Results in Nonlinear Analysis 2 (2019) No. 1, 18–24
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Common fixed point results in extended b-metric spaces endowed with a directed graph

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Abstract

The purpose of this paper is to obtain some fixed point results in extended b-metric spaces for Reich-Rus and Ciric operators.

Keywords: extended b-metric spaces, fixed point, contraction, Reich-Rus and Ciric operators
2010 MSC: 47H10, 54H25.

1. Introduction

Definition 1.1. ([3]) Let X be a nonempty set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to [0, \infty)$ is said to be a b-metric with constant s, if all axioms of the metric space take place with the following modification of the triangle axiom:

$$d(x, z) \leq s[d(x, y) + d(y, z)], \text{ for all } x, y, z \in X.$$

In this case the pair (X, d) is called a b-metric space with constant s.

Remark 1.2. The class of b-metric spaces is larger than the class of metric spaces since a b-metric space is a metric space when s=1. For more details and examples on b-metric spaces, see e.g. [2].

Example 1.3. Let $X = \mathbb{R}_+$ and $d : X \times X \to \mathbb{R}_+$ such that $d(x, y) = |x - y|^p, p > 1$. It's easy to see that d is a b-metric with $s = 2^p$, but is not a metric.

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Received january 5, 2019, February 21, 2019, Online March 12, 2019.

Definition 1.4. ([4]) Let X be a nonempty set and let $\theta : X\times X \to [1,\infty)$. A functional $d_\theta : X\times X \to [0,\infty)$ is said to be an extended b-metric, if for all $x, y, z \in X$, the following axioms are satisfied:

1. $d_\theta(x,y) = 0 \Longleftrightarrow x = y$;
2. $d_\theta(x,y) = d_\theta(y,x)$;
3. $d_\theta(x,z) \leq \theta(x,y)[d_\theta(x,y) + d_\theta(y,z)]$, for all $x, y, z \in X$

In this case the pair (X, d_θ) is called an extended b-metric space.

Remark 1.5. If $\theta(x,y) = s$, for $s \geq 1$, then we obtain the definition of $b-$metric space.

Example 1.6. ([4]) Let $X = \{1,2,3\}, \theta : X\times X \to [1,\infty)$ and $d_\theta : X\times X \to [0,\infty)$ as:

$$\begin{aligned}
\theta(x,y) &= 1+x+y \\
d_\theta(1,1) &= d_\theta(2,2) = d_\theta(3,3) = 0, \\
d_\theta(1,2) &= d_\theta(2,1) = 80, \\
d_\theta(1,3) &= d_\theta(3,1) = 1000, \\
d_\theta(2,3) &= d_\theta(3,2) = 600.
\end{aligned}$$

Then d_θ is an extended b-metric.

Example 1.7. ([1]) Let $X = [0,1], \theta : X\times X \to [1,\infty)$ and $d_\theta : X\times X \to [0,\infty)$ as:

$$\begin{aligned}
\theta(x,y) &= \frac{1+x+y}{x+y} \\
d_\theta(x,y) &= \frac{1}{xy}, x,y \in (0,1], x \neq y, \\
d_\theta(x,y) &= 0, x,y \in [0,1], x = y, \\
d_\theta(x,0) &= d_\theta(0,x) = \frac{1}{x}, x \in (0,1].
\end{aligned}$$

Then d_θ is an extended b-metric.

The following Lemma is very important in the proof of our results:

Lemma 1.8. *([1]) Let (X, d_θ) an extended b-metric space. If there exists $q \in [0,1)$, such that the sequence $(x_n)_{n\in\mathbb{N}} \subset X$, for an arbitrary $x_0 \in X$, satisfies $\lim\limits_{m,n\to\infty} \theta(x_m, x_n) < \frac{1}{q}$ and*

$$0 < d_\theta(x_n, x_{n+1}) \leq q d_\theta(x_n, x_{n+1}), \text{ for any } n \in \mathbb{N}$$

then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Let (X, d_θ) be an extended $b-$metric space and Δ be the diagonal of $X \times X$. Let G be a directed graph, such that the set $V(G)$ of its vertices coincides with X and $\Delta \subseteq E(G)$, where $E(G)$ is the set of the edges of the graph. Assume also that G has no parallel edges and, thus, one can identify G with the pair $(V(G), E(G))$.

Throughout the paper we shall say that G with the above mentioned properties *satisfies standard conditions.*

Let us denote by G^{-1} the graph obtained from G by reversing the direction of edges. Thus,

$$E(G^{-1}) = \{(x,y) \in X\times X : (y,x) \in E(G)\}.$$

Let us consider the mappings $T, S : X \to X$.

Definition 1.9. An element $x \in X$ is called *common fixed point* of the pair (T, S), if $T(x) = S(x) = x$.

We shall denote by $CFix(T, S)$ the set of all common fixed points of the pair (T, S), i.e.

$$CFix(T,S) = \{x \in X : T(x) = S(x) = x\}.$$

Definition 1.10. Suppose that $T, S : X \to X$ are two mappings on an extended b-metric space (X, d_θ) endowed with a directed graph G. We say that the pair (T, S) is G-orbital-cyclic pair, if for any $x \in X$,

$$\begin{aligned} (x, Tx) &\in E(G) \Longrightarrow (Tx, STx) \in E(G) \\ (x, Sx) &\in E(G) \Longrightarrow (Sx, TSx) \in E(G). \end{aligned}$$

Let us consider the following sets

$$\begin{aligned} X^T &= \{x \in X : (x, Tx) \in E(G)\} \\ X^S &= \{x \in X : (x, Sx) \in E(G)\} \end{aligned}$$

Remark 1.11. If the pair (T, S) is G-orbital-cyclic pair, then $X^T \neq \varnothing \Longleftrightarrow X^S \neq \varnothing$.

Proof. Let $x_0 \in X^T$. Then $(x_0, Tx_0) \in E(G) \Longrightarrow (Tx_0, STx_0) \in E(G)$.
If we denote by $x_1 = Tx_0$ we have that $(x_1, Sx_1) \in E(G)$, and thus, $X^S \neq \varnothing$. □

2. Reich-Rus type operators

Theorem 2.1. *Let T, S be two self-mappings on a complete extended b-metric space (X, d_θ) endowed with a directed graph G such that the pair (T, S) forms a G-orbital-cyclic pair. Suppose that*

(i) $X^T \neq \varnothing$;

(ii) for all $x \in X^T$ and $y \in X^S$ and $k_1, k_2, k_3 > 0$, with $k_1 + k_2 + k_3 < 1$

$$d_\theta(Tx, Sy) \leq k_1 d_\theta(x, y) + k_2 d_\theta(x, Tx) + k_3 d_\theta(y, Sy);$$

(iii) for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$, with $(x_n, x_{n+1}) \in E(G)$,

$$\lim_{n,m\to\infty} \theta(x_n, x_m) < \frac{1}{q}, \text{ where } q = \max\left\{\frac{k_1 + k_2}{1 - k_3}, \frac{k_1 + k_3}{1 - k_2}\right\};$$

(iv) S and T are continuous,

or

(iv) for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$, with $x_n \to u$ as $n \to \infty$, and $(x_n, x_{n+1}) \in E(G)$, for $n \in \mathbb{N}$, we have, $u \in X^T \cap X^S$.*

In these conditions $CFix(T, S) \neq \varnothing$.
Moreover, if we suppose

(v) if $(u, v) \in CFix(T, S)$ implies $u \in X^T$ and $v \in X^S$ then the pair (T, S) has a unique common fixed point.

Proof. Let $x_0 \in X^T$. Thus $(x_0, Tx_0) \in E(G)$.

Because the pair (T,S) is G-orbital-cyclic , we have $(Tx_0, STx_0) \in E(G)$.

If we denote by $x_1 = Tx_0$ we have $(x_1, Sx_1) \in E(G)$ and from here $(Sx_1, TSx_1) \in E(G)$. Denoting by $x_2 = Sx_1$ we have $(x_2, Tx_2) \in E(G)$.

By this procedure we construct inductively, a sequence $(x_n)_{n\in\mathbb{N}}$, with $x_{2n} = Sx_{2n-1}$ and $x_{2n+1} = Tx_{2n}$, such that $(x_{2n}, x_{2n+1}) \in E(G)$.

We shall suppose that $x_n \neq x_{n+1}$.

If, there exists $n_0 \in \mathbb{N}$, such that $x_{n_0} = x_{n_0+1}$, then, because $\Delta \subset E(G), (x_{n_0}, x_{n_0+1}) \in E(G)$ and $u = x_{n_0}$ is a fixed point of T.

In order to show that $u \in CFix(T,S)$, we shall consider two cases for n_0.

If $n_0 = 2k$, then $x_{2k} = x_{2k+1} = Tx_{2k}$ an thus, x_{2k} is a fixed point for T. Suppose that $d_\theta(Tx_{2k}, Sx_{2k+1}) > 0$, and let $x = x_{2k} \in X^T$ and $y = x_{2k+1} \in X^S$.

$$\begin{aligned} 0 &< d_\theta(x_{2k+1}, x_{2k+2}) = d_\theta(Tx_{2k}, Sx_{2k+1}) \leq k_1 d_\theta(x_{2k}, x_{2k+1}) + k_2 d_\theta(x_{2k}, Tx_{2k}) + k_3 d_\theta(x_{2k+1}, Sx_{2k+1}) \\ &= k_3 d_\theta(x_{2k+1}, Sx_{2k+1}) = k_3 d_\theta(x_{2k+1}, x_{2k+2}). \end{aligned}$$

In this way we reach to a contradiction.

In the same way we can prove the case $n_0 = 2k+1$.

In conclusion $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$.

Now we shall prove that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. In order to do this, we shall consider two possible cases:

Case 1. $x = x_{2n} \in X^T$ and $y = x_{2n+1} \in X^S$.

$$\begin{aligned} 0 &< d_\theta(x_{2n+1}, x_{2n+2}) = d_\theta(Tx_{2n}, Sx_{2n+1}) \\ &\leq k_1 d_\theta(x_{2n}, x_{2n+1}) + k_2 d_\theta(x_{2n}, Tx_{2n}) + k_3 d_\theta(x_{2n+1}, Sx_{2n+1}) \\ &= k_1 d_\theta(x_{2n}, x_{2n+1}) + k_2 d_\theta(x_{2n}, x_{2n+1}) + k_3 d_\theta(x_{2n+1}, x_{2n+2}) \end{aligned}$$

$$\begin{aligned} (1-k_3) d_\theta(x_{2n+1}, x_{2n+2}) &\leq (k_1+k_2) d_\theta(x_{2n}, x_{2n+1}) \\ d_\theta(x_{2n+1}, x_{2n+2}) &\leq \frac{k_1+k_2}{1-k_3} d_\theta(x_{2n}, x_{2n+1}) \\ d_\theta(x_{2n+1}, x_{2n+2}) &\leq q d_\theta(x_{2n}, x_{2n+1}). \end{aligned}$$

Case 2. $x = x_{2n} \in X^T$ and $y = x_{2n-1} \in X^S$.

$$\begin{aligned} 0 &< d_\theta(x_{2n+1}, x_{2n}) = d_\theta(Tx_{2n}, Sx_{2n-1}) \\ &\leq k_1 d_\theta(x_{2n}, x_{2n-1}) + k_2 d_\theta(x_{2n}, Tx_{2n}) + k_3 d_\theta(x_{2n-1}, Sx_{2n-1}) \\ &= k_1 d_\theta(x_{2n}, x_{2n-1}) + k_2 d_\theta(x_{2n}, x_{2n+1}) + k_3 d_\theta(x_{2n-1}, x_{2n}) \end{aligned}$$

$$\begin{aligned} (1-k_2) d_\theta(x_{2n}, x_{2n+1}) &\leq (k_1+k_3) d_\theta(x_{2n-1}, x_{2n}) \\ d_\theta(x_{2n}, x_{2n+1}) &\leq \frac{k_1+k_3}{1-k_2} d_\theta(x_{2n-1}, x_{2n}) \\ d_\theta(x_{2n}, x_{2n+1}) &\leq q d_\theta(x_{2n-1}, x_{2n}). \end{aligned}$$

In this way we have proved that

$$d_\theta(x_m, x_{m+1}) \leq q d_\theta(x_{m-1}, x_m), \text{ for all } m \in \mathbb{N}.$$

From Lemma 1.8., taking into account (iii), we obtain that $(x_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in a complete extended b-metric space. Therefore, there is some point $u \in X$, such that $\lim_{m\to\infty} x_m = u$.

It is obvious that $\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+1} = u$.
Using (iv) we have

$$\begin{aligned} u &= \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} T(x_{2n}) = Tu, \\ u &= \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} S(x_{2n+1}) = Su. \end{aligned}$$

Hence $u \in CFix(T,S)$.
Let us suppose now that (iv^*) take place, and let $x = u \in X^T$ and $y = x_{2n+1} \in X^S$.

$$\begin{aligned} 0 &< d_\theta(Tu, x_{2n+2}) = d_\theta(Tu, Sx_{2n+1}) \leq k_1 d_\theta(u, x_{2n+1}) + k_2 d_\theta(u, Tu) + k_3 d_\theta(x_{2n+1}, Sx_{2n+1}) \\ &= k_1 d_\theta(u, x_{2n+1}) + k_2 d_\theta(u, Tu) + k_3 d_\theta(x_{2n+1}, x_{2n+2}) = k_2 d_\theta(u, Tu). \end{aligned}$$

From here, we obtain that $d_\theta(u, Tu) = 0$.
Let now consider $x = x_{2n+1} \in X^T$ and $y = u \in X^S$.

$$\begin{aligned} 0 &< d_\theta(x_{2n+1}, Su) = d_\theta(Tx_{2n}, Su) \leq k_1 d_\theta(x_{2n}, u) + k_2 d_\theta(x_{2n}, Tx_{2n}) + k_3 d_\theta(u, Su) \\ &= k_1 d_\theta(u, x_{2n}) + k_2 d_\theta(x_{2n}, x_{2n+1}) + k_3 d_\theta(u, Su) = k_3 d_\theta(u, Su). \end{aligned}$$

From here, we obtain that $d_\theta(u, Su) = 0$, and thus, $u \in CFix(T,S)$.
Let us prove now the uniqueness of the common fixed point. Suppose that, there exist $u, v \in CFix(T,S)$, $u \neq v$. From (v) we have that $(u, Tu) \in E(G)$ and $(v, Sv) \in E(G)$. Now, using (ii) we obtain

$$0 < d_\theta(u,v) = d_\theta(Tu, Sv) \leq k_1 d_\theta(u,v) + k_2 d_\theta(u, Tu) + k_3 d_\theta(v, Sv) = k_1 d_\theta(u,v),$$

which is a contradiction. In conclusion $u = v$. □

3. Ciric type operators

Theorem 3.1. *Let T, S be two self-mappings on a complete extended b-metric space (X, d_θ) endowed with a directed graph G such that the pair (T,S) forms a G-orbital-cyclic pair. Suppose that*

(i) $X^T \neq \varnothing$;

(ii) for all $x \in X^T$ and $y \in X^S$ and $k_1, k_2, k_3 > 0$, with $k_1 + k_2 + k_3 < 1$

$$d_\theta(Tx, Sy) \leq k \max\{d_\theta(x,y), d_\theta(x,Tx), d_\theta(y,Sy)\};$$

(iii) for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$, with $(x_n, x_{n+1}) \in E(G)$, $\lim_{n,m\to\infty} \theta(x_n, x_m) < \frac{1-k}{k}$;

(iv) S and T are continuous,

or

(iv) for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$, with $x_n \to u$ as $n \to \infty$, and $(x_n, x_{n+1}) \in E(G)$, for $n \in \mathbb{N}$, we have, $u \in X^T \cap X^S$.*

In these conditions $CFix(T,S) \neq \varnothing$.
Moreover, if we suppose

(v) if $(u,v) \in CFix(T,S)$ implies $u \in X^T$ and $v \in X^S$ then the pair (T,S) has a unique common fixed point.

Proof. Let $x_0 \in X^T$. Just like in the proof of Theorem 2.1., we construct inductively, a sequence $(x_n)_{n\in\mathbb{N}}$, with $x_{2n} = Sx_{2n-1}$ and $x_{2n+1} = Tx_{2n}$, such that $(x_{2n}, x_{2n+1}) \in E(G)$.

We shall suppose that $x_n \neq x_{n+1}$.

If, there exists $n_0 \in \mathbb{N}$, such that $x_{n_0} = x_{n_0+1}$, then, because $\Delta \subset E(G), (x_{n_0}, x_{n_0+1}) \in E(G)$ and $u = x_{n_0}$ is a fixed point of T.

In order to show that $u \in CFix(T, S)$, we shall consider two cases for n_0.

If $n_0 = 2k$, then $x_{2k} = x_{2k+1} = Tx_{2k}$ an thus, x_{2k} is a fixed point for T. Suppose that $d_\theta(Tx_{2k}, Sx_{2k+1}) > 0$, and let $x = x_{2k} \in X^T$ and $y = x_{2k+1} \in X^S$.

$$\begin{aligned}
0 &< d_\theta(x_{2k+1}, x_{2k+2}) = d_\theta(Tx_{2k}, Sx_{2k+1}) \\
&\leq k \max\{d_\theta(x_{2k}, x_{2k+1}), d_\theta(x_{2k}, Tx_{2k}), d_\theta(x_{2k+1}, Sx_{2k+1})\} \\
&= kd_\theta(x_{2k+1}, Sx_{2k+1}) = kd_\theta(x_{2k+1}, x_{2k+2}).
\end{aligned}$$

In this way we reach to a contradiction.

In the same way we can prove the case $n_0 = 2k + 1$.

In conclusion $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$.

Now we shall prove that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. In order to do this, we shall consider two possible cases:

Case 1. $x = x_{2n} \in X^T$ and $y = x_{2n+1} \in X^S$.

$$\begin{aligned}
0 &< d_\theta(x_{2n+1}, x_{2n+2}) = d_\theta(Tx_{2n}, Sx_{2n+1}) \\
&\leq k \max\{d_\theta(x_{2n}, x_{2n+1}), d_\theta(x_{2n}, Tx_{2n}), d_\theta(x_{2n+1}, Sx_{2n+1})\} \\
&= k \max\{d_\theta(x_{2n}, x_{2n+1}), d_\theta(x_{2n+1}, x_{2n+2})\} \\
&\leq k[d_\theta(x_{2n}, x_{2n+1}) + d_\theta(x_{2n+1}, x_{2n+2})]
\end{aligned}$$

$$\begin{aligned}
(1-k)\, d_\theta(x_{2n+1}, x_{2n+2}) &\leq kd_\theta(x_{2n}, x_{2n+1}) \\
d_\theta(x_{2n+1}, x_{2n+2}) &\leq \frac{k}{1-k} d_\theta(x_{2n}, x_{2n+1}).
\end{aligned}$$

Case 2. $x = x_{2n} \in X^T$ and $y = x_{2n-1} \in X^S$.

$$\begin{aligned}
0 &< d_\theta(x_{2n+1}, x_{2n}) = d_\theta(Tx_{2n}, Sx_{2n-1}) \\
&\leq k \max\{d_\theta(x_{2n}, x_{2n-1}), d_\theta(x_{2n}, Tx_{2n}), d_\theta(x_{2n-1}, Sx_{2n-1})\} \\
&= k \max\{d_\theta(x_{2n}, x_{2n-1}), d_\theta(x_{2n}, x_{2n+1})\} \\
&\leq k[d_\theta(x_{2n}, x_{2n-1}) + d_\theta(x_{2n}, x_{2n+1})]
\end{aligned}$$

$$d_\theta(x_{2n}, x_{2n+1}) \leq \frac{k}{1-k} d_\theta(x_{2n-1}, x_{2n})$$

In this way we have proved that

$$d_\theta(x_m, x_{m+1}) \leq \frac{k}{1-k} d_\theta(x_{m-1}, x_m), \text{ for all } m \in \mathbb{N}.$$

From Lemma 1.8., taking into account (iii), we obtain that $(x_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in a complete extended b-metric space. Therefore, there is some point $u \in X$, such that $\lim_{m\to\infty} x_m = u$.

It is obvious that $\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+1} = u$.

Using (iv) we have

$$\begin{aligned}
u &= \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} T(x_{2n}) = Tu, \\
u &= \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} S(x_{2n+1}) = Su.
\end{aligned}$$

Hence $u \in CFix(T,S)$.

Let us suppose now that (iv^*) take place, and let $x = u \in X^T$ and $y = x_{2n+1} \in X^S$.

$$0 < d_\theta (Tu, x_{2n+2}) = d_\theta (Tu, Sx_{2n+1}) \leq k \max \{d_\theta (u, x_{2n+1}), d_\theta (u, Tu), d_\theta (x_{2n+1}, Sx_{2n+1})\} = kd_\theta (u, Tu)$$

From here, we obtain that $d_\theta (u, Tu) = 0$.

Let now consider $x = x_{2n+1} \in X^T$ and $y = u \in X^S$.

$$0 < d_\theta (x_{2n+1}, Su) = d_\theta (Tx_{2n}, Su) \leq k \max \{d_\theta (x_{2n}, u), d_\theta (x_{2n}, Tx_{2n}), d_\theta (u, Su)\} = kd_\theta (u, Su)$$

From here, we obtain that $d_\theta (u, Su) = 0$, and thus, $u \in CFix(T,S)$.

Let us prove now the uniqueness of the common fixed point. Suppose that, there exist $u, v \in CFix(T,S)$, $u \neq v$. From (v) we have that $(u, Tu) \in E(G)$ and $(v, Sv) \in E(G)$. Now, using (ii) we obtain

$$0 < d_\theta (u, v) = d_\theta (Tu, Sv) \leq k \max \{d_\theta (u, v), d_\theta (u, Tu), d_\theta (v, Sv)\} = kd_\theta (u, v),$$

which is a contradiction. In conclusion $u = v$. □

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