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COEFFICIENT BOUNDS FOR A CERTAIN SUBCLASS OF ANALYTIC AND BI-UNIVALENT FUNCTIONS

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Abstract. In this paper, we introduce and investigate a new subclass of the analytic and bi-univalent functions in the open unit disk in the complex plane. For the functions belonging to this class, we obtain estimates on the first three coefficients in their Taylor-Maclaurin series expansion. Some interesting corollaries and applications of the results obtained here are also discussed.

1. Introduction and Preliminaries

Let A denote the class of all complex-valued analytic functions in the open unit disk $U = \{z \in \mathbb{C} : \ |z| < 1\}$ in the complex plane of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n, \ z \in U. \tag{1.1}$$

Furthermore, by S we shall denote the class of all functions in A which are univalent in U. Some of the important and well-investigated subclasses of S include the class $S^*(\alpha)$ of starlike functions of order α and the class $C(\alpha)$ of convex functions of order α $(\alpha \in [0, 1))$.

By definition

$$S^*(\alpha) = \left\{ f \in S : Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in U \right\}, \ \alpha \in [0, 1)$$

and

$$C(\alpha) = \left\{ f \in S : Re\left(1 + \frac{zf'(z)}{f(z)}\right) > \alpha, z \in U \right\}, \ \alpha \in [0, 1).$$

The above mentioned function classes have been recently investigated rather extensively in [10, 20, 26, 29] and the references therein.

It is well-known that every function $f \in S$ has an inverse f^{-1}, defined by $f^{-1}(f(z)) = z$, $z \in U$ and $f(f^{-1}(w)) = w$, $w \in D = \{w \in \mathbb{C} : |w| < r_0(f)\}$, $r_0(f) \geq 1/4$ where $f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots$.

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An analytic function f is subordinate to an analytic function ϕ, written $f(z) \prec \phi(z)$, provided there is an analytic function $u : U \to U$ with $u(0) = 0$ and $|u(z)| < 1$ satisfying $f(z) = \phi(u(z))$ (see, for example, [14]).

Ma and Minda [12] unified various subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ or $1 + \frac{zf''(z)}{f'(z)}$ is subordinate to a more superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in U, with $\phi(0) = 1$, $\phi'(0) > 0$ and ϕ maps U onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike and Ma-Minda convex functions consists of functions $f \in A$ satisfying the subordination $\frac{zf'(z)}{f(z)} \prec \phi(z)$ and $1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)$, respectively. These classes denoted, respectively, by $S^*(\phi)$ and $C(\phi)$.

An analytic function $f \in S$ is said to be bi- starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are, respectively, Ma-Minda starlike or Ma-Minda convex functions. These classes are denoted, respectively, by $S^*_\Sigma(\phi)$ and $C_\Sigma(\phi)$. In the sequel, it is assumed that ϕ is an analytic function with positive real part in U, satisfying $\phi(0) = 1$, $\phi'(0) > 0$ and $\phi(U)$ is starlike with respect to 1 and symmetric with respect to the real axis. Such a function has a series expansion of the following form:

$$\phi(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots, \; b_1 > 0. \tag{1.2}$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent. Let Σ denote the class of bi-univalent functions in U given by (1.1).

Examples of functions in the class Σ are

$$\frac{z}{1-z}, \quad \ln \frac{1}{1-z}, \quad \ln \sqrt{\frac{1+z}{1-z}}.$$

However, the familiar Koebe function is not a member of Σ. Other common examples of functions in A such as

$$\frac{2z - z^2}{2} \text{ and } \frac{z}{1-z^2}$$

are also not members of Σ.

Earlier, Brannan and Taha [3] introduced certain subclasses of bi-univalent function class Σ, namely bi-starlike function of order α denoted $S^*_\Sigma(\alpha)$ and bi-convex function of order α denoted $C_\Sigma(\alpha)$ corresponding to the function classes $S^*(\alpha)$ and $C(\alpha)$, respectively. Thus, following Brannan and Taha [3], a function $f \in \Sigma$ is in the classes $S^*_\Sigma(\alpha)$ and $C_\Sigma(\alpha)$, respectively, if each of the following conditions is satisfied:

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in U, \; Re\left(\frac{zg'(w)}{g(w)}\right) > \alpha, w \in D$$

and

$$Re\left(1+\frac{zf'(z)}{f(z)}\right)>\alpha, z\in U,\ Re\left(1+\frac{zg'(w)}{g(w)}\right)>\alpha, w\in D.$$

For each of the function classes $S_{\Sigma}^{*}(\alpha)$ and $C_{\Sigma}(\alpha)$, they found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

Lewin [11] investigated bi-univalent function class Σ and showed that $|a_2|<1.51$. Subsequently, Brannan and Clunie [2] conjectured that $|a_2|<\sqrt{2}$.

For a brief history and interesting examples of functions which are in the class Σ, together with various other properties of this bi-univalent function class, one can refer the work of Srivastava et al. [22] and references therein. In [22], Srivastava et al. reviewed the study of coefficient problems for bi-univalent functions. Also, various subclasses of bi-univalent function class were introduced and non-sharp estimates on the first two coefficients in the Taylor-Maclaurin series expansion (1.1) were found in several recent investigations (see, for example, [1, 4, 5, 6, 7, 8, 9, 13, 15, 19, 21, 23, 24, 25, 27, 28]. Recently, Orhan et al. [17] reviewed the study of coefficient problems for the subclass $\mathrm{NP}_{\Sigma}^{\mu,\lambda}(\beta,h)$ of bi-univalent functions.

However, the problem to find the coefficient bounds on $|a_n|$, $n=3,4,...$ for functions $f\in\Sigma$ is presumably still an open problem (see, for example [2, 11, 16]).

Inspired by the aforementioned works, we define a subclass of Σ as follows.

Definition 1.1. A function $f\in\Sigma$ given by (1.1) is said to be in the class $M_{\Sigma}(\phi,\beta)$, $\beta\geq 0$, where ϕ is an analytic function given by (1.2), if the following conditions are satisfied:

$$\left(\frac{zf'(z)}{f(z)}\right)^{\beta}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\beta}\prec\phi(z),\ z\in U,$$

$$\left(\frac{zg'(w)}{g(w)}\right)^{\beta}\left(1+\frac{zg''(w)}{g'(w)}\right)^{1-\beta}\prec\phi(w), w\in D,$$

where $g=f^{-1}$.

Remark 1.2. Taking $\beta=1$, we have $M_{\Sigma}(\phi,1)=S_{\Sigma}^{*}(\phi)$; that is,
$\frac{zf'(z)}{f(z)}\prec\phi(z), z\in U$ and $\frac{zg'(w)}{g(w)}\prec\phi(w), w\in D$
if and only if $f\in S_{\Sigma}^{*}(\phi)$, where $g=f^{-1}$.

Remark 1.3. Taking $\beta=0$, we have $M_{\Sigma}(\phi,0)=C_{\Sigma}(\phi)$; that is,
$1+\frac{zf'(z)}{f(z)}\prec\phi(z), z\in U$ and $1+\frac{zg'(w)}{g(w)}\prec\phi(w), w\in D$ if and only if $f\in C_{\Sigma}(\phi)$, where $g=f^{-1}$.

Remark 1.4. These classes $S_{\Sigma}^{*}(\phi)$ and $C_{\Sigma}(\phi)$ were investigated by Ma and Minda [12].

The object of this paper is to introduce a new subclass $M_{\Sigma}(\phi,\beta)$ of the function class Σ that is wider (respect to β) to the subclasses examined so far and to find estimates on the first three Taylor-Maclaurin coefficients $|a_2|$, $|a_3|$ and $|a_4|$ for the functions in this class.

To prove our main results, we have to recall the following well-known Lemma [18].

Lemma 1.5. *Let* P *be the class of all analytic functions* $p(z)$ *of the form*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

satisfying $Re\,(p(z)) > 0,\ z \in U$ *and* $p(0) = 1$. *Then,*

$$2p_2 = p_1^2 + \left(4 - p_1^2\right) x,$$

$$4p_3 = p_1^3 + 2\left(4 - p_1^2\right) p_1 x - \left(4 - p_1^2\right) p_1 x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right) z,$$

for some $x,\ z$ with $|x| \le 1,\ \ |z| \le 1$ *and* $p_1 \in [0, 2]$.

2. Coefficient bounds for the function class $M_\Sigma(\phi, \beta)$

In this section, we will try to find the estimates on the coefficients $|a_2|$, $|a_3|$ and $|a_4|$ for the functions in the class $M_\Sigma(\phi, \beta)$.

Theorem 2.1. *Let the function* $f(z)$ *given by* (1.1) *be in the class* $M_\Sigma(\phi, \beta)$, $\beta \in [0, 1]$, *where* ϕ *is an analytic function given by* (1.2). *Then,*

$$|a_2| \le \frac{b_1}{2-\beta}, |a_3| \le \begin{cases} \frac{b_1^2}{(2-\beta)^2}, & \text{if } b_1 \le \frac{(2-\beta)^2}{2(3-2\beta)}, \\ \frac{b_1}{2(3-2\beta)}, & \text{if } b_1 > \frac{(2-\beta)^2}{2(3-2\beta)} \end{cases}$$

and

$$|a_4| \le \min\left\{ \frac{\left|b_1^3 \varphi(\beta) - 6(2-\beta)^3 \Lambda\right| + 6(2-\beta)^3 |2b_2 - b_1|}{18(2-\beta)^3(4-3\beta)}, \frac{b_1}{3(4-3\beta)} \right\},$$

where $\varphi(\beta) = \beta^3 - 3\beta^2 - 46\beta + 60 > 0$ and $\Lambda = \Lambda(b_1, b_2, b_3) = b_1 - 2b_2 + b_3$.

Proof. Let $f \in M_\Sigma(\phi, \beta)$, $\beta \in [0, 1]$, where ϕ is an analytic function given by (1.2) and $g = f^{-1}$. Then, there are analytic functions $u : U \to U$, $v : D \to D$ with $u(0) = 0 = v(0)$, $|u(z)| < 1$, $|v(w)| < 1$ and satisfying

$$\begin{aligned} &\left(\frac{zf'(z)}{f(z)}\right)^{\beta}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\beta} = \phi(u(z)) \\ \text{and } &\left(\frac{wg'(w)}{g(w)}\right)^{\beta}\left(1 + \frac{wg''(w)}{g'(w)}\right)^{1-\beta} = \phi(v(w)). \end{aligned} \tag{2.1}$$

Let us define the functions $p(z)$ and $q(w)$ by

$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + \sum_{n=1}^{\infty} p_n z^n, z \in U$ and $q(w) = \frac{1+v(w)}{1-v(w)} = 1 + \sum_{n=1}^{\infty} q_n w^n, w \in D$.

It follows that

$$u(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\left\{p_1z+\left[p_2-\frac{p_1^2}{2}\right]z^2+\left[p_3-p_1p_2+\frac{p_1^3}{4}\right]z^3+\cdots\right\} \quad (2.2)$$

and

$$v(w)=\frac{q(w)-1}{q(w)+1}=\frac{1}{2}\left\{q_1w+\left[q_2-\frac{q_1^2}{2}\right]w^2+\left[q_3-q_1q_2+\frac{q_1^3}{4}\right]w^3+\cdots\right\}. \quad (2.3)$$

Using (2.2) and (2.3) in (1.2), we can easily write

$$\begin{aligned}\phi\left(u(z)\right)=1&+\tfrac{b_1p_1}{2}z+\left[\tfrac{b_1}{2}\left(p_2-\tfrac{p_1^2}{2}\right)+\tfrac{1}{4}b_2p_1^2\right]z^2\\&+\left[\tfrac{b_1}{2}\left(p_3-p_1p_2+\tfrac{p_1^3}{4}\right)+\tfrac{b_2p_1}{2}\left(p_2-\tfrac{p_1^2}{2}\right)+\tfrac{b_3p_1^3}{8}\right]z^3+\cdots\end{aligned} \quad (2.4)$$

and

$$\begin{aligned}\phi\left(v(w)\right)=1&+\tfrac{b_1q_1}{2}w+\left[\tfrac{b_1}{2}\left(q_2-\tfrac{q_1^2}{2}\right)+\tfrac{1}{4}b_2q_1^2\right]w^2\\&+\left[\tfrac{b_1}{2}\left(q_3-q_1q_2+\tfrac{q_1^3}{4}\right)+\tfrac{b_2q_1}{2}\left(q_2-\tfrac{q_1^2}{2}\right)+\tfrac{b_3q_1^3}{8}\right]w^3+\cdots.\end{aligned} \quad (2.5)$$

Also, using (2.4) and (2.5) in (2.1) and equating the coefficients, we get

$$(2-\beta)\,a_2=\frac{b_1p_1}{2}, \quad (2.6)$$

$$2\,(3-2\beta)\,a_3+\frac{1}{2}\left(\beta^2+5\beta-8\right)a_2^2=\frac{b_1}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}b_2p_1^2, \quad (2.7)$$

$$\begin{aligned}3\,(4-3\beta)\,a_4+\left(4\beta^2+11\beta-18\right)a_2a_3-\tfrac{1}{6}\left(\beta^3+21\beta^2+20\beta-48\right)a_2^3\\=\tfrac{b_1}{2}\left(p_3-p_1p_2+\tfrac{p_1^3}{4}\right)+\tfrac{b_2p_1}{2}\left(p_2-\tfrac{p_1^2}{2}\right)+\tfrac{b_3p_1^3}{8}\end{aligned} \quad (2.8)$$

and

$$-\,(2-\beta)\,a_2=\frac{b_1q_1}{2}, \quad (2.9)$$

$$-\,2\,(3-2\beta)\,a_3+\frac{1}{2}\left(\beta^2-11\beta+16\right)a_2^2=\frac{b_1}{2}\left(q_2-\frac{q_1^2}{2}\right)+\frac{1}{4}b_2q_1^2, \quad (2.10)$$

$$\begin{aligned}-3\,(4-3\beta)\,a_4+\left(4\beta^2-34\beta+42\right)a_2a_3+\tfrac{1}{6}\left(\beta^3-27\beta^2+158\beta-192\right)a_2^3\\=\tfrac{b_1}{2}\left(q_3-q_1q_2+\tfrac{p_1^3}{4}\right)+\tfrac{b_2q_1}{2}\left(q_2-\tfrac{q_1^2}{2}\right)+\tfrac{b_3q_1^3}{8}\end{aligned} \quad (2.11)$$

From (2.6) and (2.9), we have

$$a_2=\frac{b_1p_1}{2\,(2-\beta)}=\frac{-b_1q_1}{2\,(2-\beta)}, \quad (2.12)$$

which is equivalent to

$$p_1=-q_1. \quad (2.13)$$

By subtracting from (2.7) to (2.10) and considering (2.12) and (2.13), we can easily obtain

$$a_3 = \frac{b_1^2 p_1^2}{4\left(2-\beta\right)^2} + \frac{b_1\left(p_2 - q_2\right)}{8\left(3-2\beta\right)}. \tag{2.14}$$

On the other hand, subtracting (2.11) from (2.8) and considering (2.12) and (2.14), we get

$$\begin{aligned} a_4 = &\tfrac{b_1^3 p_1^3 \varphi(\beta)}{144(2-\beta)^3(4-3\beta)} + \tfrac{5b_1^2 p_1 (p_2-q_2)}{32(2-\beta)(3-2\beta)} + \tfrac{b_1(p_3-q_3)}{12(4-3\beta)} \\ &+ \tfrac{(b_2-b_1)p_1(p_2+q_2)}{12(4-3\beta)} - \tfrac{p_1^3\Lambda}{24(4-3\beta)}, \end{aligned} \tag{2.15}$$

where $\varphi\left(\beta\right) = \beta^3 - 3\beta^2 - 46\beta + 60 > 0$ and $\Lambda{=}\Lambda\left(b_1, b_2, b_3\right) = b_1 - 2b_2 + b_3$.

Since $p_1 = -q_1$, according to Lemma 1.5 we write

$$p_2 - q_2 = \frac{4-p_1^2}{2}\left(x-y\right),\ p_2 + q_2 = p_1^2 + \frac{4-p_1^2}{2}\left(x+y\right) \tag{2.16}$$

and

$$\begin{aligned} p_3 - q_3 = &\tfrac{p_1^3}{2} + \tfrac{p_1\left(4-p_1^2\right)}{2}\left(x+y\right) - \tfrac{p_1\left(4-p_1^2\right)}{4}\left(x^2+y^2\right) \\ &+ \tfrac{4-p_1^2}{2}\left[\left(1-|x|^2\right)z - \left(1-|y|^2\right)w\right]. \end{aligned} \tag{2.17}$$

for some x, y, z, w with $|x| \le 1$, $|y| \le 1, |z| \le 1, |w| \le 1$. In this case, since $p_1 \in [0,2]$, we may assume without any restriction that $t \in [0,2]$, where $t = |p_1|$. Hence, we find from (2.12) that

$$|a_2| \le \frac{b_1}{2-\beta}.$$

Substituting the first expression (2.16) in (2.14), we obtain

$$a_3 = \frac{b_1^2 p_1^2}{4\left(2-\beta\right)^2} + \frac{b_1\left(4-p_1^2\right)}{16\left(3-2\beta\right)}\left(x-y\right).$$

Applying triangle inequality on the last equation and taking $\xi = |x|, \eta = |y|$, we have

$$|a_3| \le c_1(t) + c_2(t)\left(\xi + \eta\right), \tag{2.18}$$

where

$$c_1(t) = \frac{b_1^2 t^2}{4\left(2-\beta\right)^2} \ge 0,\ c_2(t) = \frac{b_1\left(4-t^2\right)}{16\left(3-2\beta\right)} \ge 0, t \in [0,2].$$

Let us define the function $F : \mathbb{R}^3 \to \mathbb{R}$ as follows:

$$F(\xi, \eta, t) = c_1(t) + c_2(t)\left(\xi + \eta\right),\ \ (\xi, \eta) \in \Omega,\ t \in [0,2], \tag{2.19}$$

where $\Omega = \{(\xi, \eta) : \xi, \eta \in [0,1]\}$.

From (2.18) and (2.19), we can write

$$|a_3| \le \min \{\max \{F(\xi,\eta,t) : (\xi,\eta) \in \Omega\} : t \in [0,2]\}. \tag{2.20}$$

We can easily show that

$$\max \{F(\xi,\eta,t) : (\xi,\eta) \in \Omega\} = F(1,1,t) = c_1(t) + 2c_2(t), \ t \in [0,2]. \tag{2.21}$$

Now, let us define the function $H : \mathbb{R} \to \mathbb{R}$ as follows:

$$H(t) = c_1(t) + 2c_2(t), \ t \in [0,2].$$

Substituting the value of $c_1(t)$ and $c_2(t)$ in the above function, we have

$$H(t) = \frac{b_1}{2(3-2\beta)} + \frac{\Delta(\beta,b_1)}{8(3-2\beta)(2-\beta)^2}t^2, \tag{2.22}$$

where $\Delta(\beta,b_1) = 2(3-2\beta)b_1^2 - (2-\beta)^2 b_1$.
Differentiating both sides of (2.22), we get

$$H'(t) = \frac{\Delta(\beta,b_1)}{4(3-2\beta)(2-\beta)^2}t.$$

It is clear that $H'(t) \le 0$ if $0 < b_1 \le \frac{(2-\beta)^2}{2(3-2\beta)}$; that is, $H(t)$ is a decreasing function. Therefore,

$$\min \{H(t) : \ t \in [0,2]\} = H(2) = \frac{b_1^2}{(2-\beta)^2}. \tag{2.23}$$

Let $b_1 > \frac{(2-\beta)^2}{2(3-2\beta)}$, then $H'(t) > 0$, so $H(t)$ is a strictly increasing function.
Therefore,

$$\min \{H(t) : \ t \in [0,2]\} = H(0) = \frac{b_1}{2(3-2\beta)}. \tag{2.24}$$

Consequently, from (2.21)-(2.24) and (2.20), we have

$$|a_3| \le \begin{cases} \frac{b_1^2}{(2-\beta)^2}, & \text{if } b_1 \le \frac{(2-\beta)^2}{2(3-2\beta)}, \\ \frac{b_1}{2(3-2\beta)}, & \text{if } b_1 > \frac{(2-\beta)^2}{2(3-2\beta)}. \end{cases} \tag{2.25}$$

Substituting the expressions (2.16) and (2.17) in (2.15), we obtain

$$\begin{aligned} a_4 = {} & \tfrac{b_1\left(4-p_1^2\right)}{24(4-3\beta)}\left[\left(1-|x|^2\right)z - \left(1-|y|^2\right)w\right] - \tfrac{b_1\left(4-p_1^2\right)p_1}{48(4-3\beta)}\left(x^2+y^2\right) \\ & + \tfrac{b_2\left(4-p_1^2\right)p_1}{24(4-3\beta)}(x+y) + \tfrac{5b_1^2\left(4-p_1^2\right)p_1}{64(2-\beta)(3-2\beta)}(x-y) \\ & + \tfrac{b_1^3\varphi(\beta)-6(2-\beta)^3\Lambda+6(2-\beta)^3(2b_2-b_1)}{144(2-\beta)^3(4-3\beta)}p_1^3. \end{aligned}$$

Applying triangle inequality on the last equation, we have

$$|a_4| \leq d_1(t)\left(\xi^2+\eta^2\right)+d_2(t)(\xi+\eta)+d_3(t), \tag{2.26}$$

where

$$d_1(t)=\frac{b_1\left(4-t^2\right)(t-2)}{48(4-3\beta)} \leq 0,$$

$$d_2(t)=\frac{\left(4-t^2\right)t\left[8|b_2|(2-\beta)(3-2\beta)+15b_1^2(4-3\beta)\right]}{192(2-\beta)(3-2\beta)(4-3\beta)} \geq 0,$$

$$d_3(t)=\frac{\left|b_1^3\varphi(\beta)-6(2-\beta)^3\Lambda\right|+6(2-\beta)^3|2b_2-b_1|}{144(2-\beta)^3(4-3\beta)}t^3+\frac{b_1\left(4-t^2\right)}{12(4-3\beta)} \geq 0.$$

Let us define the function $G:\mathbb{R}^3 \to \mathbb{R}$ as follows:

$$G(\xi,\eta,t)=d_1(t)\left(\xi^2+\eta^2\right)+d_2(t)(\xi+\eta)+d_3(t),\ (\xi,\eta)\in\Omega,\ t\in[0,2]. \tag{2.27}$$

From (2.26) and (2.27), we can write

$$|a_4| \leq \min\{\max\{G(\xi,\eta,t):(\xi,\eta)\in\Omega\}:t\in[0,2]\}. \tag{2.28}$$

Firstly, we need investigate maximum of the function $G(\xi,\eta,t)$ on the closed square Ω for each $t\in[0,2]$. Since the coefficients of the function $G(\xi,\eta,t)$ is dependent to variable t, we must investigate this maximum respect to t taking into account these cases: $t=0,\ t\in(0,2)$ and $t=2$.

For $t=0$ we have

$$G_0(\xi,\eta)=G(\xi,\eta,0)=\frac{-b_1}{6(4-3\beta)}\left(\xi^2+\eta^2\right)+\frac{b_1}{3(4-3\beta)},\ (\xi,\eta)\in\Omega.$$

We can easily show that the maximum of the function $G_0(\xi,\eta)$ occurs at $(\xi,\eta)=(0,0)$, and

$$\max\{G_0(\xi,\eta):(\xi,\eta)\in\Omega\}=G_0(0,0)=\frac{b_1}{3(4-3\beta)}. \tag{2.29}$$

In the case $t\in(0,2)$, by simple differentiation, we get

$$G_\xi^{'}(\xi,\eta,t)=2d_1(t)\xi+d_2(t),\ G_\eta^{'}(\xi,\eta,t)=2d_1(t)\eta+d_2(t),$$
$$G_{\xi\xi}^{''}(\xi,\eta,t)=G_{\eta\eta}^{''}(\xi,\eta,t)=2d_1(t),\ G_{\xi\eta}^{''}(\xi,\eta,t)=G_{\eta\xi}^{''}(\xi,\eta,t)=0.$$

From the first and second equations above, we see that (ξ_0,η_0), where $\xi_0=\eta_0=\frac{-d_2(t)}{2d_1(t)}$, is critical and likely a extremal point for of the function $G(\xi,\eta,t)$.

Since

$$\Delta(\xi_0,\eta_0)=G_{\xi\xi}^{''}(\xi_0,\eta_0,t)G_{\eta\eta}^{''}(\xi_0,\eta_0,t)-\left[G_{\xi\xi}^{''}(\xi_0,\eta_0,t)\right]^2=4d_1^2(t)>0$$

and $G''_{\xi\xi}(\xi,\eta,t) = G''_{\eta\eta}(\xi,\eta,t) = 2d_1(t) < 0$, (ξ_0,η_0) is a likely maximum point for the function $G(\xi,\eta,t)$. But, it is clear that (ξ_0,η_0) is not a local maximum point if $\frac{-d_2(t)}{2d_1(t)} > 0$; that is if $(\xi_0,\eta_0) \notin \Omega$. We assume that $(\xi_0,\eta_0) \in \Omega$. In this case (ξ_0,η_0) is a local maximum point for the function $G(\xi,\eta,t)$.

Therefore,

$$\max\{G(\xi,\eta,t) : (\xi,\eta) \in \Omega\} = G(\xi_0,\eta_0,t) = d_3(t) - \frac{-d_2^2(t)}{2d_1(t)}.$$

Let us define the function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(t) = d_3(t) - \frac{-d_2^2(t)}{2d_1(t)},\ t \in (0,2).$$

Substituting the value $d_1(t)$, $d_2(t)$ and $d_3(t)$ in the above function, we have

$$h(t) = h_1 t^3 + h_2 t^2 + h_3,\ t \in (0,2), \tag{2.30}$$

where

$$h_1 = \frac{\left|b_1^3\varphi(\beta) - 6(2-\beta)^3\Lambda\right| + 6(2-\beta)^3|2b_2 - b_1|}{144(2-\beta)^3(4-3\beta)} + \frac{\left[8|b_2|(2-\beta)(3-2\beta) + 15b_1^2(4-3\beta)\right]^2}{1536(2-\beta)^2(3-2\beta)^2(4-3\beta)b_1} > 0,$$

$$h_2 = \frac{\left[8|b_2|(2-\beta)(3-2\beta) + 15b_1^2(4-3\beta)\right]^2}{768(2-\beta)^2(3-2\beta)^2(4-3\beta)b_1} - \frac{b_1}{12(4-3\beta)},$$

$$h_3 = \frac{b_1}{3(4-3\beta)} > 0.$$

Also, we consider the function $\bar{h} : \mathbb{R} \to \mathbb{R}$ as follows:

$$\bar{h}(t) = h_1 t^3 + \bar{h}_2 t^2 + h_3,\ t \in (0,2), \tag{2.31}$$

where

$$\bar{h}_2 = h_2 + \frac{b_1}{12(4-3\beta)} = \frac{\left[8|b_2|(2-\beta)(3-2\beta) + 15b_1^2(4-3\beta)\right]^2}{768(2-\beta)^2(3-2\beta)^2(4-3\beta)b_1} > 0.$$

Since $h(t) < \bar{h}(t)$ for all $t \in (0,2)$, we can write

$$\min\{h(t) : t \in (0,2)\} \le \min\{\bar{h}(t) : t \in (0,2)\}. \tag{2.32}$$

Now, we will investigate minimum of the function $\bar{h}(t)$ on the open interval $(0,2)$.

Differentiating both sides of (2.31), we have

$$\bar{h}'(t) = \left(3h_1 t + 2\bar{h}_2\right)t,\ t \in (0,2).$$

Since $h_1 > 0$, $\bar{h}_2 > 0$, the function $\bar{h}(t)$ is a strictly increasing function on $(0,2)$. Therefore,

$$\min\{\bar{h}(t) : t \in (0,2)\} = \bar{h}(0+) = \lim_{t\to 0+}\bar{h}(t) = \frac{b_1}{3(4-3\beta)}. \tag{2.33}$$

Finally, let $t = 2$. In this case the function $G(\xi, \eta, 2)$ is a constant as follows:

$$G_2(\xi,\eta) = G(\xi,\eta,2) = d_3\,(2) = \frac{\left|b_1^3\varphi\,(\beta) - 6\,(2-\beta)^3\,\Lambda\right| + 6\,(2-\beta)^3\,|2b_2 - b_1|}{18\,(2-\beta)^3\,(4-3\beta)}. \tag{2.34}$$

Thus, from (2.29)-(2.34) and (2.28), we obtain

$$|a_4| \le \min\left\{\frac{\left|b_1^3\varphi\,(\beta) - 6\,(2-\beta)^3\,\Lambda\right| + 6\,(2-\beta)^3\,|2b_2 - b_1|}{18\,(2-\beta)^3\,(4-3\beta)}, \frac{b_1}{3\,(4-3\beta)}\right\}.$$

With this, the proof of Theorem 2.1 is completed. □

The following theorems are direct results of Theorem 2.1.

Theorem 2.2. *Let the function $f(z)$ given by* (1.1) *be in the class $S_\Sigma^*(\phi)$, where ϕ is an analytic function given by* (1.2). *Then,*

$$|a_2| \le b_1, |a_3| \le \begin{cases} b_1^2, & \text{if } b_1 \le \frac{1}{2}, \\ \frac{b_1}{2}, & \text{if } b_1 > \frac{1}{2} \end{cases}$$

and

$$|a_4| \le \min\left\{\frac{\left|2b_1^3 - \Lambda\right| + |2b_2 - b_1|}{3}, \frac{b_1}{3}\right\},$$

where $\Lambda = \Lambda(b_1, b_2, b_3) = b_1 - 2b_2 + b_3$.

Theorem 2.3. *Let the function $f(z)$ given by* (1.1) *be in the class $C_\Sigma(\phi)$, where ϕ is an analytic function given by* (1.2). *Then,*

$$|a_2| \le \frac{b_1}{2}, |a_3| \le \begin{cases} \frac{b_1^2}{4}, & \text{if } b_1 \le \frac{2}{3}, \\ \frac{b_1}{6}, & \text{if } b_1 > \frac{2}{3} \end{cases}$$

and

$$|a_4| \le \min\left\{\frac{\left|5b_1^3 - 4\Lambda\right| + 4\,|2b_2 - b_1|}{48}, \frac{b_1}{12}\right\},$$

where $\Lambda = \Lambda(b_1, b_2, b_3) = b_1 - 2b_2 + b_3$.

3. Concluding remarks

If the function $\phi(z)$, aforementioned in study, is given by

$$\phi(z) = \frac{1+az}{1+bz} = 1 + (a-b)z - b(a-b)z^2 + b^2(a-b)z^3 + \cdots \quad (-1 \le b < a \le 1), \tag{3.1}$$

then $b_1 = (a-b)$, $b_2 = -b(a-b)$ and $b_3 = b^2(a-b)$.

Taking $a = 1 - 2\alpha,\ b = -1$ in (3.1), we have

$$\phi(z) = \frac{1+(1-2\alpha)z}{1-z} = 1 + 2(1-\alpha)z + 2(1-\alpha)z^2 + 2(1-\alpha)z^3 + \cdots \quad (0 \le \alpha < 1). \tag{3.2}$$

Hence, $b_1 = b_2 = b_3 = 2(1-\alpha)$.

Choosing $\phi(z)$ of the form (3.1) and (3.2) in Theorem 2.1, we can readily deduce the following results, respectively.

Corollary 3.1. *Let the function* $f(z)$ *given by* (1.1) *be in the class* $M_\Sigma\left(\frac{1+az}{1+bz}, \beta\right)$ $(-1 \le b < a \le 1,\ 0 \le \beta \le 1)$. *Then,*

$$|a_2| \le \frac{a-b}{2-\beta},\ |a_3| \le \begin{cases} \frac{(a-b)^2}{(2-\beta)^2}, & \text{if } a-b \le \frac{(2-\beta)^2}{2(3-2\beta)}, \\ \frac{a-b}{2(3-2\beta)}, & \text{if } a-b > \frac{(2-\beta)^2}{2(3-2\beta)} \end{cases} \quad \textit{and } |a_4| \le \frac{a-b}{3(4-3\beta)}.$$

Corollary 3.2. *Let the function* $f(z)$ *given by* (1.1) *be in the class* $M_\Sigma\left(\frac{1+(1-2\alpha)z}{1-z}, \beta\right)$ $= M_\Sigma(\alpha, \beta),\ \alpha \in [0,1),\ \beta \in [0,1]$. *Then,*

$$|a_2| \le \frac{2(1-\alpha)}{2-\beta},\ |a_3| \le \begin{cases} \frac{1-\alpha}{3-2\beta}, & \text{if } 0 \le \alpha < 1-\alpha_0, \\ \frac{4(1-\alpha)^2}{(2-\beta)^2}, & \text{if } 1-\alpha_0 \le \alpha < 1, \end{cases} \quad \textit{and } |a_4| \le \frac{2(1-\alpha)}{3(4-3\beta)},$$

where $\alpha_0 = \frac{(2-\beta)^2}{4(3-2\beta)}$.
Also, taking $\alpha = 0$ *in* (3.2), *we get*

$$\phi(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + 2z^3 + \cdots\ . \tag{3.3}$$

Hence, $b_1 = b_2 = b_3 = 2$.

Choosing $\phi(z)$ of the form (3.3) in Theorem 2.1, we arrive at the following corollary.

Corollary 3.3. *Let the function* $f(z)$ *given by* (1.1) *be in the class* $M_\Sigma\left(\frac{1+z}{1-z}, \beta\right)$, $\beta \in [0,1]$. *Then,*
$|a_2| \le \frac{2}{2-\beta},\ |a_3| \le \frac{1}{3-2\beta}$ *and* $|a_4| \le \frac{2}{3(4-3\beta)}$.

Choosing $\phi(z)$ of the form (3.1) and (3.2) in Theorem 2.2, we can readily deduce the following results, respectively.

Corollary 3.4. *Let the function* $f(z)$ *given by* (1.1) *be in the class* $S^*_\Sigma\left(\frac{1+az}{1+bz}\right)$ $(-1 \le b < a \le 1)$. *Then,*

$$|a_2| \le a-b,\ |a_3| \le \begin{cases} (a-b)^2, & \text{if } a-b \le \frac{1}{2}, \\ \frac{a-b}{2}, & \text{if } a-b > \frac{1}{2} \end{cases} \quad \textit{and } |a_4| \le \frac{a-b}{3}.$$

Corollary 3.5. *Let the function* $f(z)$ *given by* (1.1) *be in the class*$S^*_\Sigma\left(\frac{1+(1-2\alpha)z}{1-z}\right)$ $= S^*_\Sigma(\alpha),\ \alpha \in [0,1)$. *Then,*

$$|a_2| \le 2(1-\alpha),\ |a_3| \le \begin{cases} 1-\alpha, & \text{if } 0 \le \alpha < \frac{3}{4}, \\ 4(1-\alpha)^2, & \text{if } \frac{3}{4} \le \alpha < 1 \end{cases} \quad \textit{and } |a_4| \le \frac{2(1-\alpha)}{3}.$$

Remark 3.6. In the special case, we can also obtain Corollary 3.4 from Corollary 3.1 and Corollary 3.5 from Corollary 3.2 for $\beta = 1$.

Moreover, taking, for example, $\alpha = \frac{3}{4}$ in (3.2), we have

$$\phi(z) = \frac{2-z}{2\,(1-z)} = 1 + \frac{1}{2}z + \frac{1}{2}z^2 + \frac{1}{2}z^3 + \cdots \ . \tag{3.4}$$

Hence, $b_1 = b_2 = b_3 = \frac{1}{2}$.

Choosing $\phi(z)$ of the form (3.4) in Theorem 2.2, we arrive at the following corollary.

Corollary 3.7. *Let the function $f(z)$ given by (1.1) be in the class $S^*_\Sigma\left(\frac{2-z}{2(1-z)}\right)$. Then,*
$|a_2| \le \frac{1}{2}$, $|a_3| \le \frac{1}{4}$ *and* $|a_4| \le \frac{1}{6}$.

Remark 3.8. In the special case, we can also obtain Corollary 3.7 from Corollary 3.5 for $\alpha = \frac{3}{4}$.

Choosing $\phi(z)$ of the form (3.1) and (3.2) in Theorem 2.3, we can readily deduce the following results, respectively.

Corollary 3.9. *Let the function $f(z)$ given by (1.1) be in the class $C_\Sigma\left(\frac{1+az}{1+bz}\right)$ $(-1 \le b < a \le 1)$. Then,*
$|a_2| \le \frac{a-b}{2}$, $|a_3| \le \begin{cases} \frac{(a-b)^2}{4}, & \text{if } a-b \le \frac{2}{3}, \\ \frac{a-b}{6}, & \text{if } a-b > \frac{2}{3} \end{cases}$ *and* $|a_4| \le \frac{a-b}{12}$.

Corollary 3.10. *Let the function $f(z)$ given by (1.1) be in the class $C_\Sigma\left(\frac{1+(1-2\alpha)z}{1-z}\right)$ $= C_\Sigma(\alpha),\ \alpha \in [0,1)$. Then,*
$|a_2| \le 1-\alpha$, $|a_3| \le \begin{cases} \frac{1-\alpha}{3}, & \text{if } \ 0 \le \alpha < \frac{2}{3}, \\ (1-\alpha)^2, & \text{if } \ \frac{2}{3} \le \alpha < 1 \end{cases}$ *and* $|a_4| \le \frac{1-\alpha}{6}$.

Moreover, taking, for example, $\alpha = \frac{2}{3}$ in (3.2), we get

$$\phi(z) = \frac{3-z}{3\,(1-z)} = 1 + \frac{2}{3}z + \frac{2}{3}z^2 + \frac{2}{3}z^3 + \cdots \ . \tag{3.5}$$

Hence, $b_1 = b_2 = b_3 = \frac{2}{3}$.

Choosing $\phi(z)$ of the form (3.5) in Theorem 2.3, we arrive at the following corollary.

Corollary 3.11. *Let the function $f(z)$ given by (1.1) be in the class$C_\Sigma\left(\frac{3-z}{3(1-z)}\right)$. Then,*
$|a_2| \le \frac{1}{3}$, $|a_3| \le \frac{1}{9}$ *and* $|a_4| \le \frac{1}{18}$.

Remark 3.12. In the special case, we can also obtain Corollary 3.11 from Corollary 3.10 for $\alpha = \frac{2}{3}$.

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