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A new family of $k-$ Gaussian Fibonacci numbers

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Abstract

In this manuscript, a new family of $k-$ Gaussian Fibonacci numbers has been identified and some relationships between this family and known Gaussian Fibonacci numbers have been found. Also, I the generating functions of this family for $k = 2$ has been obtained.

Key Words: *Fibonacci numbers, Gaussian fibonacci numbers, Gaussian numbers.*

k- Gaussian Fibonacci sayılarının yeni bir ailesi

Özet

Bu yazıda, yeni bir k - Gaussian Fibonacci sayıları ailesi tanımlanmış ve bu aile ile bilinen Gaussian Fibonacci sayıları arasında bazı ilişkiler bulunmuştur. Ayrıca, k=2 için bu ailenin üreteç fonksiyonlarını elde edilmiştir.

Anahtar Kelimeler: *Fibonacci sayıları, Gaussian Fibonacci sayıları, Gaussian sayıları.*

1. Introduction

Horadam [1] in 1963 and Berzsenyi [2] in 1977 defined complex Fibonacci numbers. Horadam introduced the concept the complex Fibonacci numbers as the Gaussian Fibonacci numbers. Moawwad El-Mikkawy and Tomohiro Sogabe [3] in 2015 defined a new family of k- Fibonacci numbers and they gave $F_n^{(k)}$ and establish some properties of the relation to the F_n. There are many studies on Fibonacci and Gaussian Fibonacci numbers. See, e.g. [4-15].

The Binet's formula of the Fibonacci numbers are defined as follows:

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$$F_n = \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1}), \quad n = 0,1,2,\cdots$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. The first few Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, $\cdots$. For more detailed information on these numbers see [16]. The numbers F_n with the initial conditions $F_0 = 0$ and $F_1 = 1$ satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad n \geq 0.$$

The Gaussian Fibonacci numbers: GF_n for $n \geq 0$ are defined

$$GF_n = GF_{n-1} + GF_{n-2}$$

where $GF_0 = i, GF_1 = 1$. The first few Gaussian Fibonacci numbers are $i, 1, i+1, i+2, 2i+3, 3i+5, 5i+8, \cdots$.

2. A new family of k − Gaussian Fibonacci numbers

Definition 2.1. Let n and k $(k \neq 0)$ be natural numbers, then according to the division algorithm, there are m and r such that $n = mk + r,\ 0 \leq r < k$. According to this, we define a new family of generalized k −Gaussian Fibonacci numbers $GF_n^{(k)}$ by

$$GF_n^{(k)} = \left[\left(\frac{\sqrt{5}}{5} + \left(\frac{5-\sqrt{5}}{10}\right)i\right)\alpha^m + \left(-\frac{\sqrt{5}}{5} + \left(\frac{5+\sqrt{5}}{10}\right)i\right)\beta^m\right]^{k-r}$$
$$\left[\left(\frac{\sqrt{5}}{5} + \left(\frac{5-\sqrt{5}}{10}\right)i\right)\alpha^{m+1} + \left(-\frac{\sqrt{5}}{5} + \left(\frac{5+\sqrt{5}}{10}\right)i\right)\beta^{m+1}\right]^{r}.$$

For $k = 2,3$ are as follows:

$$\left\{GF_n^{(2)}\right\} = \{-1, i, 1, i+1, 2i, 3i+1, 4i+3, 7i+4, 12i+5, \cdots\},$$
$$\left\{GF_n^{(3)}\right\} = \{-i, -1, i, 1, i+1, 2i, 2i-2, 4i-2, 7i-1, \cdots\}.$$

From Definition 2.1, $GF_n^{(k)}$ and GF_n related by

$$GF_n^{(k)} = (GF_m)^{k-r}(GF_{m+1})^r.$$

If $k = 1$, we see that $m = n$ and $r = 0$. So, $GF_n^{(1)}$ is well-known Gaussian Fibonacci numbers GF_n.

3. Main results

Theorem 3.1. Let $k, m \in \{1, 2, 3, 4, \cdots\}$. For k and m, $GF_n^{(k)}$ and GF_n numbers satisfy

i) $\sum_{j=0}^{k-1}(-1)^j \binom{k-1}{j} GF_{mk+j}^{(k)} = (-1)^{k-1} GF_m GF_{(m-1)(k-1)}^{(k-1)}$
ii) $\sum_{j=0}^{k-1} \binom{k-1}{j} GF_{mk+j}^{(k)} = GF_m GF_{(m+2)(k-1)}^{(k-1)}$
iii) $\sum_{j=0}^{k-1} GF_{mk+j}^{(k)} = \frac{GF_m}{GF_{m-1}}\left[GF_{(m+1)k}^{(k)} - GF_{mk}^{(k)}\right]$

Proof.

i) I have

$$\begin{aligned}\sum_{j=0}^{k-1}(-1)^j \binom{k-1}{j} GF_{mk+j}^{(k)} &= (-1)^{k-1}\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}(GF_m)^{k-j}(GF_{m+1})^j \\ &= (-1)^{k-1}GF_m \sum_{j=0}^{k-1}\binom{k-1}{j}(GF_{m+1})^j(-GF_m)^{k-1-j} \\ &= (-1)^{k-1}GF_m[(GF_{m+1}-GF_m)^{k-1}] \\ &= (-1)^{k-1}GF_m[(GF_{m-1})^{k-1}] \\ &= (-1)^{k-1}GF_m GF_{(m-1)(k-1)}^{(k-1)}\end{aligned}$$

ii) In a similar manner, I have

$$\begin{aligned}\sum_{j=0}^{k-1}\binom{k-1}{j} GF_{mk+j}^{(k)} &= \sum_{j=0}^{k-1}\binom{k-1}{j}(GF_m)^{k-j}(GF_{m+1})^j \\ &= GF_m \sum_{j=0}^{k-1}\binom{k-1}{j}(GF_{m+1})^j(GF_m)^{k-1-j} \\ &= GF_m[(GF_{m+1}+GF_m)^{k-1}] \\ &= GF_m[(GF_{m+2})^{k-1}] \\ &= GF_m GF_{(m+2)(k-1)}^{(k-1)}\end{aligned}$$

iii) It follows I have

$$GF_{mk+j}^{(k)} = (GF_m)^{k-j}(GF_{m+1})^j = \left(\frac{GF_{m+1}}{GF_m}\right)^j (GF_m)^k.$$

Using the above equation and some algebraic operations, the desired result is obtained,

Theorem 3.2. For the $GF_n^{(2)}$, I have the following relations:

i. $GF_{2(m-1)}^{(2)} - GF_m GF_{m-2} = (-1)^{m-1}(i-2), \quad m \geq 1$
ii. $GF_n^{(2)} = GF_{n-1}^{(2)} + GF_{n-3}^{(2)} + GF_{n-4}^{(2)}, \quad n \geq 4$

Proof. i) Let A be the Fibonacci matrix of the form

$$A = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}.$$

Then, from the matrix A and definition the Gaussian Fibonacci numbers I have

$$\begin{bmatrix}GF_m & GF_{m-1}\\ GF_{m-1} & GF_{m-2}\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}^{m-1}\begin{bmatrix}GF_1 & GF_0\\ GF_0 & GF_{-1}\end{bmatrix}.$$

The determinants of both sides of the above equation are taken

$$GF_mGF_{m-2} - GF_{m-1}^2 \quad = (-1)^{m-1}(-i+2)$$

$$GF_{m-1}^2 - GF_mGF_{m-2} \quad = (-1)^{m-1}(i-2)$$

$$GF_{2(m-1)}^{(2)} - GF_mGF_{m-2} = (-1)^{m-1}(i-2), \ \left(from\ GF_{m-1}^2 = GF_{2(m-1)}^{(2)} \right).$$

ii) If n is even, i.e,
$GF_{2m}^{(2)} = GF_{2m-1}^{(2)} + GF_{2m-3}^{(2)} + GF_{2m-4}^{(2)}$.

To illustrate the above equations, I will use the following relations:

$$GF_{2m}^{(2)} = (GF_m)^2$$
$$GF_{2m+1}^{(2)} = GF_mGF_{m+1}.$$

The relations are readily obtained from Definition 2.1. Now, It can be written as

$$\begin{aligned} GF_{2m}^{(2)} &= (GF_m)^2 \\ &= GF_mGF_m \\ &= GF_m(GF_{m-1} + GF_{m-2}) \\ &= GF_{m-1}GF_m + GF_{m-2}GF_m \\ &= GF_{m-1}GF_m + GF_{m-2}(GF_{m-1} + GF_{m-2}) \\ &= GF_{m-1}GF_m + GF_{m-2}GF_{m-1} + (GF_{m-2})^2 \\ &= GF_{2m-1}^{(2)} + GF_{2m-3}^{(2)} + GF_{2m-4}^{(2)} \end{aligned}$$

Similarly, if n is odd, i.e.,

$$GF_{2m+1}^{(2)} = GF_{2m}^{(2)} + GF_{2m-2}^{(2)} + GF_{2m-3}^{(2)},$$

the desired result is obtained.

Theorem 3.3. The generating function of $GF_n^{(2)}$ are given by

$$C_n^{(2)}(x) = \frac{-1+(i+1)x+(1-i)x^2+(1+i)x^3}{1-x-x^3-x^4}.$$

Proof. I have $C_n^{(2)}(x) = \sum_{n=0}^{\infty} GF_n^{(2)} x^n$. Then

$$\begin{aligned} C_n^{(2)}(x) &= \sum_{n=0}^{\infty} GF_n^{(2)} x^n \\ -xC_n^{(2)}(x) &= \sum_{n=0}^{\infty} GF_{n-1}^{(2)} x^n \end{aligned}$$

$$-x^3C_n^{(2)}(x)=\sum_{n=0}^{\infty} GF_{n-3}^{(2)}x^n$$
$$-x^4C_n^{(2)}(x)=\sum_{n=0}^{\infty} GF_{n-4}^{(2)}x^n,$$

equations can be written. In this case

$$\begin{aligned}(1-x-x^3-x^4)C_n^{(2)}(x) &= (1-x-x^3-x^4)C_n^{(2)}(x)\\ &-\left(GF_0^{(2)}x+GF_1^{(2)}x^2+GF_2^{(2)}x^3\right)-GF_0^{(2)}x^3\\ &+\sum_{n=4}^{\infty}\left(GF_n^{(2)}-GF_{n-1}^{(2)}-GF_{n-3}^{(2)}-GF_{n-4}^{(2)}\right)x^n\\ &=GF_0^{(2)}+\left(GF_1^{(2)}-GF_0^{(2)}\right)x+\left(GF_2^{(2)}-GF_1^{(2)}\right)x^2\\ &\quad+\left(GF_3^{(2)}-GF_2^{(2)}-GF_0^{(2)}\right)x^3+0\\ &=-1+(i+1)x+(1-i)x^2+(1+i)x^3.\end{aligned}$$

Hence, $C_n^{(2)}(x)$ of $GF_n^{(2)}$ is

$$C_n^{(2)}(x)=\frac{-1+(i+1)x+(1-i)x^2+(1+i)x^3}{1-x-x^3-x^4}.$$

Finally, I give two identities without proofs:

- $\sum_{j=0}^{n} GF_{2j-1}^{(1)}=GF_{2n}^{(1)}+(1-2i),$
- $\sum_{j=0}^{n} GF_{2j-1}^{(2)}=\begin{cases}GF_{2n}^{(2)}+(i+2) & ; \quad if\ n\ is\ even\\ GF_{2n}^{(2)}+(2i) & ; \quad if\ n\ is\ odd\end{cases}.$

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