(\psi^{-1}|f|)$ is well defined it follows that $F(\psi^{-1}f)$ is well defined. Also, from the inequality above it follows that $F(\psi^{-1}f) \in C_0[0,1]$. □

According to notations used in the previous sections we have $A_n = \frac{1}{n}G_{B_n}$, $n \in \mathbb{N}$.

Theorem 4.3. *We have*

$$\lim_{n\to\infty} \|n^{-1}G_{B_n}(f) - 2F(\psi^{-1}f)\| = 0,\ \textit{for all } f \in C_0[0,1]. \tag{4.15}$$

Proof. Let $f \in C_0[0,1]$. Let $\varepsilon > 0$ be arbitrarily chosen. Since the space $\psi C[0,1]$ is dense in $C_0[0,1]$ (Lemma 2.1), we can find $g \in \psi C[0,1]$ such that $\|f - g\| < \varepsilon$. From Corollary A there is $n_\varepsilon \in \mathbb{N}$ such that $\|n^{-1}G_{B_n}(g) - 2F(\psi^{-1}g))\| < \varepsilon$, for $n \geq n_\varepsilon$. Then, for such index n we obtain

$$\begin{aligned}
&\|n^{-1}G_{B_n}(f) - 2F(\psi^{-1}f)\| \\
&\leq \|n^{-1}G_{B_n}(f-g)\| + \|n^{-1}G_{B_n}(g) - 2F(\psi^{-1}g)\| + \|2F(\psi^{-1}(f-g))\| \\
&\leq n^{-1}\|G_{B_n}\|_{\mathcal{L}(C_0[0,1],C_0[0,1])}\|f-g\| + \varepsilon + \|f-g\| \cdot \|2F(\psi^{-1})\| \\
&\leq (n^{-1} + 3\ln 2)\varepsilon + \varepsilon + 2\ln 2 \cdot \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the proof is finished. □

5. Voronovskaya type result

Recall that $A_n = \frac{1}{n}\sum_{k=0}^{\infty}(B_n)^k$, where B_n is the Bernstein operator of order n.

Theorem 5.4. *If $f \in C^4[0,1]$, then*

$$\lim_{n\to\infty} n(A_n(\psi f)(x) - F(f)(x)) = \frac{1}{2}\psi(x)f(x) - \frac{1}{3}F(f)(x), \tag{5.16}$$

uniformly with regard to $x \in [0,1]$.

Proof. Fix $f \in C^4[0,1]$ and denote $F(t) = F(f)(t)$, $t \in [0,1]$. Because $F''(t) = -f(t)$, we have from Taylor formula, for $s,t \in [0,1]$:

$$\begin{aligned} F(s) &= F(t) + F'(t)(s-t) - \frac{1}{2}f(t)(s-t)^2 - \frac{1}{6}f'(t)(s-t)^3 \\ &\quad -\frac{1}{24}f''(t)(s-t)^4 - \frac{1}{120}f'''(t)(s-t)^5 - R_5(t,s), \end{aligned} \tag{5.17}$$

where

$$R_5(t,s) = \frac{1}{5!}\int_t^s (s-u)^5 f^{IV}(u)du.$$

Denote $m_k(t) = B_n((e_1-t)^k)(t)$, for $k = 0,1,2,\ldots$, $t \in [0,1]$. In [9] the following relation is given:

$$m_{s+1}(t) = \frac{\psi(t)}{n}\left[m_s'(t) + s m_{s-1}(t)\right],\ s = 1,2,\ldots,\ t \in [0,1].$$

we obtain

$$\begin{aligned} m_2(t) &= \frac{1}{n}\psi(t) \\ m_3(t) &= \frac{1}{n^2}\psi(t)\psi'(t), \\ m_4(t) &= \frac{3}{n^2}\psi^2(t) + \frac{1}{n^3}\psi(t)(1-6\psi(t)), \\ m_5(t) &= \frac{10}{n^3}\psi^2(t)\psi'(t) + \frac{1}{n^4}(\psi(t)\psi'(t) - 12\psi^2(t)\psi'(t)), \\ m_6(t) &= \frac{15}{n^3}\psi^3(t) + \frac{1}{n^4}(24\psi^2(t) - 124\psi^3(t)) + \frac{1}{n^5}(\psi(t) - 28\psi^2(t) + 120\psi^3(t)). \end{aligned}$$

Applying operator B_n from relation (5.17) we obtain

$$\begin{aligned} (I-B_n)(F)(t) &= \frac{1}{2}f(t)m_2(t) + \frac{1}{6}f'(t)m_3(t) + \frac{1}{24}f''(t)m_4(t) + \frac{1}{120}f'''(t)m_5(t) \\ &\quad +B_n(R_5(t,\cdot))(t). \end{aligned}$$

Note that $F \in \psi C[0,1]$. Also $m_k \in \psi C[0,1]$, $k = 2,3,4,5$. From the above equality it follows that also $B_n(R_5(t,\cdot))(t) \in \psi C[0,1]$. So that we can apply operator $G_{B_n} = nA_n$ to the tems of the both side of above equality and from Lemma 2.3 - iv), we obtain

$$\begin{aligned} F(x) &= \frac{1}{2}A_n(f\psi)(x) + \frac{n}{6}A_n(f'm_3)(x) + \frac{n}{24}A_n(f''m_4)(x) + \frac{n}{120}A_n(f'''m_5)(x) \\ &\quad +nA_n(B_n(R_5(t,\cdot))(t))(x), \end{aligned}$$

Finally we obtain

$$\begin{aligned} n(A_n(f\psi)(x) - 2F(x)) &= -\frac{1}{3}n^2A_n(f'm_3)(x) - \frac{1}{12}n^2A_n(f''m_4)(x) - \frac{n^2}{60}A_n(f'''m_5)(x) \\ &\quad -2n^2A_n(B_n(R_5(t,\cdot))(t))(x), \end{aligned}$$

Using Corollary A we obtain

$$\begin{aligned} -\frac{1}{3}n^2A_n(f'm_3)(x) &= -\frac{1}{3}A_n(f'\psi\psi')(x) \\ &= -\frac{2}{3}F(f'\psi')(x) + o(1); \\ -\frac{1}{12}n^2A_n(f''m_4)(x) &= -\frac{1}{4}A_n(f''\psi^2)(x) - \frac{1}{12n}A_n(f''\psi(1-6\psi))(x) \\ &= -\frac{1}{4}(2F(f''\psi)(x) + o(1)) - \frac{1}{12n}(2F(f''(1-6\psi))(x) + o(1)) \\ &= -\frac{1}{2}F(f''\psi)(x) + o(1); \\ -\frac{n^2}{60}A_n(f'''m_5)(x) &= -\frac{1}{6n}A_n(f'''\psi^2\psi')(x) - \frac{1}{60n^2}A_n(f'''(\psi\psi' - 12\psi^2\psi'))(x) \\ &= -\frac{1}{6n}(2F(f'''\psi\psi')(x) + o(1)) - \frac{1}{60n^2}(2F(f'''(\psi' - 12\psi\psi')) + o(1)) \\ &= o(1). \end{aligned}$$

In all these relations $o(1)$ is uniform with regard to $x \in [0,1]$. Also we have

$$|R_5(t,s)| \le \frac{\|f^{IV}\|}{5!}\int_t^s (s-u)^5 du = \frac{(s-t)^6}{6!}\|f^{IV}\|.$$

Therefore

$$B_n(|R_5(t,\cdot)|)(t) \le \frac{1}{6!}m_6(t)\|f^{IV}\|.$$

It follows

$$\begin{aligned} &|-2n^2A_n(B_n(R_5(t,\cdot))(t))(x)| \le \frac{2n^2}{6!}\|f^{IV}\|A_n(m_6)(x) \\ \le\ & \frac{2\|f^{IV}\|}{6!}A_n\left(\frac{15}{n}\psi^3 + \frac{1}{n^2}(24\psi^2 - 124\psi^3) + \frac{1}{n^3}(\psi - 28\psi^2 + 120\psi^3\right)(x) \\ =\ & \frac{2\|f^{IV}\|}{6!}\left[2F\left(\frac{15}{n}\psi^2 + \frac{1}{n^2}(24\psi - 124\psi^2) + \frac{1}{n^3}(e_0 - 28\psi + 120\psi^2\right)(x) + o(1)\right] \\ =\ & o(1). \end{aligned}$$

From the relation above we coclude that

$$\lim_{n\to\infty} n(A_n(f\psi)(x) - 2F(x)) = -\frac{2}{3}F(f'\psi')(x) - \frac{1}{2}F(f''\psi)(x). \tag{5.18}$$

Next integrating by parts we obtain

$$\begin{aligned} -\frac{2}{3}F(f'\psi')(x) &= -\frac{2}{3}\left[(1-x)\int_0^x t(1-2t)f'(t)dt + x\int_x^1 (1-t)(1-2t)f'(t)dt\right] \\ &= \frac{2}{3}(1-x)\int_0^x (1-4t)f(t)dt + \frac{2}{3}x\int_x^1 (4t-3)f(t)dt. \end{aligned}$$

$$
\begin{aligned}
-\frac{1}{2}F(f''\psi)(x) &= -\frac{1}{2}\left[(1-x)\int_0^x t^2(1-t)f''(t)dt + x\int_x^1 t(1-t)^2 f''(t)dt\right] \\
&= \frac{1}{2}(1-x)\int_0^x (2t-3t^2)f'(t)dt + \frac{1}{2}x\int_x^1 (1-4t+3t^2)f'(t)dt \\
&= \frac{1}{2}f(x)\psi(x) + (1-x)\int_0^x (3t-1)f(t)dt + x\int_x^1 (2-3t)f(t)dt.
\end{aligned}
$$

Hence

$$
-\frac{2}{3}F(f'\psi')(x) - \frac{1}{2}F(f''\psi)(x) = \frac{1}{2}\psi(x)f(x) - \frac{1}{3}\left[(1-x)\int_0^x (1-t)f(t)dt + x\int_x^1 tf(t)dt\right].
$$

□

References

[1] U. Abel, *Geometric series of Bernstein-Durrmeyer operators*, East J. on Approx. Vol. 15, No. 4 (2009) 439–450.

[2] U. Abel, M. Ivan, R. Păltănea, *Geometric series of Bernstein operators revisited*, J. Math. Anal. Appl. Vol. 400. No. 1 (2013) 22-24.

[3] U. Abel, M. Ivan, R. Păltănea, *Geometric series of positive linear operators and the inverse Voronovskaya theorem on a compact interval*, J. Approx. Theory Vol. 184 (2014), 163-175.

[4] F. Altomare, S. Diomede, *Asymptotic formulae for positive linear operators: direct and converse results*, Jaen J. Approx. Vol. 2, No. 2 (2010) 255–287.

[5] J. Bustamante, *Bernstein Operators and Their Properties*, Birkhäuser, 2017.

[6] R. A. DeVore, G. G. Lorentz, *Constructive approximation*, Springer, Berlin, 1993.

[7] H Gonska, R. Păltănea, *General Voronovskaja and asymptotic theorems in simultaneous approximation*, Mediterranean J. Math. Vol. 7 (2010) 37-49.

[8] V. Gupta, G. Tachev, *Approximation with Positive Linear Operators and Linear Combinations*, Springer, 2017.

[9] G. G. Lorenz, *Bernstein polynomials*, Univ. Toronto Press, 1953.

[10] R. Păltănea, *Approximation Theory Using Positive Linear Operators*, Birkhäuser, Boston, 2004.

[11] R. Păltănea, *The power series of Bernstein operators*, Automation Computers Applied Mathematics Vol. 15, No. 1 2006, 7-14.

[12] I. Raşa, *Power series of Bernstein operators and approximation resolvents* Mediterr. J. Math. Vol. 9 (2012) 635-644.

[13] I. A. Rus, *Iterates of Bernstein operators, via contraction principle*, J. Math. Anal. Appl. Vol 292, No. 1 (2004) 259-261.

TRANSILVANIA UNIVERSITY OF BRAŞOV,
FACULTY OF MAHEMATICS AND INFORMATICS,
RO- 500091, BRAŞOV,
ROMANIA
E-mail address: radupaltanea@yahoo.com