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Eigenvalues and eigenvectors of a certain complex tridiagonal matrix family

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Abstract

In this paper, we obtain the eigenvalues and eigenvectors of a certain complex tridiagonal matrix family in terms of the Chebyshev polynomials of the first kind.

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1. Introduction

Tridiagonal matrices frequently arise in many areas of mathematics and engineering, such as boundary value problems, parallel computing and telecommunication system analysis. Solving some difference, differential and delay differential equations we meet the necessity to compute the arbitrary positive integer powers of square matrices. Therefore, calculating eigenvalues of special square matrices is a very popular problem. Rimas investigated positive integer powers of certain tridiagonal matrices of odd and even order depending on the Chebyshev polynomials [1-4]. Some authors also investigated eigenvalues and eigenvectors of certain tridiagonal matrices [5-12].

In this paper, we obtain the eigenvalues and eigenvectors of one type of n-square complex tridiagonal matrix family, which is a generalization of [1-4],

$$(1.1)\qquad B_n = \begin{bmatrix} a & 2b & & & & \\ c & a & b & & 0 & \\ & c & a & \ddots & & \\ & & \ddots & \ddots & b & \\ & 0 & & c & a & b \\ & & & & 2c & a \end{bmatrix}$$

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where $bc \neq 0$.

Now, we are beginning with following lemma.

1.1. Lemma. [13] *Let* $\{H_n,\ n = 1, 2, \ldots\}$ *be sequence of tridiagonal matrices of the form*

$$H_n = \begin{bmatrix} h_{1,1} & h_{1,2} & & & \\ h_{2,1} & h_{2,2} & h_{2,3} & 0 & \\ & h_{3,2} & h_{3,3} & \ddots & \\ & 0 & \ddots & \ddots & h_{n-1,n} \\ & & & h_{n,n-1} & h_{n,n} \end{bmatrix}.$$

Then the succesive determinants of H_n *are given by the recursive formula:*

$$\begin{aligned} |H_1| &= h_{1,1}, \\ |H_2| &= h_{1,1}h_{2,2} - h_{1,2}h_{2,1}, \\ |H_n| &= h_{n,n}\,|H_{n-1}| - h_{n-1,n}h_{n,n-1}\,|H_{n-2}|. \end{aligned}$$

Let $\left\{H_n^{\dagger},\ n = 1, 2, \ldots\right\}$ be a sequence of tridiagonal matrices of the form

$$H_n^{\dagger} = \begin{bmatrix} h_{1,1} & -h_{1,2} & & & \\ -h_{2,1} & h_{2,2} & -h_{2,3} & 0 & \\ & -h_{3,2} & h_{3,3} & \ddots & \\ & 0 & \ddots & \ddots & -h_{n-1,n} \\ & & & -h_{n,n-1} & h_{n,n} \end{bmatrix}.$$

Since the determinant of the sequences H_n and $H_n^{\dagger}$ have the same recurrence formula, it can be written that

$$|H_n| = \left|H_n^{\dagger}\right|. \tag{1.2}$$

2. Eigenvalues and eigenvectors of B_n

In this section, we investigate the eigenvalues and eigenvectors of B_n, given in (1.1). Let U_n be the following n-square tridiagonal matrix

$$U_n = \begin{bmatrix} 0 & 2 & & & & \\ 1 & 0 & 1 & & 0 & \\ & 1 & 0 & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & 0 & & 1 & 0 & 1 \\ & & & & 2 & 0 \end{bmatrix}.$$

By using (1.2), we write its characteristic polynomial as:

$$|tI_n - U_n| = \begin{vmatrix} t & 2 & & & & \\ 1 & t & 1 & & 0 & \\ & 1 & t & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & 1 & t & 1 \\ & & & & 2 & t \end{vmatrix}. \tag{2.1}$$

By using [2], we obtain the eigenvalues of U_n as

$$t_k = 2\cos\frac{(k-1)\pi}{n-1}, \text{ for } k = 1, 2, \ldots, n \tag{2.2}$$

where t_k denotes kth eigenvalue of U_n.

2.1. Lemma. *Let Q_n be n-square tridiagonal matrix as in the following*

$$Q_n = \begin{bmatrix} a & 2 & & & & \\ 1 & a & 1 & & 0 & \\ & 1 & a & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & 0 & & 1 & a & 1 \\ & & & & 2 & a \end{bmatrix} \tag{2.3}$$

where $a \in \mathbb{C}$. Then the eigenvalues of Q_n are

$$\mu_k = a + 2\cos\frac{(k-1)\pi}{n-1}, \text{ for } k = 1, 2, \ldots, n. \tag{2.4}$$

Proof. By using (1.2), the characteristic polynomial of Q_n can be written as

$$|\mu I_n - Q_n| = \begin{vmatrix} \mu - a & 2 & & & & \\ 1 & \mu - a & 1 & & 0 & \\ & 1 & \mu - a & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & 0 & & 1 & \mu - a & 1 \\ & & & & 2 & \mu - a \end{vmatrix}.$$

Substituting $t = \mu - a$ and taking (2.1) and (2.2) into account, we find the eigenvalues of Q_n as

$$\mu_k = a + 2\cos\frac{(k-1)\pi}{n-1}, \text{ for } k = 1, 2, \ldots, n$$

□

2.2. Theorem. *Let B_n be n-square matrix as in (1.1). Then the eigenvalues of B_n are*

$$\lambda_k = a + 2\sqrt{bc}\cos\frac{(k-1)\pi}{n-1}, \text{ for } k = 1, 2, \ldots, n. \tag{2.5}$$

Proof. In order to prove the theorem, we need a relation between the B_n and Q_n. Let M_n be a complex tridiagonal matrix as in the following

$$M_n = \begin{bmatrix} a/\sqrt{bc} & 2 & & & & \\ 1 & a/\sqrt{bc} & 1 & & 0 & \\ & 1 & a/\sqrt{bc} & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & 0 & & 1 & a/\sqrt{bc} & 1 \\ & & & & 2 & a/\sqrt{bc} \end{bmatrix}$$

where $bc \neq 0$. Taking (2.3) and (2.4) into account, we find the eigenvalues of M_n as

$$\frac{a}{\sqrt{bc}} + 2\cos\frac{(k-1)\pi}{n-1}, \text{ for } k = 1, 2, \ldots, n.$$

By dividing all entries of B_n by $\sqrt{bc}$, we get a new n-square matrix $\widetilde{M_n}$ as

$$\widetilde{M_n} = \begin{bmatrix} a/\sqrt{bc} & 2b/\sqrt{bc} & & & & \\ c/\sqrt{bc} & a/\sqrt{bc} & b/\sqrt{bc} & & 0 & \\ & c/\sqrt{bc} & a/\sqrt{bc} & \ddots & & \\ & & \ddots & \ddots & b/\sqrt{bc} & \\ & 0 & & c/\sqrt{bc} & a/\sqrt{bc} & b/\sqrt{bc} \\ & & & & 2c/\sqrt{bc} & a/\sqrt{bc} \end{bmatrix}.$$

From Lemma 1, the characteristic polynomials of M_n and $\widetilde{M_n}$ are equal. Therefore, the eigenvalues of these matrices are the same. Furthermore, the eigenvalues of B_n are just $\sqrt{bc}$ times the eigenvalues of $\widetilde{M_n}$. Consequently, we get

$$\lambda_k = a + 2\sqrt{bc}\cos\frac{(k-1)\pi}{n-1}, \text{ for } k = 1, 2, \ldots, n,$$

and the proof is complete. □

Now, let us find the eigenvectors corresponding to each eigenvalue of B_n.

Each eigenvector of B_n is the solution of the following homogeneous linear equation system

$$(2.6) \qquad (\lambda_j I_n - B_n)\, x = 0,$$

where λ_j is the jth eigenvalue of B_n $(1 \le j \le n)$. We clearly write the expression (2.6) as follows:

$$(2.7) \qquad \begin{aligned} (\lambda_j - a)\, x_1 - 2bx_2 &= 0 \\ -cx_1 + (\lambda_j - a)\, x_2 - bx_3 &= 0 \\ -cx_2 + (\lambda_j - a)\, x_3 - bx_4 &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -cx_{n-2} + (\lambda_j - a)\, x_{n-1} - bx_n &= 0 \\ -2cx_{n-1} + (\lambda_j - a)\, x_n &= 0. \end{aligned}$$

By dividing all terms of equations in (2.7) by $\sqrt{bc}$, choosing $x_1 = 1$ arbitrarily and solving the set of systems (2.7) according to x_1, we find the eigenvectors of B_n as

$$(2.8) \qquad x_{ij} = \left(\sqrt{\frac{c}{b}}\right)^{i-1} T_{i-1}\left(\frac{\lambda_j - a}{2\sqrt{bc}}\right) \text{ for } i, j = 1, 2, \ldots, n,$$

where $T_k(x)$ is the kth degree Chebyshev polynomial of the first kind [14]:

$$T_k(x) = \cos k\,(\arccos x)\,,\ -1 \le x \le 1.$$

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