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Application variational iteration method with studying the convergence to nonlinear PDEs

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Abstract: This article is devoted to implement variational iteration method (VIM) for solving nonlinear partial differential equations. This method is based on the use of Lagrange multiplier for identification of optimal value of a parameter in a functional. This procedure is a powerful tool for solving large amount of problems. Using VIM, it is possible to find a sequence of functions which converges to the exact solution or an approximate solution of the problem. Our emphasis will be on study the convergence of the proposed method. Convergence analysis is reliable enough to estimate the maximum absolute error of the solution given by VIM.

Keywords: Variational iteration method, convergence analysis.

1 Introduction

Many different methods have recently introduced to solve nonlinear problems ([4], [11]), such as, VIM ([1]-[3], [7]-[18]), Adomian decomposition method, and homotopy perturbation method ([12], [15]). VIM is strongly and simply capable for solving a large class of linear or nonlinear differential equations without the tangible restriction of sensitivity to the degree of the nonlinear term and also it reduces the size of calculations besides, its interactions are direct and straightforward.

The main aim in this work is to effectively employ VIM to establish exact solutions of nonlinear partial differential equations (NPDEs) and study the convergence of the method. To guarantee this study we present example of NPDE (Burger's equation).

2 Analysis of the VIM

To illustrate the analysis of VIM, we limit ourselves to consider the following nonlinear differential equation in the type:

$$Lu+Ru+N(u)=0, \tag{1}$$

with specified initial conditions, where L and R are linear bounded operators i.e., it is possible to find numbers $m_1, m_2 > 0$ such that $||Lu|| \leq m_1||u||$, $||Ru|| \leq m_2||u||$. The nonlinear term $N(u)$ is Lipschitz continuous with $|N(u)-N(v)| \leq m|u-v|, \forall t \in J=[0,T]$, for any constant $m>0$.
The VIM gives the possibility to write the solution of Eq.(1) with the aid of the correction functional:

$$u_p = u_{p-1} + \int_0^t \lambda(\tau)[Lu_{p-1} + R\tilde{u}_{p-1} + N(\tilde{u}_{p-1})]d\tau, \quad p \geq 1. \tag{2}$$

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It is obvious that the successive approximations u_p, $p \geq 0$ (the subscript p denotes the p^{th} order approximation), can be established by determining λ, a general Lagrange multiplier, which can be identified optimally via the variational theory. The function $\tilde{u}_p$ is a restricted variation, which means $\delta\tilde{u}_p = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations u_p, $p \geq 1$, of the solution u will be readily obtained upon using the Lagrange multiplier obtained and by using any selective function u_0. The initial values of the solution are usually used for selecting the zeroth approximation u_0. With λ determined, then several approximations u_p, $p \geq 1$, follow immediately. Consequently, the exact solution may be obtained by using.

$$u = \lim_{p\to\infty} u_p. \tag{3}$$

In what follows, we will apply VIM to Burger's equation to illustrate the strength of the method and to establish exact solutions for this nonlinear problem.

Now, to illustrate how to find the value of the Lagrange multiplier λ, we will consider the following case, which dependent on the order of the operator L in Eq.(1). We study the case of the operator $L = \frac{\partial}{\partial t}$ (without loos of generality).

Making the above correction functional stationary, and noticing that $\delta\tilde{u}_p = 0$, we obtain:

$$\delta u_p = \delta u_{p-1} + \delta \int_0^t \lambda(\tau)[\frac{\partial u_{p-1}}{\partial \tau} + R\tilde{u}_{p-1} + N(\tilde{u}_{p-1})]d\tau$$

$$= \delta u_{p-1} + [\lambda(\tau)\delta u_{p-1}]_{\tau=t} - \int_0^t \dot{\lambda}(\tau)[\delta u_{p-1}]d\tau = 0,$$

where $\delta\tilde{u}_p$ is considered as a restricted variation i.e., $\delta\tilde{u}_p = 0$, yields the following stationary conditions:

$$\dot{\lambda}(\tau) = 0, \qquad 1 + \lambda(\tau)|_{\tau=t} = 0. \tag{4}$$

The first equation in (4) is called Lagrange-Euler equation and the second equation in (4) is called natural boundary condition, the Lagrange multiplier, therefore, can be readily identified:

$$\lambda(\tau) = -1.$$

Now, the following variational iteration formula can be obtained:

$$u_p = u_{p-1} - \int_0^t [Lu_{p-1} + Ru_{p-1} + N(u_{p-1})]d\tau. \tag{5}$$

We start with an initial approximation, and by using the above iteration formula (5), we can obtain directly the other components of the solution.

3 Convergence analysis of VIM

In this section, the sufficient conditions are presented to guarantee the convergence of VIM, when applied to solve NPDEs, where the main point is that we prove the convergence of the recurrence sequence, which is generated by using VIM.

Lemma 1.*Let $A : U \to V$ be a bounded linear operator and let $\{u_p\}$ be a convergent sequence in U with limit u, then $u_p \to u$ in U implies that $A(u_p) \to A(u)$ in V.*

Proof. Since

$$\|Au_p - Au\|_V = \|A(u_p - u)\|_V \leq \|A\| \, \|u_p - u\|_U,$$

hence:

$$\lim_{p\to\infty} \|Au_p - Au\|_V \leq \|A\| \lim_{p\to\infty} \|u_p - u\|_U = 0,$$

implies that $A(u_p) \to A(u)$.

3.1 Uniqueness theorem

Theorem 1. *The nonlinear problem (1) has an unique solution, whenever* $0 < \alpha < 1$, *where,* $\alpha = (m_2 + m)T$, *where the constants* m_2 *and* m *are defined above.*

Proof. We will consider the case, which depends on the order of the operator L in the Eq.(1), i.e., we study the case of the operator $L = \frac{\partial}{\partial t}$ (without loos of generality). Therefore, the solution of Eq.(1) can be writhen the following form.

$$u = f(x) - L^{-1}[Ru + N(u)],$$

where the function $f(x)$ is the solution of the homogeneous equation $Lu = 0$, and the inverse operator L^{-1} defined by $L^{-1}(.) = \int_0^t (.)dt$.

Now let, u and u^* be two different solutions to (1) then by using the above equation, we get.

$$\begin{aligned} |u - u^*| &= |-\int_0^t [\,R(u-u^*) + N(u) - N(u^*)\,]\,dt| \\ &\leq \int_0^t [\,|R(u-u^*)| + |N(u) - N(u^*)|\,]dt \\ &\leq (\,m_2|u-u^*| + m\,|u-u^*|\,)T \\ &\leq \alpha\,|u-u^*|, \end{aligned}$$

from which we get $(1-\alpha)\,|u-u^*| \leq 0$. Since $0 < \alpha < 1$, then $|u-u^*| = 0$ implies, $u = u^*$ and this complete the proof.

Now, to prove the convergence of the variational iteration method, we will rewrite the equation (5) in the operator form as follows.

$$u_p = A[u_{p-1}], \tag{6}$$

where the operator A takes the following form:

$$A[u] = -\int_0^t [Lu + Ru + N(u)]\,d\tau. \tag{7}$$

3.2 Convergence theorem

Theorem 2. ***(Banach's fixed point theorem)*** *Assume that* X *be a Banach space and* $A : X \to X$ *is a nonlinear mapping, and suppose that*

$$||A[u] - A[v]\,|| \leq \gamma||u - v||, \quad \forall\ u, v \in X, \tag{8}$$

for some constant $\gamma = (\alpha + m_1 T) < 1$*. Then A has an unique fixed point. Furthermore, the sequence (6) using VIM with an arbitrary choice of* $u_0 \in X$*, converges to the fixed point of A and*

$$||u_p - u_q|| \leq \frac{\gamma^q}{1-\gamma}||u_1 - u_0||. \tag{9}$$

Proof. Denoting $(C[J], ||.||)$ Banach space of all continuous functions on J with the norm defined by:

$$||f(t)|| = \max_{t \in J} |f(t)|.$$

We are going to prove that the sequence $\{u_p\}$ is a Cauchy sequence in this Banach space,

$$\begin{aligned} \|u_p - u_q\| &= \max_{t\in J} |\, u_p - u_q \,| \\ &= \max_{t\in J} | - \int_0^t [L(u_{p-1} - u_{q-1}) + R(u_{p-1} - u_{q-1}) + N(u_{p-1}) - N(u_{q-1})]\, d\tau| \\ &\leq \max_{t\in J} \int_0^t [\,|L(u_{p-1} - u_{q-1})| + |R(u_{p-1} - u_{q-1})| + |N(u_{p-1}) - N(u_{q-1})|\,]\, d\tau \\ &\leq \max_{t\in J} \int_0^t [\,(m_1 + m_2 + m)(u_{p-1} - u_{q-1})\,]\, d\tau \\ &\leq \gamma ||u_{p-1} - u_{q-1}||. \end{aligned}$$

Let, $p = q+1$ then

$$\|u_{q+1} - u_q\| \leq \gamma \|u_q - u_{q-1}\| \leq \gamma^2 \|u_{q-1} - u_{q-2}\| \leq ... \leq \gamma^q \|u_1 - u_0\|.$$

From the triangle inequality we have

$$\begin{aligned} ||u_p - u_q|| &\leq ||u_{q+1} - u_q|| + ||u_{q+2} - u_{q+1}|| + ... + ||u_p - u_{p-1}|| \\ &\leq [\gamma^q + \gamma^{q+1} + ... + \gamma^{p-1}]\,||u_1 - u_0|| \\ &\leq \gamma^q [1 + \gamma + \gamma^2 + ... + \gamma^{p-q-1}]\,||u_1 - u_0|| \\ &\leq \gamma^q [\frac{1 - \gamma^{p-q-1}}{1-\gamma}]\,||u_1 - u_0||. \end{aligned}$$

Since $0 < \gamma < 1$ so, $(1 - \gamma^{p-q}) < 1$ then:

$$||u_p - u_q|| \leq \frac{\gamma^q}{1-\gamma}||u_1 - u_0||.$$

But $||u_1 - u_0|| < \infty$ so, as $q \to \infty$ then $||u_p - u_q|| \to 0$. We conclude that $\{u_p\}$ is a Cauchy sequence in $C[J]$ so, the sequence converges and the proof is complete.

3.3 Error estimate

The maximum absolute error of the approximate solution u_p to problem (1) is estimated to be:

$$\max_{t\in J} |u_{exact} - u_p| \leq \beta, \tag{10}$$

where

$$\beta = \frac{\gamma^q\, T\, [\,(m_1 + m_2)\,||u_0|| + k\,]}{1-\gamma}, \qquad k = \max_{t\in J} |N(u_0)|.$$

Proof. From Theorem 2 and inequality (9) we have

$$||u_p - u_q|| \leq \frac{\gamma^q}{1-\gamma}||u_1 - u_0||,$$

as $p \to \infty$ then $u_p \to u_{exact}$ and

$$\begin{aligned} ||u_1 - u_0|| &= \max_{t \in J}\left| - \int_0^t [Lu_0 + Ru_0 + N(u_0)]\, d\tau \right| \\ &\leq \max_{t \in J} \int_0^t [\,|Lu_0| + |Ru_0| + |N(u_0)|\,]\, d\tau \\ &\leq T\,[\,(m_1 + m_2)\,||u_0|| + k],\end{aligned}$$

so, the maximum absolute error in the interval J is:

$$||u_{exact} - u_p|| = \max_{t \in J}|u_{exact} - u_p| \leq \beta.$$

This completes the proof.

For more about the convergence of VIM, see, ([16], [18]).

4 Numerical example

Consider the following one-dimension Burger's equation:

$$u_t + uu_x = u_{xx}, \tag{11}$$

with an initial condition

$$u(x,0) = \frac{1}{2}(1 - tanh(\frac{x}{4})).$$

To solve Eq.(11) by means of VIM, we construct a correction functional which reads

$$u_{p+1}(x,t) = u_p(x,t) + \int_0^t \lambda(\tau)[u_{p\tau} + \tilde{u}_p\tilde{u}_{px} - \tilde{u}_{pxx}]\, d\tau,\ \ p \geq 0. \tag{12}$$

Making the above correction functional stationary, and noticing that $\delta u(x,0) = 0$, we obtain

$$\delta u_{p+1}(x,t) = \delta u_p(x,t) + \delta \int_0^t \lambda(\tau)[u_{p\tau} + \tilde{u}_p\tilde{u}_{px} - \tilde{u}_{pxx}]\, d\tau$$

$= \delta u_p + [\lambda(\tau)\delta u_p]_{\tau=t} - \int_0^t \dot{\lambda}(\tau)[\delta u_p]\, d\tau = 0$, where $\delta\tilde{u}_p$ is considered as a restricted variation i.e., $\delta\tilde{u}_p = 0$, yields the following stationary conditions

$$\dot{\lambda}(\tau) = 0, \quad 1 + \lambda(\tau)|_{\tau=t} = 0. \tag{13}$$

The Lagrange multiplier λ, therefore, can be readily identified

$$\lambda(\tau) = -1.$$

Now, the following variational iteration formula can be obtained:

$$u_{p+1}(x,t) = u_p(x,t) - \int_0^t [u_{p\tau} + u_p u_{px} - u_{pxx}]\, d\tau. \tag{14}$$

We start with an initial approximation $u_0(x,t) = u(x,0)$, and by using the above iteration formula (14), we can obtain directly the other components as

$$u_0(x,t) = \frac{1}{2}(1 - \tanh(\frac{x}{4})),$$
$$u_1(x,t) = u_0(x,t) - (\frac{-t}{16}\,\text{sech}^2(\frac{x}{4})),$$
$$u_2(x,t) = u_1(x,t) - (\frac{-t^2}{1536}(6 + t + 6\cosh(\frac{x}{2})),$$
$$\text{sech}^4(\frac{x}{4})\tanh(\frac{x}{2}), \ldots.$$

In order to verify numerically whether the proposed methodology lead to higher accuracy, we can evaluate the numerical solutions using $p = 2$ terms approximation. Table 1 shows the difference between the analytical solution and the numerical solution at the value of $t = 0.5$. We achieved a very good approximation with the actual solution of Eq.(11) by using two terms only of the iteration equation derived above. It is evident that the overall errors can be made smaller by adding new terms of the iteration formula. The numerical approximation shows a high degree of accuracy and in most case $u_p(x,t)$, the p-term approximation is accurate for quite low values of p, the solutions are very rapidly convergent by utilizing VIM. The numerical results obtained justify the advantage of this method, even in the few terms approximation is accurate.

Table 1: Comparison between the exact solution & the approximate solution.

x	u_p	u_{exact}	$\lvert u_p - u_{exact}\rvert$
1.0	0.407383	0.407333	5.00073e-05
1.5	0.348690	0.348645	4.52270e-05
2.0	0.294251	0.294215	3.60271e-05
2.5	0.245110	0.245085	2.48932e-05
3.0	0.201827	0.201813	1.40846e-05
3.5	0.164522	0.164516	5.08692e-06
4.0	0.132963	0.132964	1.48126e-06
4.5	0.106685	0.106691	5.66538e-06
5.0	0.085091	0.085099	7.87967e-06

It must be noted that VIM used here gives the possibility of obtaining an analytical satisfactory solution for which the other techniques of calculation are more laborious and the results contain a great complexity.

From the above solution process, we can see that the approximate solutions converge to its exact solution relatively slowly due to the approximate identification of the multiplier. It should be specially pointed out that the more accurate the identification of the multiplier, faster the approximations converge to its exact solutions.

5 Conclusion

In this paper, the He's variational iteration method has been successfully applied to find the solution of NPDEs. The presented example shows that the results of the proposed method are in excellent agreement with those of Adomian decomposition method. In our work, we used the Mathematica Package. An interesting point about VIM is that only few iterations or, even in some special cases, one iteration, lead to exact solution or solution with high accuracy. The main merits of VIM are:

(1) VIM overcomes the difficulties arising in calculation of Adomian's polynomials in ADM.
(2) VIM does not require small parameters which are needed in perturbation method.
(3) No linearization is needed; the method is very promising for solving wide application in nonlinear differential equations.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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