A NOVEL CLASS OF MULTIVALENTLY ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS AND ITS APPLICATIONS

NEGATİF KATSAYILI ÇOK DEĞERELİ ANALİTİK FONKSİYONLARIN YENİ BİR SINIFI VE UYGULAMALARI

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ABSTRACT

A novel class $T_\lambda(p,n,\alpha)$ of multivalently analytic functions wiht negative coeffiicients, and some interesting properties belonging to the this class is obtained.

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1. Introduction, Definitıons

Let $T(n, p)$ denote the class of functions $f(z)$ of the form:

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, \qquad (a_k \geq 0; n, p \in N = \{1, 2, 3, ...\}), \tag{1.1}$$

which are *analytic* and *p-valent* in the *open unit* disk

$$U = \{z;\, z \in C \text{ and } |z| < 1\}.$$

A functions $f(z) \in T(n, p)$ is said to be *p-valently starlike of order* α if it satisfies the inequality:

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (0 \leq \alpha < p;\ p \in N;\ z \in U). \tag{1.2}$$

On the order hand, a function $f(z) \in T(n, p)$ is said to be *p-valently convex of order* α if it satisfies the inequality:

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$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha,\ \left(0\le\alpha<p;\ p\in N;\ z\in U\right). \qquad (1.3)$$

Furtermore, a function $f(z) \in T(n, p)$ is said to be *p-valently close-to-convex of order* α if it satisfies the inequality:

$$\Re\left\{z^{1-p}\ f'(z)\right\}>\alpha,\ \left(0\le\alpha<p;\ p\in N;\ z\in U\right). \qquad (1.4)$$

It is easily seen that a function $f(z)$ *p-valently convex of order* α ($0 \le \alpha < p$; $p \in N$) *if and only if* $zf'(z)$ is *p-valently starlike of order* α ($0 \le \alpha < p$; $p \in N$). (See, for example, [1-3].)

A function $f(z)$ in $T(n, p)$ is said to be in the class $T_\lambda(p,n,\alpha)$ if it also satify the inequality:

$$\left|\frac{zf'(z)+\lambda\ z^2 f''(z)}{(1-\lambda)f(z)+\lambda\ zf'(z)}-p\right|<\alpha, \qquad (1.5)$$

$$\left(0\le\lambda\le 1; 0<\alpha\le p; n,p\in N\right),$$

In view of the inequallity (1.5), it is easily verified that the class $T_\lambda(p,n,\alpha)$ can be identified with the class of

(i) p-valently close-to-convex functions of order $p-\alpha\ \left(0<\alpha\le p;\quad p\in N\right)$ when $\lambda=0$

(ii) p-valently close-to-convex functions of order $p-\alpha\ \left(0<\alpha\le p;\quad p\in N\right)$ when $\lambda=1$

(iii) starlike function of order $1-\alpha\ \left(0<\alpha\le 1\right)$ when $\lambda=0$ *and* $p=1$

(iv) convex function of order $1-\alpha\ \left(0<\alpha\le 1\right)$ when $\lambda=0$ *and* $p=1$.

Other interesting works involving functions $f(z)$ of the form (1.1) were studied Altıntaş *et al.* [4], Chen *et al.* [5,6], Irmak *et al.* [7,8], and Aouf *et al.* [9].

2. A Theorem on Coefficient Bounds

A necessary and sufficient condition for a function $f(z) \in T(n, p)$ to be in the class $T_\lambda(p, n, \alpha)$ is provided by.

Theorem 1. Let a function $f(z)$ defined by (1.1) be in the class $T(n, p)$. Then, the function $f(z)$ belongs to the class $T_\lambda(p, n, \alpha)$ ***if and only if***

$$\sum_{k=n+p}^{\infty}(k-q+\alpha)(k\lambda-\lambda+1)a_k \le \alpha(p\lambda-\lambda+1), \tag{2.1}$$

$$(0 \le \lambda \le 1; 0 < \alpha \le p; n, p \in N).$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{\alpha(p\lambda-\lambda+1)}{(n+\alpha)[1+\lambda(n+p-1)]} z^{n+p}, \quad (n, p \in N). \tag{2.2}$$

Proof. We assume that the inequality (2.1) holds true, if we let $z \in \partial U$, we find from (1.1) and (2.1) that

$$\left|zf'(z)+\lambda z^2 f''(z)-p[(1-\lambda)f(z)+\lambda zf'(z)]\right| - \alpha\left|(1-\lambda)f(z)+\lambda zf'(z)\right|$$

$$\le \sum_{k=n+p}^{\infty}(k-q+\alpha)(k\lambda-\lambda+1)a_k - \alpha(p\lambda-\lambda+1) \le 0,$$

$$= \left|\sum_{k=n+p}^{\infty}(k-q)(k\lambda-\lambda+1)a_k z^{k-p}\right| - \alpha\left|(p\lambda-\lambda+1) - \sum_{k=n+p}^{\infty}(k\lambda-\lambda+1)a_k z^{k-p}\right| \tag{2.3}$$

$$(0 \le \lambda \le 1; 0 < \alpha \le p; n, p \in N).$$

Hence, by the maximum modulus theorem, the function $f(z)$ defined by (1.1) belongs to the class $T_\lambda(p, n, \alpha)$.

In order hand to prove the converse, we suppose that $f(z) \in T_\lambda(p, n, \alpha)$, that is that

$$\left|\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}-p\right|=\left|\frac{-\sum_{k=n+p}^{\infty}(k-p)(\lambda k-\lambda+1)a_k z^{k-p}}{(\lambda p-\lambda+1)-\sum_{k=n+p}^{\infty}(\lambda k-\lambda+1)a_k z^{k-p}}\right|<\alpha \qquad (2.4)$$

$$\left(z\in U;\,0\le\lambda\le 1;0<\alpha\le p;n,p\in N\right).$$

Since $|\mathrm{Re}(z)|\le|z|$ for any z, choosing z to be real and letting $z\to 1^-$ through real values, (2.5) yields

$$\sum_{k=n+p}^{\infty}(k-q)(k\lambda-\lambda+1)a_k\le\alpha\left\{(p\lambda-\lambda+1)-\sum_{k=n+p}^{\infty}(k\lambda-\lambda+1)a_k\right\},$$

which leads us immediately to the desired inequality (2.1).

3. Growth and Distortion Theorems Involving Operators of Fractional Calculus

Im this section, we shall prove several growth and distortion theorems for functions belonging to the genaral class $T_\lambda(p,n,\alpha)$. Each of these theorems would involve certain operators of fractional calculus, which are defined as follows (cf., e.g., [10,11]).

Definition 1 (Fractional Integral Operator). The fractional integral of order δ is defined by, for a function *f* (z), by

$$D_z^{-\delta}f(z)=\frac{1}{\Gamma(\delta)}\int_0^z\frac{f(\xi)}{(z-\xi)^{1-\delta}}d\xi,\ \ (\delta>0), \qquad (3.1)$$

where *f* (z) is an analytic function in a simply-conneted region of the z-plane containing the origion, and the multiplicity of $(z-\xi)^{\delta-1}$ is removed by requiring $\log(z-\xi)$ to be real $(z-\xi)>0$.

Definition 2 (Fractional Derivative Operator). The fractional derivative of order q+δ is defined by, for a function *f* (z), by

$$D_z^{q+\delta} f(z) = \begin{cases} \dfrac{1}{\Gamma(1-\delta)} \dfrac{d}{dz} \displaystyle\int_0^z \dfrac{f(\xi)}{(z-\xi)^{\delta}} d\xi, & (0 \le \delta < 1;\, q = 0) \\ \dfrac{d^q}{dz^q} D_z^{\delta} f(z), & (0 \le \delta < 1;\, q \in N), \end{cases} \tag{3.2}$$

where *f* (z) is constrained, and the multiplicity of $(z-\xi)^{\delta}$ is removed, as in Definition 1.

Theorem2. If $f(z) \in T_{\lambda}(p,n,\alpha)$, then

$$\left| \Gamma(p+\mu+1) \left| D_z^{-\mu} f(z) \right| - p! |z|^{p+\mu} \right| \le \frac{\alpha(p\lambda - \lambda + 1)(n+p)!}{(n+\alpha)[1+\lambda\,(n+p-1)]\Gamma(n+p+\mu+1)} |z|^{n+p+\mu}, \tag{3.3}$$

$$(z \in U; \mu > 0;\, 0 \le \lambda \le 1; 0 < \alpha \le p; n, p \in N).$$

The result sharp for a function *f* (z) given by (2.2).

Proof. Suppose $f(z) \in T_{\lambda}(p,n,\alpha)$. Then, we find from the inequality (2.1) of Theorem 1, that

$$\sum_{k=n+p}^{\infty} k! a_k \le \frac{\alpha(p\lambda - \lambda + 1)}{(n+\alpha)[1+\lambda(n+p-1)]}, \quad (n,\ p \in N). \tag{3.4}$$

Making use of the inequality (3.4) and Definitions 1, we have

$$D_z^{-\delta} f(z) = \frac{p!}{\Gamma(p+\mu+1)} z^{p+\mu} - \sum_{k=n+p}^{\infty} k!\Psi(k) a_k z^{k+\mu}, \tag{3.5}$$

where, for converience,

$$\Psi(k) = \frac{k!}{\Gamma(k+\mu+1)}, \quad (k = n+p;\ p \in N; \quad \mu > 0).$$

Clearly, the function $\Psi(k)$ is decreasing in k, and we have

$$0 < \Psi(k) \leq \Psi(n+p) = \frac{(n+p)!}{\Gamma(n+p+\mu+1)}, \quad (k = n+p;\ p \in N; \quad \mu > 0). \tag{3.6}$$

Thus, we find from (3.2)-(3.4) that

$$\left| \left| D_z^{-\delta} f(z) \right| - \frac{p!}{\Gamma(p+\mu+1)} |z|^{p+\mu} \right| \leq \frac{(n+p)!}{\Gamma(n+p+\mu+1)} |z|^{n+p+\mu} \sum_{k=n+p}^{\infty} a_k$$
$$\leq \frac{\alpha(p\lambda - \lambda + 1)(n+p)!}{(n+\alpha)[1+\lambda\,(n+p-1)]\Gamma(n+p+\mu+1)} z^{n+p+\mu},$$

which completes the proof of Theorem 2.

Theorem 3. If $f(z) \in T_\lambda(p,n,\alpha)$, then

$$\left| \Gamma(p-q+\mu+1) \left| D_z^{q+\mu} f(z) \right| - p! |z|^{p-q-\mu} \right|$$
$$\leq \frac{\alpha(p\lambda - \lambda + 1)(n+p)!}{(n+\alpha)[1+\lambda\,(n+p-1)]\Gamma(n+p-q-\mu+1)} |z|^{n+p-q-\mu}, \tag{3.7}$$

$(z \in U;\ p > q;\ 0 \leq \mu < 1;\ 0 \leq \lambda \leq 1; 0 < \alpha \leq p; n, p \in N;\ q \in N \cup \{0\})$.
The result sharp for a function *f* (z) given by (2.2).

Proof. Under the hypothesis $f(z) \in T_\lambda(p,n,\alpha)$, we find from Theorem 1, that

$$\frac{(n+\alpha)[1+\lambda(n+p-1)]}{(n+p)!} \sum_{k=n+p}^{\infty} k!a_k \leq \sum_{k=n+p}^{\infty} (k-p+\alpha)(k\lambda - \lambda + 1)a_k, \tag{3.8}$$
$$\leq \alpha\,(p\lambda - \lambda + 1),$$

which readily yields

$$\sum_{k=n+p}^{\infty} k!a_k \leq \frac{\alpha(n+p)!(p\lambda - \lambda + 1)}{(n+\alpha)[1+\lambda(n+p-1)]}, \quad (n,\ p \in N). \tag{3.9}$$

Now, making use of the inequality (3.9) and Definition 2, we have

$$D_z^{q+\mu} f(z) = \frac{p!}{\Gamma(p-q-\mu+1)} z^{p-q-\mu} - \sum_{k=n+p}^{\infty} k!\Theta(k) a_k z^{k-q-\mu}, \qquad (3.10)$$

where, for converience,

$$\Theta(k) = \frac{1}{\Gamma(k-q-\mu+1)}, \quad (k = n+p;\, n, p \in N; 0 \le \mu < 1).$$

Since, the function $\Theta(k)$ is decreasing in k, and we have

$$0 < \Theta(k) \le \Theta(n+p) = \frac{1}{\Gamma(n+p-q-\mu+1)}, \quad . \qquad (3.11)$$

Thus, we find from (3.9)-(3.11) that

$$\left| \left| D_z^{q+\mu} f(z) \right| - \frac{p!}{\Gamma(p-q-\mu+1)} |z|^{p-q-\mu} \right| \le \frac{1}{\Gamma(n+p-q-\mu+1)} |z|^{n+p-q-\mu} \sum_{k=n+p}^{\infty} k! a_k$$

$$\le \frac{\alpha(p\lambda-\lambda+1)(n+p)!}{(n+\alpha)[1+\lambda(n+p-1)]\Gamma(n+p-q-\mu+1)} z^{n+p-q-\mu},$$

$$(z \in U;\, p > q;\, 0 \le \mu < 1;\, 0 \le \lambda \le 1; 0 < \alpha \le p; n, p \in N;\, q \in N \cup \{0\}).$$

Which completes the proof of Theorem 3.

4. Radii of *p*-Valently Close-To-Convexity,
***p*-Valently Starlikenes and *p*-Valently Convexity**

Finally, we determine the radii of *p*-valently starlikenes, *p*-valently convexity, and *p*-valently close-to-convexity, for function in the class $T_\lambda(p,n,\alpha)$, which are given by

Theorem 4. If $f(z) \in T_\lambda(p,n,\alpha)$, then $f(z)$ is *p*-valently close-to-convexity of order $\beta\ (0 \le \beta < p;\ p \in N)$ *in* $|z| < r_1$, *p*-valently starlike of order

$\beta\ (0 \le \beta < p;\ p \in N)\ in\ |z| < r_2$, and p-valently convex of order $\beta\ (0 \le \beta < p;\ p \in N)\ in\ |z| < r_3$, where

$$r_1 = r_1(p;\beta,\lambda,\alpha) = \inf_k \left[\frac{(p-\beta)(k-p+\alpha)(k\lambda-\lambda+1)}{\alpha k (p\lambda-\lambda+1)} \right]^{\frac{1}{k-p}}, \qquad (4.1)$$

$$(k \ge n+p;\ n,p \in N;\ 0 < \alpha \le p;\ 0 \le \beta < p;\ 0 \le \lambda \le 1),$$

and

$$r_2 = r_2(p;\beta,\lambda,\alpha) = \inf_k \left[\frac{(p-\beta)(k-p+\alpha)(k\lambda-\lambda+1)}{\alpha (k-\beta)(p\lambda-\lambda+1)} \right]^{\frac{1}{k-p}}, \qquad (4.2)$$

$(k \ge n+p;\ n,p \in N;\ 0 < \alpha \le p;\ 0 \le \beta < p;\ 0 \le \lambda \le 1)$,

and that

$$r_3 = r_3(p;\beta,\lambda,\alpha) = \inf_k \left[\frac{(p-\beta)(k-p+\alpha)(k\lambda-\lambda+1)}{\alpha k (k-\beta)(p\lambda-\lambda+1)} \right]^{\frac{1}{k-p}}, \qquad (4.3)$$

$(k \ge n+p;\ n,p \in N;\ 0 < \alpha \le p;\ 0 \le \beta < p;\ 0 \le \lambda \le 1)$,

Each of these result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p\lambda-\lambda+1)}{(k-p-\alpha)(k\lambda-\lambda+1)} z^k,$$

$(k \ge n+p;\ n,p \in N)$. (4.4)

Proof. It is sufficient to show that

$$\left| z^{1-p} f'(z) - p \right| < p - \beta, \quad (|z| < r_1;\ 0 \le \beta < p;\ z \in U;\ p \in N), \qquad (4.5)$$

and

$$\left|1+\frac{zf'(z)}{f(z)}-p\right|<p-\beta, \quad \left(|z|<r_2;\, 0\leq\beta<p;\, z\in U;\; p\in N\right), \qquad (4.6)$$

and that

$$\left|\frac{zf''(z)}{f'(z)}-p\right|<p-\beta, \quad \left(|z|<r_3;\, 0\leq\beta<p;\, z\in U;\; p\in N\right), \qquad (4.7)$$

for a function $f(z) \in T_\lambda(p,n,\alpha)$, where $r_1, r_2, and\ r_3$ are defined by (4.1), (4.2), and (4.3) respectively. The details involved are fairly straightforward and may be ommited.

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