ON SUBSEQUENTIALLY CONVERGENT SEQUENCES

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ABSTRACT. In this study we obtain some sufficient conditions under which subsequential convergence of a sequence of real numbers follows from its boundedness. Eventually, we obtain crucial information about the subsequential behavior of sequences.

1. INTRODUCTION

It is well known that convergence of a sequence \( \{s_n\} \) of real numbers implies its boundedness, yet the converse is not necessarily true is clear from the example of \( \{\sin(n\pi/2)\} \). Since boundedness is a necessary condition for convergence of \( \{s_n\} \), we put the following question: Under which conditions we get information on the convergence behavior of bounded sequences. In the case where \( \{s_n\} \) is monotonic and bounded, we have its convergence. On the other hand, Bolzano-Weierstrass theorem states that every bounded sequence has at least one accumulation point. However, there are some bounded sequences such as \( \{\sin(\log n)\} \) whose accumulation points lie on a finite interval and all points in this interval are accumulation points of the sequence. In this case we just have convergence of some subsequences of \( \{s_n\} \). Motivated by this idea, Stanojević [10] defined a new kind of convergence as follows.

Definition 1. A sequence \( \{s_n\} \) is said to be subsequentially convergent if there exists a finite interval \( I \) such that all accumulation points of the sequence \( \{s_n\} \) are in \( I \) and every point of \( I \) is an accumulation point of \( \{s_n\} \).

Throughout this paper, we adopt the following familiar conventions:

(i) \( a_n = o(b_n) \) means \( a_n/b_n \to 0 \) as \( n \to \infty \),
(ii) \( a_n = O(b_n) \) means \( |a_n| \leq Hb_n \) for sufficiently large \( n \), where \( H \) is a positive constant,

Received by the editors: July 31, 2018; Accepted: January 15, 2019.

2010 Mathematics Subject Classification. Primary 40A05; Secondary 40E05.

Key words and phrases. Subsequential convergence, slowly oscillating sequences, logarithmic summability.

Submitted via 2nd International Conference of Mathematical Sciences (ICMS 2018).
(iii) $a_n \sim b_n$ means $a_n/b_n \to 1$ as $n \to \infty$.

Note that every convergent sequence is subsequentially convergent. Further, it is obvious that subsequential convergence implies boundedness. But the converse is not always valid, provided by the example $\{(-1)^n\}$. The first theorem which reveals that the converse is valid under certain conditions was obtained by Dik [3] as stated below.

**Theorem 2.** If $\{s_n\}$ is a bounded sequence such that $\Delta s_n = o(1)$ as $n \to \infty$, then $\{s_n\}$ is subsequentially convergent.

Using Theorem 2 we can easily show that $s_n = \{\sin(\log n)\}$ is subsequentially convergent. Indeed, since $\{s_n\}$ is bounded and

$$|\Delta s_n| = |\Delta \sin(\log n)| = |\sin(\log n) - \sin(\log(n-1))| \leq |\log n - \log(n-1)| = o(1), \quad n \to \infty,$$

$\{s_n\}$ is subsequentially convergent by Theorem 2.

Subsequential convergence was studied in a number of papers such as Çanak and Totur [1, 2], Dik [3], Dik et al. [4] and Sezer and Çanak [8]. In this paper we investigate conditions under which subsequential convergence of $\{s_n\}$ follows from its boundedness.

2. Preliminaries

In this section, we present some fundamental definitions, identities and lemmas which will be needed in the sequel.

The logarithmic mean of $\{s_n\}$ is defined by

$$t_n^{(1)}(s) = \frac{1}{\ell_n} \sum_{k=0}^{n} \frac{s_k}{k+1}, \quad \text{where} \quad \ell_n = \sum_{k=0}^{n} \frac{1}{k+1} \sim \log n, \quad n = 0, 1, 2, \ldots \tag{1}$$

**Definition 3.** A sequence $\{s_n\}$ is said to be summable to a finite number $L$ by the logarithmic mean method $(\ell, 1)$ if $\lim_{n \to \infty} t_n^{(1)}(s) = L$. In this case, we write $s_n \to \xi(\ell, 1)$.

The difference between a sequence $s_n$ and its logarithmic mean $t_n^{(1)}(s)$, that is known as the logarithmic Kronecker identity (see [9]) is given by

$$s_n - t_n^{(1)}(s) = v_n^{(0)}(\Delta s) \tag{2}$$

where $v_n^{(0)}(\Delta s) = \frac{1}{\ell_n} \sum_{k=1}^{n} \ell_{k-1} \Delta s_k$ and $\Delta s_n = s_n - s_{n-1}$ with $s_{-1} = 0$.

Since identity (2) can be rewritten as

$$s_n = v_n^{(0)}(\Delta s) + \sum_{k=1}^{n} \frac{v_k^{(0)}(\Delta s)}{(k+1)\ell_{k-1}} + s_0, \tag{3}$$

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$$s_n = v_n^{(0)}(\Delta s) + \sum_{k=1}^{n} \frac{v_k^{(0)}(\Delta s)}{(k+1)\ell_{k-1}} + s_0, \tag{3}$$
The classical logarithmic control modulo of the oscillatory behavior of \( \{s_n\} \) is given by
\[
\omega_n^{(0)}(s) = \alpha_n \Delta s_n \sim n \log n \Delta s_n, \quad (4)
\]
where \( \alpha_n = (n + 1)\ell_{n-1} \). The general logarithmic control modulo of the oscillatory behavior of \( \{s_n\} \) of integer order \( r \geq 1 \) is recursively defined by
\[
\omega_n^{(r)}(s) = \omega_n^{(r-1)}(s) - t_n^{(r)}(s) - t_n^{(r-1)}(s), \quad (5)
\]
For every nonnegative integer \( r \), we have
\[
(\alpha_n \Delta)_r s_n = (\alpha_n \Delta)_{r-1}^r (\alpha_n \Delta s_n) = \alpha_n \Delta ((\alpha_n \Delta)_{r-1}^r s_n),
\]
where \( (\alpha_n \Delta)_{r}^r s_n = s_n \) and \( (\alpha_n \Delta)_{r}^r s_n = \alpha_n \Delta s_n \).

The next lemma provides a different representation of \( \{\omega_n^{(r)}(s)\} \).

**Lemma 4.** (Sezer and Çanak, [9]) For every integer \( r \geq 1 \), the assertion
\[
\omega_n^{(r)}(s) = (\alpha_n \Delta)_r v_n^{(r-1)}(\Delta s)
\]
is valid.

**Definition 5.** A sequence \( \{s_n\} \) is called slowly oscillating with respect to summability \((\ell,1)\) if
\[
\lim_{\lambda \to 1^+} \limsup_{n \to \infty} \max_{n < k \leq [n^\lambda]} |s_k - s_n| = 0 \quad (6)
\]
or equivalently
\[
\lim_{\lambda \to 1^-} \limsup_{n \to \infty} \max_{[n^\lambda] < k < n} |s_n - s_k| = 0, \quad (7)
\]
where \([n^\lambda]\) denotes the integer part of \( n^\lambda \).

Note that if the two-sided condition \( n \log n \Delta s_n = O(1) \) is satisfied, then \((6)\) holds.

There are subsequentially convergent sequences which are not slowly oscillating with respect to summability \((\ell,1)\), and vice versa. For instance, \( \{\log(\log n)\} \) is subsequentially convergent but not slowly oscillating with respect to summability.
The following lemma indicates that slow oscillation of \( \{s_n\} \) is a Tauberian condition for \((\ell, 1)\) summability.

**Lemma 6.** If \( \{s_n\} \) is \((\ell, 1)\) summable to \( L \) and slowly oscillating with respect to summability \((\ell, 1)\), then it converges to the same value.

### 3. Main Results

In this section, we present our main theorems.

**Theorem 7.** If \( s_n \) is bounded and \( \{\Delta s_n\} \) is slowly oscillating with respect to summability \((\ell, 1)\), then \( s_n \) is subsequentially convergent.

**Proof.** Considering identity (2), we have

\[
\Delta s_n = \frac{v_n^{(0)}(\Delta s)}{\alpha_n} + \Delta v_n^{(0)}(\Delta s).
\]

Since \( \{s_n\} \) be bounded, then so is \( \frac{v_n^{(0)}(\Delta s)}{\alpha_n} \). By identity (8) and slow oscillation of \( \{\Delta s_n\}, \{\Delta v_n^{(0)}(\Delta s)\} \) is slowly oscillating with respect to summability \((\ell, 1)\). Also, since

\[
\frac{1}{\ell_n} \sum_{k=0}^{n} \frac{\Delta v_k^{(0)}(\Delta s)}{k+1} = \frac{1}{\ell_n} \sum_{k=0}^{n} \frac{v_k^{(0)}(\Delta s)}{k+1} + \frac{1}{\ell_n} \frac{v_n^{(0)}(\Delta s)}{n+2} \rightarrow 0 \text{ as } n \rightarrow \infty,
\]

\( \{\Delta v_n^{(0)}(\Delta s)\} \) is \((\ell, 1)\) summable to 0. Hence, we obtain \( \Delta v_n^{(0)}(\Delta s) = o(1) \) by using Lemma 6. Also, by (8), \( \Delta s_n = o(1) \). Therefore, proof of Theorem 7 follows from Theorem 2. \( \square \)

**Remark 8.** Notice that the following conditions are some of the classical Tauberian conditions for the \((\ell, 1)\) summability which imply slow oscillation of \( \{\Delta s_n\} \):

(i) \( \{s_n\} \) is slowly oscillating with respect to summability \((\ell, 1)\), (Kwee, [6])

(ii) \( \omega_n^{(0)}(s) = O(1) \), (Kwee, [6])

(iii) \( \omega_n^{(0)}(s) = o(1) \), (Ishiguro, [5])

(iv) \( \{v_n^{(0)}(\Delta s)\} \) is slowly oscillating with respect to summability \((\ell, 1)\), (Sezer and Çanak, [9])

(v) \( v_n^{(0)}(\Delta s) = o(1) \) (Kwee, [4])

In the next theorems, we propose new conditions imposed on the general logarithmic control modulo of the oscillatory behavior of \( \{s_n\} \).

**Theorem 9.** If \( s_n \) is bounded and \( \{\Delta(t_n^{(1)}(\omega^{(r)}(s)))\} \) is slowly oscillating with respect to summability \((\ell, 1)\) for some nonnegative integer \( r \), then \( s_n \) is subsequentially convergent.
Proof. Suppose \( s_n = O(1) \). We see by using (2) that \( v_n^{(0)}(\Delta s) = t_n^{(1)}(\omega^{(0)}(s)) = O(1) \). From the identity

\[
t_n^{(1)}(\omega^{(0)}(s)) - t_n^{(2)}(\omega^{(0)}(s)) = t_n^{(1)}(\omega^{(1)}(s)),
\]

we get \( t_n^{(1)}(\omega^{(1)}(s)) = O(1) \). Continuing in the same fashion, we obtain

\[
t_n^{(1)}(\omega^{(r)}(s)) = O(1)
\]

for all integer \( r \geq 0 \), which is equivalent to

\[
(\alpha_n \Delta) v_n^{(r)}(\Delta s) = O(1).
\] (9)

Hence, we observe

\[
t_n^{(1)}(\Delta(t_n^{(1)}(\omega^{(r)}(s)))) = t_n^{(1)}(\Delta((\alpha_n \Delta) v_n^{(r)}(\Delta s))) = 1 \left( \sum_{k=0}^{n} \frac{(\alpha_k \Delta) v_k^{(r)}(\Delta s) - (\alpha_{k-1} \Delta) v_{k-1}^{(r)}(\Delta s)}{k+1} \right) + 1 \left( \frac{(\alpha_n \Delta) v_n^{(r)}(\Delta s)}{n+2} \right) \to 0
\]

as \( n \to \infty \). Combining the hypothesis of Theorem 9 and Lemma 6 yields

\[
\Delta(t_n^{(1)}(\omega^{(r)}(s))) = O((\alpha_n \Delta) v_n^{(r)}(\Delta s)) = o(1).
\] (10)

Considering identity

\[
\omega_n^{(r)}(s) - t_n^{(1)}(\omega^{(r)}(s)) = \omega_n^{(r+1)}(s),
\]

we have

\[
\Delta((\alpha_n \Delta) v_n^{(r-1)}(\Delta s)) = \frac{(\alpha_n \Delta) v_n^{(r)}(\Delta s)}{\alpha_n} + \Delta((\alpha_n \Delta) v_n^{(r)}(\Delta s)).
\]

Now, using (9) and (10), we have

\[
\Delta((\alpha_n \Delta) v_n^{(r-1)}(\Delta s)) = o(1).
\] (11)

In the light of (10) and (11), if we continue in the same manner, then we get

\[
\Delta v_n^{(0)}(\Delta s) = o(1).
\]

Therefore, taking the identity

\[
\Delta s_n = \frac{v_n^{(0)}(\Delta s)}{\alpha_n} + \Delta v_n^{(0)}(\Delta s)
\]

into account together with the assumption \( s_n = O(1) \), we conclude \( \Delta s_n = o(1) \). This completes the proof. \( \square \)

Remark 10. The following results are noteworthy.
(i) If \( \{t^{(1)}_n(\omega(r)(s))\}\) is slowly oscillating with respect to summability \((\ell, 1)\), then so is \( \{\Delta(t^{(1)}_n(\omega(r)(s)))\}\) of its backward difference.

(ii) Set \( r = 0 \) in \( \{t^{(1)}_n(\omega(r)(s))\}\). Then slow oscillation of \( \{v^{(0)}_n(\Delta s)\} = \{t^{(0)}_n(\omega(0)(s))\}\) is sufficient for subsequential convergence of a bounded sequence.

(iii) Two-sided condition \( n \log n \Delta v^{(0)}_n(\Delta s) = O(1) \) implies slow oscillation of \( \{v^{(0)}_n(\Delta s)\}\).

**Theorem 11.** Let \( \{s_n\} \) be a bounded sequence and \( \{A_n\} \) be a sequence satisfying

\[
\frac{1}{\ell_n} \sum_{k=0}^{n} |A_k|^p = O(1), \quad p > 1.
\] (12)

If

\[
\omega^{(r)}_n(s) = O(A_n)
\] (13)

for some nonnegative integer \( r \), then \( \{s_n\} \) is subsequentially convergent.

**Proof.** By (12), we see that \( \left\{ \sum_{j=0}^{n} \frac{A_j}{\alpha_j} \right\} \) is slowly oscillating with respect to summability \((\ell, 1)\). Indeed,

\[
\max_{n < k \leq [n^\lambda]} \left| \sum_{j=n+1}^{k} \frac{A_j}{\alpha_j} \right| \leq \max_{n < k \leq [n^\lambda]} \sum_{j=n+1}^{k} \frac{|A_j|}{\alpha_j} \leq \sum_{j=n+1}^{[n^\lambda]} \frac{|A_j|}{(j+1)\ell_{j-1}}
\]

\[
\leq \frac{1}{\ell_n} \sum_{j=n+1}^{[n^\lambda]} \frac{|A_j|}{(j+1)}
\]

\[
\leq \frac{1}{\ell_n} \left( \sum_{j=n+1}^{[n^\lambda]} \frac{1}{j+1} \right)^\frac{1}{p} \left( \sum_{j=n+1}^{[n^\lambda]} \frac{1}{j+1} |A_j|^p \right)^\frac{1}{p}
\]

\[
\leq \left( \frac{\ell_{[n^\lambda]} - \ell_n}{\ell_n} \right)^\frac{1}{p} \left( \frac{\ell_{[n^\lambda]}}{\ell_n} \right)^\frac{1}{p} \left( \frac{1}{\ell_{[n^\lambda]}} \sum_{j=0}^{[n^\lambda]} |A_j|^p \right)^\frac{1}{p}
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \). Taking the limit supremum as \( n \to \infty \) of both sides of the last inequality

\[
\limsup_{n \to \infty} \max_{n < k \leq [n^\lambda]} \left| \sum_{j=n+1}^{k} \frac{A_j}{\alpha_j} \right|
\]

\[
\leq \limsup_{n \to \infty} \left( \frac{\ell_{[n^\lambda]} - \ell_n}{\ell_n} \right)^\frac{1}{p} \left( \frac{\ell_{[n^\lambda]}}{\ell_n} \right)^\frac{1}{p} \limsup_{n \to \infty} \left( \frac{1}{\ell_{[n^\lambda]}} \sum_{j=0}^{[n^\lambda]} |A_j|^p \right)^\frac{1}{p}
\]
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\[
\lim_{n \to \infty} \left( \frac{\ell_{[n^\lambda]} - \ell_n}{\ell_n} \right)^{\frac{1}{\lambda}} \lim_{n \to \infty} \left( \frac{\ell_{[n^\lambda]}}{\ell_n} \right)^{\frac{1}{\lambda}} \lim_{n \to \infty} \sup_{j \in \mathbb{N}} \left( \frac{1}{\ell_{[n^\lambda]}} \sum_{j=0}^{[n^\lambda]} |A_j|^p \right)^{\frac{1}{p}}.
\]

Hence, from (12) we get

\[
\limsup_{n \to \infty} \max_{n < k \leq [n^\lambda]} \left| \sum_{j=n+1}^{k} \frac{A_j}{\alpha_j} \right| \leq (\lambda - 1)^{\frac{1}{\lambda}} \lambda^\lambda H
\]

for \( H > 0 \). Now, letting \( \lambda \to 1^+ \) in (14) gives

\[
\lim_{\lambda \to 1^+} \limsup_{n \to \infty} \max_{n < k \leq [n^\lambda]} \left| \sum_{j=n+1}^{k} \frac{A_j}{\alpha_j} \right| \leq \lim_{\lambda \to 1^+} (\lambda - 1)^{\frac{1}{\lambda}} \lambda^\lambda H = 0.
\]

Since slow oscillation of \( \left\{ \sum_{j=0}^{n} \frac{A_j}{\alpha_j} \right\} \) implies \( \frac{A_{[n]}}{\alpha_{[n]}} = o(1) \), it follows from

\[
\omega_{(r)}(s) = \alpha_n \Delta((\alpha_n \Delta)_{r-1} v_n^{(r-1)}(s)) = O(A_n)
\]

that

\[
\Delta((\alpha_n \Delta)_{r-1} v_n^{(r-1)}(s)) = o(1).
\]

By the boundedness of \( \{s_n\} \), we also have

\[
\ell_{(r)}^{(1)}(\omega^{(m)}(s)) = O(1) \text{ for each integer } m \geq 0.
\]

Considering (16) for \( m = r - 1 \), we have

\[
\ell_{(r)}^{(1)}(\omega^{(r-1)}(s)) = (\alpha_n \Delta)_{r-1} v_n^{(r-1)}(s) = O(1).
\]

Now, construct identity below using the definition of general logarithmic control modulo

\[
\Delta((\alpha_n \Delta)_{r-2} v_n^{(r-2)}(s)) = (\alpha_n \Delta)_{r-1} v_n^{(r-1)}(s) \Delta((\alpha_n \Delta)_{r-1} v_n^{(r-1)}(s)).
\]

Thus, by (15) and (17), we have

\[
\Delta((\alpha_n \Delta)_{r-2} v_n^{(r-2)}(s)) = o(1).
\]

Taking (15) and (18) into account and proceeding likewise, we accomplish

\[
\Delta v_n^{(0)}(s) = o(1).
\]

Then, since

\[
\Delta s_n = \frac{v_n^{(0)}(s)}{\alpha_n} + \Delta v_n^{(0)}(s)
\]

and \( \{s_n\} \) is bounded, we find \( \Delta s_n = o(1) \), which completes the proof.

**Remark 12.** Considering special cases of Theorem 11, we obtain the following corollaries.
(i) Take $A_n = 1$ for all integer $n \geq 0$. Then (12) and (13) reduce to $\omega_n^{(r)}(s) = O(1)$.

(ii) Take $A_n = \alpha_n \Delta_n^{(1)}(\omega^{(r)}(s))$, then by the condition

$$\frac{1}{t_n} \sum_{k=0}^{n} |\alpha_k \Delta_k(\omega^{(r)}(s))|^{p} = O(1), \quad p > 1,$$

we get subsequential convergence of a bounded sequence $\{s_n\}$ using Remark 10, since (20) necessitate that $\{t_n^{(1)}(\omega^{(r)}(s))\}$ is slowly oscillating with respect to summability $(\ell,1)$.

**Acknowledgment**

This paper is presented at the 2nd International Conference of Mathematical Sciences (ICMS 2018), 31 July 2018-06 August 2018, Maltepe University, Istanbul, Turkey.

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